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Control of Underactuated Driftless Systems Using Higher-Order Averaging Theory

Patricio Vela[†‡], Joel W. Burdick[‡] [1]

[†]Control and Dynamical Systems [‡]Mechanical Engineering
California Institute of Technology, Mail Code 107-81, Pasadena, CA, 91125
`pvela@cds.caltech.edu, jwb@robotics.caltech.edu`

Abstract This papers applies a recently developed "generalized averaging theory" to construct stabilizing feedback control laws for underactuated driftless systems. These controls exponentialy stabilize in the average. Conditions relating the properties of the averaged and unaveraged systems are given. An example validates the theory, demonstrating its utility.

1 Introduction

A tremendous amount of research has gone into understanding controllability and determining conditions under which a system is controllable. There is still a gap, however, between the tests that determine controllablility and the actual feedback laws that realize control. This paper demonstrates how a recently developed "generalized averaging theory" [13] may be used in conjunction with controllability tests to realize feedback control for underactuated driftless systems. Our approach is an easily implementable and understandable strategy for designing exponentially stabilizing controllers for such systems. The results hold to general orders of Lie bracketing. The method does not use a homogeneous norm to demonstrate the exponential stabilization, does not require complicated coordinate expansions, the construction of Lyapunov functions, or the pre-existence of stabilizing controllers. The complexity of the nonlinear analysis grows with the order of Lie bracketing. Due to space constraints, only lower orders will be discussed. Extensions to higher order follow the same principles, but may be more involved computationally.

We seek to unite averaging theory and nonlinear control design. Our generalized averaging theory, which captures the dynamics of a system to arbitrary orders of approximation [13], has its roots in prior work on series expansions due to Magnus, Chen, and Agrač̌hev and Gamkrelidze [23, 8, 1]. Sussman and Kawski have analyzed series expansions with respect to controllability. The strongest results tying controllability analysis to control design is found in the work of Sussman and Liu [6, 7, 24].

Moving from anaylsis to the design of stabilizing feedback laws is a challenge. Bullo [4, 5] has developed series expansions for similar flows and their concomitant approximate inversion formulas in the case of simple mechanical systems and kinematic actuation.

The control of underactuated driftless systems has been widely studied. Generically, smooth state-feedback will not stabilize these systems [20], leading to the use of time-varying or non-smooth feedback techniques. In the time-varying control domain, the use of sinusoidal functions with appropriate phasing leads to motion in Lie bracket directions [17]. The analysis of open-loop response to sinusoidal inputs has been well studied [19]. Some have used averaging methods to understand the nonlinear response under time-periodic inputs [9, 24]. The closed loop problem for time-varying controls is more difficult and fewer general solutions exist. In [18], a method to transform asymptotically stabilized homogeneous systems into an exponentially stabilizing homogeneous system is given. Stabilization is with regards to a homogeneous norm. In [11], a solution requiring the use of homogeneity is found for global or local stabilization, but the algorithm is computationally difficult and stabilization is with respect to a homogeneous norm. Alternatively, methods based on non-smooth or discontinuous feedback may be used. However, these solutions are largely found in a specific application domain [2, 3, 10].

Section 2 reviews the key elements of the generalized averaging theory developed in the companion paper [13]. Section 3 applies this theory to the control of underactuated driftless systems. Section 4 demonstrates the utility of the approach via an example.

2 A Generalized Averaging Theory

The flow of the differential equation,

$$\dot{x} = X(x,t;\epsilon) = \epsilon \widehat{X}(x,t)\,, \quad x(0) = x_0, \qquad (1)$$

with X, T-periodic, i.e., $X(x,t;\epsilon) = X(x,t+T;\epsilon)$ can be analyzed by a non-linear version of Floquet theory. This approach represents the flow as the composition of a periodic flow and the evolution of an averaged vector field. These can be approximated to arbitrary order by appropriate series expansions.

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Theorem 1 (Nonlinear Floquet Theorem) *[13] Let $\Phi^X_{0,t}$ be the flow of the time-periodic differential equation (1). If the monodramy map has a logarithm, then the flow $\Phi^X_{0,t}$ can be represented as a composition of flows $\Phi^X_{0,t} = P(t) \circ \exp(Yt)\mathrm{Id}$, where P is T-periodic.*

The monodramy map is the flow of X at time T, e.g., $\Phi^X_{0,T}$. It coincides with the flow of the *averaged* autonomous vector field, Y, at time T, e.g., $\exp(YT)$.

