Averaging Methods for Control Part I: Driftless Systems

Patricio A. Vela and Joel W. Burdick

June 19, 2004

Abstract This paper applies a recently developed "generalized averaging theory" to construct stabilizing feedback control laws for underactuated driftless nonlinear systems. The technique uses state feedback and time-periodic inputs for stabilization. Our averaging technique produces an intrinsic description of the averaged control system under periodic forcing. Assuming controllability of the system, this description leads to several related methods of control design based on periodic controls, some of which result in control laws that exponentialy stabilize in the average. The strength of the strategy lies in the fact that standard linear or nonlinear control techniques can be applied to the fully actuated averaged system. Many known stabilization techniques either conform to - or are shown to be special cases of - the general strategy outlined in this paper. More explicit understanding of the link with averaging theory leads to improved controllers. The technique is easy to apply as illustrated by an example. A companion paper extends the methods to systems with drift.

1 Introduction

This paper applies a recently developed "generalized averaging theory" [58] to realize feedback control of underactuated driftless systems. The approach leads to an easily implementable and understandable strategy for designing exponentially stabilizing controllers for these systems. The method does not need a homogeneous norm to demonstrate exponential stabilization, the construction of Lyapunov functions, nor the pre-existence of stabilizing controllers, although such elements can be used to facilitate the process [55]. The complexity of the nonlinear analysis grows with the order of Jacobi-Lie bracketing. Only lower-order Jacobi-Lie bracket controllable systems will be discussed; the extension to higher-order Jacobi-Lie bracket controllable systems follows the same principles. The method discussed not only improves upon known stabilizing controllers from the literature, but can also be directly extended to develop a method applicable to systems with drift. A companion paper will provide more details on the case of underactuated systems with drift [57]. Before describing our controller design method, we first review the relevant literature.

Averaging Theory and Series Expansions. We seek to unite averaging theory and nonlinear control design. The generalized averaging theory [58], which captures the dynamics of a system to arbitrary order of approximation, has its roots in prior work on series expansions by Magnus, Chen, and Agračhev and Gamkrelidze [1, 2, 11, 31]. Series expansion methods for time-varying vector fields and their flows have yielded a greater understanding of the control-theoretic properties of nonlinear systems. Kawski and Sussmann have analyzed the algebraic structure of the series expanions and its relation to controllability of nonlinear systems [15, 17, 18, 19]. Results that tie controllability analysis to control design can be found in the work of Sussmann and Liu [51, 52, 53, 26, 27]. Building upon Kurzweil and Jarnik [21, 22], they studied the limiting behavior of highly oscillatory inputs to driftless control systems. The limiting process leads to a fully actuated autonomous system. Sussmann and Liu demonstrated that, in the very high frequency limit, the flow of the fully actuated autonomous system is faithful to the flow of the actual underactuated time-varying system.

Moving from analysis to design of stabilizing feedback laws is a challenge. Bullo [7] has developed series expansions for the flows of *simple mechanical systems*, Lagrangian systems with kinetic and potential energy only. Further research led to a constructive algorithm for computing the series expansions and approximate inversions under kinematic actuation of systems with drift [6, 8]. One important element of analysis is missing: although time-periodic inputs are required to generate many of the motion primitives, averaging theory was not explicitly referenced. By explicitly understanding the role of averaging within this context, stronger conclusions are made regarding the motion control algorithms and novel stabilizing controllers are provided in this paper. In this paper, analysis is restricted to driftless systems only. The case of systems with drift is considered in a companion paper [57].

Feedback control of driftless systems. Smooth state-feedback will not exponentially stabilize underactuated driftless nonlinear systems [4], leading to the use of time-varying or non-smooth feedback techniques. Coron [12] has demonstrated that time-varying controllers can asymptotically stabilize underactuated nonlinear control systems. Work on non-smooth stabilizing controllers has varied from continuous asymptotically stabilizing controllers to discontinuous and hybrid controllers [20]. The focus of this paper is on deriving exponentially stabilizing controllers, for which the time-varying approach appears most promising.

Prior to developing stabilizing feedback, early research focused on the open-loop response to sinusoidal and oscillatory inputs [5, 41], leading to control methods for specific classes of control systems. Teel et al. [54] demonstrated how sinusoids may be used to obtain stabilizing controllers for chained form systems. Leonard and Krishnaprasad used averaging methods, together with series expansions, to approximate the flow of systems evolving on Lie groups [25]. The averaged response is computed up to third order, and is used for open-loop motion planning. Lafferrierre and Sussmann [23, 24] proposed an open-loop path planning methodology based on the averaging process of Sussmann and Liu, which requires highly oscillatory control inputs. Motions planned for the fully actuated averaged system are transformed to oscillatory control inputs for the actual control system.

The closed loop problem for time-varying control of general underactuated driftless systems is difficult and few general solutions exist. One of most general methods with strong results utilizes homogeneous approximations to underactuated driftless control systems. The resulting controllers are exponentially stabilizing with regards to a homogeneous norm, and are termed *ρ-exponential stabilizing controllers*. In M'Closkey and Murray [35], a method to transform asymptotically stabilized homogeneous systems into ρ-exponentially stabilized homogeneous systems is given. Although the method is quite general, it does require a pre-existing controller which is typically constructed using a method that does not result in ρ-exponential stabilization. In Pomet [45], homogeneous controllers for driftless systems are algorithmically constructed by solving a linear partial differential equation. The method is quite general, though often dificult to apply in practice. Pomet addresses how additional problem-specific heuristics may streamline the controller design process. Further work by M'Closkey [34] extended Pomet's algorithm for computing ρ-exponentially stabilizing feedbacks, thereby avoiding the *a priori* assumptions of M'Closkey and Murray [35].

Morin et al. [37] successfully extended the open loop path planning ideas of Lafferrierre, Liu, and Sussmann to obtain closed loop ρ-exponentially stabilizing controllers. The feedback solution, requiring homogeneous approximations, is based on an algorithm that is computationally difficult in application. Additional investigation by Morin and Samson has sought to better understand the properties of the time-periodic control inputs required for stabilization [39], and on how to derive a constructive controller method that is provably robust [38]. A drawback to following the approach pioneered by Sussman and Liu is the requirement of highly oscillatory inputs. The oscillatory controllers are designed with a frequency inversely proportional to an ϵ parameter. The averaging process requires the limit $\epsilon \rightarrow 0$, leading to highly oscillatory inputs, although it is understood that the method may work for ϵ small. The generalized averaging methods used in this paper build upon the work of Agračhev and Gamkrelidze [1], who consider the effect of a small ϵ parameter, as opposed to an infinitesimally small ϵ parameter that tends to zero. Error estimates due to trunctation of the infinite series expansion for ϵ small may be obtained [1, 58]. Consequently, the need to approach the limit $\epsilon \rightarrow 0$ is avoided, while it is still possible to obtain stabilization with no error. The results found in the aforementioned citations of Sussmann and Liu