Theorem 2 *[13] If the monodramy map has a fixed point, the actual flow has a periodic orbit whose stability is determined by the stability of the monodramy map.*

Corollary 1 *[13] If the flow of system (1) has a fixed point x^*, as does the monodramy map, then stability of the actual flow may be determined using the monodramy map. A linearly (asymptotically) stable fixed point for the monodramy map implies a linearly (asymptotically) stable fixed point for the system (1).*

Averaging theory seeks to find suitable approximations to the infinite series expansions given by $P(t)$ and Y. The theorems below relate the properties of the truncated series versions of these maps to the full expansions.

Theorem 3 *[13] The m^{th}-order truncation of the logarithm of the monodramy map gives an $(\mathrm{m}+1)^{th}$-order flow approximation for finite time, i.e., for time $O(1)$*

$$\exp(Yt) = \exp(Y^m t) + O(\epsilon^{m+1}). \tag{2}$$

Theorem 4 *[13] An m^{th}-order truncation of the time-periodic Floquet mapping is of order $(m+1)$-close to the time-periodic Floquet mapping on the time scale $o(1)$.*

$$P(t) = \mathrm{Trunc}_m\left(P(t)\right) + O(\epsilon^{m+1})$$

Proposition 1 *[13] If the Floquet mapping has a time-independent bias, i.e., $P(t) = \widetilde{P}(t) \circ P_0$. Then the new averaged vector field may be written $Z = (P_0)_* Y$.*

The evolution of the Floquet solution becomes, $x(t) = \widetilde{P}(t) \circ \exp(Zt)$, with $z_0 = P_0^{-1}(x_0)$.

3 Control of Kinematic Systems

The standard form for an underactuated driftless affine control system is

$$\dot{q} = Y_a(q) u^a(q,t) \tag{3}$$

defined on a domain $D \subset \mathbb{R}^n$ for $a = 1, \ldots, m < n$. For driftless affine control systems, small-time local controllability (see [22] for details) is based on the Lie Algebra Rank Condition (LARC),

$$\dim \overline{\Delta}(q) = \dim T_q D, \quad \forall q \in D \tag{4}$$

where $\overline{\Delta}$ is the involutive closure of the control vector field distribution.

Theorem 5 (Chow's Theorem) *[22] The system (3) is small-time locally controllable if and only if the Lie Algebra Rank Condition holds.*

We assume that this theorem is satisfied, and now focus on relating the terms in $\overline{\Delta}$ to control system design.

3.1 Averaging Theory for Control

The controls $u^a(q,t)$ in Eq. (3) can be decomposed into state feedback and time-dependent oscillatory terms,

$$\dot{q} = Y_a(q) f^a(q) + Y_a(q) v^a(t/\epsilon). \tag{5}$$

The functions $v^a(t)$ are T-periodic functions (typically with zero average) and the functions $f^a(q)$ stabilize the directly controlled states. A transformation of time,

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = \epsilon Y_a(q) f^a(q) + \epsilon Y_a(q) v^a(t). \tag{6}$$

takes (5) into a form compatible with averaging theory.

The averaged system vector fields will contain combinations of time integrals and Lie brackets. Since the periodic inputs act as coefficients to the input vector fields, and iterated Lie brackets are multi-linear, the integrals can be factored. These terms represent the net effect of the inputs on the Lie brackets terms. Define the following notation for the *averaging coefficients*:

$$\mathrm{V}^{(a)}_{(n)}(t) = \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_2} v^a(s_1) ds_1 \cdots ds_{n-1}. \tag{7}$$

The time-averaged terms are *averaged coefficients*. Cases of multiple upper and lower indices denote products of this type of integral. E.g., $\mathrm{V}^{(a,b)}_{(1,1)}(t)$ has the form

$$\mathrm{V}^{(a,b)}_{(1,1)}(t) = \mathrm{V}^{(a)}_{(1)}\,\mathrm{V}^{(b)}_{(1)} = \left(\int_0^t v^a(s_1) ds_1\right)\left(\int_0^t v^b(s_1) ds_1\right).$$

Additionally define the following, $\tilde{\mathrm{V}}^{(a)}_{(n)} \equiv \mathrm{V}^{(a)}_{(n)} - \overline{\mathrm{V}^{(a)}_{(n)}}$ and for the multi-index version $\tilde{\mathrm{V}}^{(A)}_{(N)} \equiv \mathrm{V}^{(A)}_{(N)} - \overline{\mathrm{V}^{(A)}_{(N)}}$ where $(A) = (a_1, a_2, \ldots, a_{|A|})$ and $(N) = (n_1, n_2, \ldots, n_{|N|})$. The $\widehat{\ }$ symbol will denote integrals within the product structure. For example,

$$\mathrm{V}^{(\widehat{a,b},c)}_{(\widehat{0,0},1)}(t) = \left(\int_0^t \mathrm{V}^{(a,b)}_{(0,0)}(\tau)\,\mathrm{d}\tau\right)\left(\mathrm{V}^{(c)}_{(1)}(t)\right)$$

1^{st} and 2^{nd}-order averaging. The 1^{st} order averaged verion of system (5) is,

$$\dot{z} = Y_a(z) f^a(z) + Y_a(z)\overline{\mathrm{V}^{(a)}_{(0)}(t)}.$$

Second order terms are typically used when the 1^{st}-order average vanishes. Let us assume that: $\overline{\mathrm{V}^{(a)}_{(0)}(t)} = 0$. All higher-order averaging use this assumption. The 2^{nd} order averaged system has the form

$$\begin{aligned}\dot{z} = Y_a(z) f^a(z) &+ \epsilon\overline{\mathrm{V}^{(a)}_{(1)}(t)}\left[Y_a(z), Y_b(z) f^a(z)\right] \\ &+ \frac{1}{2}\epsilon\overline{\mathrm{V}^{(a,b)}_{(1,0)}(t)}\left[Y_a(z), Y_b(z)\right].\end{aligned}$$

$$
\begin{aligned}
\dot{z} = {} & Y_a(q) f^a(q) + \epsilon \overline{\mathrm{V}^{(a)}_{(1)}(t)} \left[Y_a(z), Y_b(z) f^b(z) \right] \\
& + \frac{1}{2} \epsilon \overline{\mathrm{V}^{(a,b)}_{(1,0)}(t)} \left[Y_a(z), Y_b(z) \right] \\
& + \epsilon^2 \left(-\frac{1}{2} T \overline{\mathrm{V}^{(c)}_{(1)}(t)} + \overline{\mathrm{V}^{(c)}_{(2)}} \right) \\
& \qquad \left[Y_a(z) f^a(z), \left[Y_b(z) f^b(z), Y_c(z) \right] \right] \\
& + \epsilon^2 \left(\frac{1}{2} \overline{\mathrm{V}^{(b,c)}_{(1,0)}(t)} + \frac{1}{3} T \overline{\mathrm{V}^{(b,c)}_{(1,0)}(t)} - \frac{1}{3} \overline{\mathrm{V}^{(\widehat{a,b})}_{(\widehat{1,0})}(t)} \right) \\
& \qquad \left[Y_a(z) f^a(z), \left[Y_b(z), Y_c(z) \right] \right] \\
& + \frac{1}{3} \epsilon^2 \left(\overline{\mathrm{V}^{(a,b)}_{(1,1)}(t)} - T \overline{\mathrm{V}^{(a,b)}_{(1,0)}(t)} + \overline{\mathrm{V}^{(\widehat{a,b})}_{(\widehat{1,0})}} \right) \\
& \qquad \left[Y_a(z), \left[Y_b(z), Y_c(z) f^c(z) \right] \right] \\
& + \frac{1}{3} \epsilon^2 \overline{\mathrm{V}^{(a,b,c)}_{(1,1,0)}(t)} \left[Y_a(z), \left[Y_b(z), Y_c(z) \right] \right] .
\end{aligned}
$$

Table 1: 3^{rd}-order average

3^{rd}-order averaging If the LARC is satisfied via higher levels of iterated Lie brackets, then higher-order averaging is required. The averaged vector field, including third level iterated Lie brackets, is found in table 1.

The above analysis demonstrates that although Lie brackets determine possible flow directions, the averaged coefficients dictate the degree of flow in those directions. Since the LARC predicts the controllable directions, one would like to have a similar procedure to determine when input functions contribute to critical bracket directions.

3.2 Sinusoidal Inputs for Indirect Actuation

Ref.s [17] and [2] have demonstrated the use of sinusoidal inputs for motion generation in Lie bracket directions. By approximating the flow, one can compute the amplitudes of the sinusoidal functions for a given direction. This *approximate inversion* technique is successfully used in [5] and [21] to derive motion control algorithms for underactuated mechanical systems. We generalize this work and provide constructive control laws for underactuated driftless affine control systems.

Recent work on the 2^{nd}-order averaged case has shown how to construct sinusoidal inputs with proper amplitude modulation and frequency spacing relations so as to isolate various Lie bracket contributions [14, 16]. We now show how the averaged coefficients lead to these kinds of relations. Unless noted, the inputs are either cosines or sines with whole number frequency coefficients.

Preliminary Investigation, 2^{nd}-order Averaging Determining the inputs for arbitrary systems at any averaging order is difficult. However, guidelines can be established by investigating simple, lower-order cases. Abstraction to higher order follows naturally.

For 2^{nd}-order averaging, the averaged coefficient is $\overline{\mathrm{V}^{(a,b)}_{(0,1)}(t)}$. The possible inputs for $v^a(t)$, $a = 1 \dots m$, are $\alpha_a \sin(\omega_a t)$ or $\alpha_a \cos(\omega_a t)$. Of the four options for a desired input pair, simultaneous sinusoidal inputs do not work. Additionally, the algebraic equality

$$\omega_a - \omega_b = 0. \tag{8}$$

must hold for the desired input pair and for no other inputs. The net result will be a set of inputs that operate at unique carrier frequencies; a commonly known fact.

Lemma 1 *For the case of two vector fields entering into the 2^{nd}-order Lie brackets $[Y_a, Y_b]$, if the inputs*

$$v^a(t) = \alpha_{ab} \cos(\omega t), \;\; v^b(t) = \omega \sin(\omega t) \tag{9}$$

are chosen for some principle carrier frequency, ω, then only the bracket $[Y_a, Y_b]$ will be excited.

3.2.1 Higher Order Sinusoidal Actuation: Higher-order expansions will have additional algebraic restrictions analogous to (8) in order to keep the effect of the inputs isolated. These restrictions will also affect the previous construction if both 2^{nd} and 3^{rd} order effects are desired simultaneously.

3^{rd}-order averaging Consider the third order averaged vector field with coefficient $\overline{\mathrm{V}^{(a,b,c)}_{(1,1,0)}(t)}$. The 8 possible input permutations lead to the contributions found in Table 2, with potential coupling found in Table 3.

$v^i(t)$	=	$\overline{\mathrm{V}(t)}$
sin, sin, sin	=	0
sin, sin, cos	=	$\begin{cases} \frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} & \text{if } \begin{array}{r} \omega_1+\omega_2-\omega_3=0, \\ \omega_1-\omega_2-\omega_3=0, \\ \omega_1-\omega_2+\omega_3=0 \end{array} \\ 0 & \text{otherwise} \end{cases}$
sin, cos, sin	=	$\begin{cases} \frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} & \text{if } \omega_1+\omega_2-\omega_3=0, \\ & \quad \omega_1-\omega_2+\omega_3=0 \\ -\frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} & \text{if } \omega_1-\omega_2-\omega_3=0 \\ 0 & \text{otherwise} \end{cases}$
sin, cos, cos	=	0
cos, sin, sin	=	$\begin{cases} -\frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} & \text{if } \omega_1-\omega_2-\omega_3=0, \\ & \quad \omega_1-\omega_2+\omega_3=0 \\ \frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} & \text{if } \omega_1+\omega_2-\omega_3=0 \\ 0 & \text{otherwise} \end{cases}$
cos, sin, cos	=	0
cos, cos, sin	=	0
cos, cos, cos	=	$\begin{cases} \frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} & \text{if } \omega_1-\omega_2-\omega_3=0, \\ & \quad \omega_1-\omega_2+\omega_3=0 \\ -\frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} & \text{if } \omega_1+\omega_2-\omega_3=0 \\ 0 & \text{otherwise} \end{cases}$

Table 2: Averaged coefficients for third order

$v^i(t)$	$\omega_1 = \omega_2$	$\omega_1 = \omega_3$	$\omega_2 = \omega_3$
sin, sin, sin	0	0	0
sin, sin, cos	0	$-\frac{\alpha_1\alpha_2\alpha_3}{2\omega_1\omega_2}$	$-\frac{\alpha_1\alpha_2\alpha_3}{2\omega_1\omega_2}$
sin, cos, sin	0	0	$-\frac{\alpha_1\alpha_2\alpha_3}{2\omega_1\omega_2}$
sin, cos, cos	0	0	0
cos, sin, sin	0	$\frac{\alpha_1\alpha_2\alpha_3}{2\omega_1\omega_2}$	0
cos, sin, cos	0	0	0
cos, cos, sin	0	0	0
cos, cos, cos	0	0	0

Table 3: Coupling of averaged coefficients for third order

Thus, the important algebraic equalities are

$$\omega_1+\omega_2-\omega_3 = 0,\ \omega_1-\omega_2-\omega_3 = 0,\ \omega_1-\omega_2+\omega_3 = 0. \tag{10}$$

In order to avoid coupling between terms, the following inequalities may also need to hold,

$$\omega_i - 2\omega_j \neq 0,\ \text{or}\ \omega_i \neq \omega_j. \tag{11}$$

With the above conditions met, the only non-zero combinations involve an odd number of cosines and an even number of sines.

This example highlights a few critical issues when moving to higher order. First, the coupling due to integrally related choices of frequency may fail to satisfy the algebraic inequalities (11). Secondly more than one Lie bracket may be simultaneously excited.

Lemma 2 *When two distinct vector fields enter into the 3^{rd}-order Lie brackets $[Y_b,[Y_a,Y_b]]$, if the inputs*

$$v^a(t) = \omega\cos(2\omega_t),\ v^b(t) = \alpha_{bab}\omega\sin(\omega t) \tag{12}$$

are chosen for some principle carrier frequency, ω, then only the bracket $[Y_b,[Y_a,Y_b]]$ will be excited.

Proof: Assume for now that these are the only nonzero inputs to the system. Without loss of generality, let $a = 1$, $b = 2$. Set the input functions to be,

$$v^1(t) = \alpha_1\cos(2\omega t),\ v^2(t) = \alpha_2\sin(\omega t). \tag{13}$$

The critical elements of the averaged coefficient are,

$$\overline{\mathrm{V}_{(1,1,0)}^{(c,d,c)}(t)} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ -\frac{\alpha_1\alpha_2^2}{8\omega^2} \end{bmatrix} \\ \begin{bmatrix} 0 \\ -\frac{\alpha_1\alpha_2^2}{8\omega_2} \end{bmatrix} & \begin{bmatrix} \frac{\alpha_1\alpha_2^2}{4\omega^2} \\ 0 \end{bmatrix} \end{bmatrix} \tag{14}$$

The corresponding contribution is then,

$$\overline{\mathrm{V}_{(1,1,0)}^{(c,d,d)}(t)}\,[Y_a,[Y_b,Y_a]] = -\frac{3\alpha_1\alpha_2^2}{8\omega^2}\,[Y_2,[Y_1,Y_2]] \tag{15}$$

With the choice of $\alpha_1 = \alpha_{212}$ and $\alpha_2 = \omega$,

$$\overline{\mathrm{V}_{(1,1,0)}^{(c,d,d)}(t)}\,[Y_a,[Y_b,Y_a]] = -\frac{3\alpha_{212}}{8}\,[Y_2,[Y_1,Y_2]], \tag{16}$$

and, the sum over all averaged coefficients is,

$$\overline{\mathrm{V}_{(1,1,0)}^{(c,d,e)}(t)}\,[Y_c,[Y_d,Y_e]] = -\frac{3\alpha_{212}}{8}\,[Y_b,[Y_a,Y_b]] \tag{17}$$

If there are other input functions, then ω must be chosen according to the algebraic equalities (10) and (11). ■

Lemma 3 *When 3 distinct vector fields entering into a 3^{rd}-order Lie bracket ($a \neq b$, $a \neq c$, and $b \neq c$), no choice of inputs results in a single bracket. If the inputs*

$$\begin{gathered} v^a(t) = \omega\cos(\omega t),\ v^b(t) = \omega\sin(3\omega t) \\ v^c(t) = \alpha_{abc}\sin(2\omega t) \end{gathered} \tag{18}$$

a re chosen for some principle carrier frequency, ω, then only the bracket $[Y_a,[Y_b,Y_c]]$ and a cyclicy related bracket, $[Y_c,[Y_a,Y_b]]$ or $[Y_b,[Y_c,Y_a]]$, will be excited.

Proof: Assume that these are the only nonzero system inputs. Without loss of generality, let $a = 1$, $b = 2$, and $c = 3$, and also $\omega_2 > \omega_3 > \omega_1$, with the input functions,

$$\begin{gathered} v^1(t) = \alpha_1\cos(\omega_1 t),\ v^2(t) = \alpha_2\sin(\omega_2 t), \\ v^3(t) = \alpha_3\sin(\omega_3 t). \end{gathered}$$

The critical elements of the averaged coefficient are:

$$\overline{\mathrm{V}_{(1,1,0)}^{(c,d,e)}(t)} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} \end{bmatrix} & \begin{bmatrix} 0 \\ -\frac{\alpha_1\alpha_2\alpha_3}{\omega_1\omega_3} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_2} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{\alpha_1\alpha_2\alpha_3}{4\omega_2\omega_3} \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -\frac{\alpha_1\alpha_2\alpha_3}{4\omega_1\omega_3} \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{\alpha_1\alpha_2\alpha_3}{24\omega_2\omega_3} \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

After grouping like Jacobi-Lie bracket terms,

$$\begin{aligned} &\overline{\mathrm{V}_{(1,1,0)}^{(a,b,a)}(t)}\,[Y_a,[Y_b,Y_a]] \\ &\quad= \frac{\alpha_1\alpha_2\alpha_3}{4}\left(\left(\frac{1}{\omega_1\omega_2} - \frac{1}{\omega_1\omega_3}\right)[Y_1,[Y_3,Y_2]]\right. \\ &\qquad + \left(\frac{1}{\omega_2\omega_3} - \frac{1}{\omega_1\omega_2}\right)[Y_2,[Y_3,Y_1]] \\ &\qquad \left. - \left(\frac{1}{\omega_1\omega_3} + \frac{1}{\omega_2\omega_3}\right)[Y_3,[Y_1,Y_2]]\right) \end{aligned} \tag{19}$$

Denote $\hat{\omega}_i$ by $\hat{\omega}_1 = \frac{1}{\omega_1}\left(\frac{1}{\omega_2} + \frac{1}{\omega_3}\right)$, $\hat{\omega}_2 = \frac{1}{\omega_2}\left(\frac{1}{\omega_3} - \frac{1}{\omega_1}\right)$, and $\hat{\omega}_3 = \frac{1}{\omega_3}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)$. With these definitions,

$$\hat{\omega}_1 > 0,\ \hat{\omega}_2 < 0,\ \hat{\omega}_3 > 0$$

The Jacobi-Lie identity results in,

$$\begin{aligned} &\overline{\mathrm{V}_{(1,1,0)}^{(a,b,a)}(t)}\,[Y_a,[Y_b,Y_a]] = \\ &\frac{\alpha_1\alpha_2\alpha_3}{4}\left((\hat{\omega}_1+\hat{\omega}_3)[Y_1,[Y_3,Y_2]] + (\hat{\omega}_2+\hat{\omega}_3)[Y_2,[Y_3,Y_1]]\right) \end{aligned}$$

The second Lie bracket can be cancelled only if there exists a choice of ω_i satisfying one of the equalities (10), such that, $\hat{\omega}_2 + \hat{\omega}_3 = 0$. This requires finding ω_1 and ω_3, acheiving the equality,

$$\frac{2}{(\omega_1+\omega_3)\omega_3} - \frac{1}{(\omega_1+\omega_3)\omega_1} + \frac{1}{\omega_1\omega_3} = 0, \tag{20}$$

or equivalently, $2\omega_1 - \omega_3 + (\omega_1 + \omega_3) = 3\omega_1 = 0$, which is not a valid solution. Thus, there is no choice of $\omega_2 > \omega_3 > \omega_1$ that leads to a single Lie bracket contribution.

As for the choice of inputs given above, notice that selecting any $\omega_2 > \omega_3 > \omega_1$ such that the equality $\omega_2 = \omega_1 + \omega_3$ holds will give a contribution with the two Lie brackets. Choosing $\omega_3 = 2\omega_1 = 2\omega$ will do the trick.

$$\overline{\mathrm{V}^{(a,b,a)}_{(1,1,0)}(t)}\,[Y_a,[Y_b,Y_a]] = \frac{\alpha_1\alpha_2\alpha_3}{8\omega^2}\left(3\,[Y_1,[Y_3,Y_2]] + [Y_2,[Y_3,Y_1]]\right)$$

With the choice of $\alpha_1 = \omega$, $\alpha_2 = \omega$, and $\alpha_3 = \alpha_{123}$,

$$\overline{\mathrm{V}^{(a,b,a)}_{(1,1,0)}(t)}\,[Y_a,[Y_b,Y_a]] = \frac{\alpha_{123}}{8}\left(3\,[Y_1,[Y_3,Y_2]] + [Y_2,[Y_3,Y_1]]\right) \tag{21}$$

Abstracting back to arbitrary inputs functions, the analysis implies,

$$\overline{\mathrm{V}^{(c,d,e)}_{(1,1,0)}(t)}\,[Y_c,[Y_d,Y_e]] = -\frac{\alpha_{abc}}{8}\left(3\,[Y_a,[Y_b,Y_c]] + [Y_b,[Y_c,Y_a]]\right) \tag{22}$$

Like the previous Lemma, with other input functions present, ω must meet equalities (10) and (11). ■

This theorem is not as restrictive as it may seem. To excite only one of the terms, it will be necessary to expand the set of available input functions. Alternatively, one might have a vanishing bracket, e.g., $[Y_b,[Y_c,Y_a]] = 0$.

For higher order expansions, myriad algebraic identities must hold. Each averaged coefficient must be examined to determine its contribution, and the limitations arising from the chosen set of input functions. Note that once a particular calculation is done, it need not be repeated for another problem with the same Lie bracket structure.

3.3 Stabilization Using Sinusoids

It was previously shown how to obtain the system response of an oscillatory control to some arbitrary order. Subsequently, the required inputs for some of the expansions were analyzed. These inputs are α-parametrized so that the effect of the control could be arbitraily selected according to one's choice of the α. The idea, now, is to determine a a feedback strategy for stabilization.

A lexigraphical ordering will be introduced once more to deal with the Jacobi-Lie brackets. A multi-index is defined to be a sequence of whole numbers, as per the definitions of the averaged coefficients, $a_i = \{\, a_1, a_2, \ldots a_{k-1}, a_k \,\}$. whose length $|a_i| = k$, the number of terms in the multi-index. The ordering will follow the rules below.

$$\{\, a_i \,\} < \{\, b_i \,\} \text{ if } \begin{cases} |a_i| < |b_i| \text{ or} \\ |a_i| = |b_i| \text{ and } \exists k:\ a_i \le b_i, \\ \quad \forall i \in \{1,\ldots,k-1\} \text{ and } a_k < b_k \end{cases}$$

List the Jacobi-Lie brackets showing up in the averaged vector field according to this ordering. Once ordered, let $\mathrm{T}^i(\alpha)$ denote the averaged coefficients corresponding to the ith Lie bracket, $\widehat{\mathrm{Y}}_i$. This will convert the averaged equation into,

$$\dot{z} = Y_a(z)f^a(z) + \mathrm{T}^i(\alpha)\widehat{\mathrm{Y}}_i(z). \tag{23}$$

The averaged system will now be written as,

$$\dot{z} = Y_a(x)f^a(x) + B(z)H(\alpha) \tag{24}$$

where the matrices B and H are,

$$B(z) = [\widehat{\mathrm{Y}}_1 \ldots \widehat{\mathrm{Y}}_N] \text{ and } H(\alpha) = [\mathrm{T}^1 \ldots \mathrm{T}^N]. \tag{25}$$

Ideally, the only non-zero averaged coefficient terms are those that complement $\{\, Y_1, \ldots, Y_m \,\}$ and contribute to a Lie algebra basis for the tanget spaces over the entire configuration space.

Oscillatory Control via Discretized Feedback The feedback that will be introduce is periodic discrete feedback. It is very similar to the motion control algorithms using approximate inversion for open loop stabilization and trajectory tracking. The idea is to use the state error as feedback to modulate the parameters α in such a way that stabilization occurs.

Theorem 6 *Consider a system of the form (5) which satisfies the LARC. Let $u^a(t)$ be the corresponding set of α parametrized, T-periodic input functions where $a = 1 \ldots m$ and $\alpha \in \mathbb{R}^{n-m}$. Lastly denote by $z(t)$, the averaged system response to the inputs. Given the averaged system (24), assuming that the m directly controlled states have been linearly stabilized and that the linearization of H with respect to α at $\alpha = 0$ and $z = z^*$ is invertible on the $(n-m)$ dimensional subspace to control, then there exists a $K \in \mathbb{R}^{(n-m)\times n}$ such that for*

$$\alpha = -\Lambda K z(T\lfloor t/T \rfloor)$$

where $\Lambda^{(n-m)\times(n-m)}$ is invertible and $\lfloor\cdot\rfloor$ denotes the floor function, we have stabilized the average system response.

Proof: The proof was already given in [16], but will be quickly sketched. Given the assumptions on the system, the averaged system (24) is fully controllable. Linearization of the system with respect to z and α yields

$$\dot{z} = Az + B\frac{\partial H}{\partial \alpha}\alpha = Az + B\Gamma\alpha. \tag{26}$$

Choosing α constant over a period, the above system can be directly integrated to obtain a discrete, linear system

$$z(k+1) = \hat{A}z(k) + \hat{B}\alpha \tag{27}$$

The control assumptions on the system imply that $\hat{B}$ has a pseudo-inverse, Λ, for the $(n-m)$ dimensional subspace to stabilize. Choose K such that the eigenvalues of $\hat{A} - \Lambda K$ lie within the unit circle. With this choice of Λ and K, the discrete system has been stabilized, as is the continuous system with piecewise constant feedback. ■

Comments. This theorem stabilizes an equilibrium point of the averaged system. To track a trajectory, replace $x(t)$ with $x(t) - x_d(t)$. Whether the system stabilizes to an orbit or to a fixed point will depend on whether Theorem 2 or Corollary 1 is used in the above proof. Typically this will be decided by the inputs (see section 4). Due to the periodic nature of the feedback, the Nyquist criteria is a limiting factor in tracking a trajectory for the indirectly controlled states.

Lastly, when applying this to an actual system, the feedback values are averages of the indirectly controlled states over the previous period as opposed to instantaneous values of the states.

4 Example

The nonholonomic integrator in $\mathbb{R}^3$ is a driftless system with two control inputs given by,

$$Y_1(x) = \left[1,\, 0,\, -x2\right]^T \text{ and } Y_2(x) = \left[0,\, 1,\, x1\right]^T$$

The system is not linearly controllable, but does satisfy the LARC. To see this, take the Lie bracket of the two input vector fields,

$$[Y_1(x), Y_2(x)] = \left[0,\, 0,\, 2\right]^T,$$

and only second order averaging will be necessary. As per the discussion in subsection 3.2, to excite the critical averaged coefficient out of phase sinusoids will be used. Define,

$$v^1(t) = \alpha \sin(t) \quad , \quad v^2(t) = \cos(t), \tag{28}$$

and use state-feedback to stabilize the first two states, $f^a(q) = -3q^a$. This choice of control inputs results in the averaged vector field, after linearization,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\begin{matrix} z_1 \\ z_2 \\ z_3 \end{matrix}\right\} = \left\{\begin{matrix} -3z_1 \\ -3z_2 \\ 0 \end{matrix}\right\} + \epsilon a \left\{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right\}, \tag{29}$$

and can be stabilized according to theorem 6. To determine the range of feedback gains that will stabilize the system, it suffices to work out the derivation of the feedback law in the proof. Integrating over one time period, gives the following discrete time system:

$$\begin{aligned} z_1(k+1) &= \exp(-3T)z_1(k) \\ z_2(k+1) &= \exp(-3T)z_2(k) \\ z_3(k+1) &= z_3(k) + \epsilon T\alpha \end{aligned}$$

The first two states are exponentially stabilized (in the average), so the primary concern is the last state. By choosing $\alpha = -Kz_3(k)$, $K > 0$, exponential stabilization occurs for $|1 + \epsilon TK| < 1$. The range of stabilizing feedback gains is $K \in \left(0, \frac{1}{\pi\epsilon^2}\right)$. Notice that it depends on ϵ (smaller ϵ, larger range).

The most effective gain lies at the midpoint of the range, whereas the boundary coincides with nominal stability. In fact, simulations of gains set slightly outside of the boundary points do diverge. A plot of the step response is given in figure 1 for $\epsilon = 1/5$ and $K = 3$. The averaged system response stabilizes exponentially, whereas the actual system orbits about the origin. For kinematic systems deactivation of the control inputs halts the system. Stopping at the end of a cycle will leave one $\delta(\epsilon)$-close to the origin.

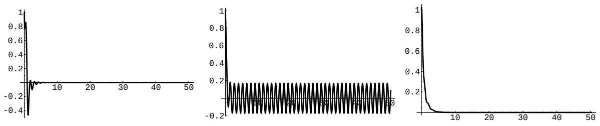

Figure 1: Orbit Stabilization for (28)

If, one does not mind periodic discontinuities in the control inputs, then the equivalent control law will exponentially stabilize the system

$$v^1(t) = \operatorname{sign}(\alpha)\sqrt{|\alpha|}\sin(t) \quad , \quad v^2(t) = \sqrt{|\alpha|}\cos(t). \tag{30}$$

The discontinuity occurs for the second input, $v^2(t)$, at the end of each period, as the feedback gain is adjusted. Notice that the exponential envelope has the characteristic exponent of the slowest convergent mode, which would be that of the last state.

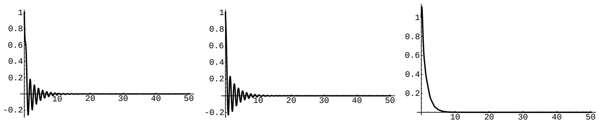

Figure 2: Point Stabilization for (30)

5 Conclusion

A generalized averaging theory was applied to underactuated driftless affine control systems, resulting in an exponentially stabilizing control strategy. Since the control strategy and stabilization theorem are constructive, any system satisfying the conditions can be stabilized by this theory. This approach may have other attractive features. Since the time dependent portions of the inputs are integrated (thereby smoothing out discontinuities), this approach can be used for legged locomotion control design. Recent research [12] studies this idea for a simple bipdel model. Although there are unique problems inherent to systems with drift, the general strategy set forth in this paper still holds in this case. For example, averaging methods have recently been applied to the stabilization of a carangiform fish [16] and the snakeboard [15]. After averaging, both of these systems with drift have remarkably similar control laws. Lastly, the proof of stabilization for the feedback control law use a discretized linear model. The same model could be used to perform robust control under parametric uncertainty.

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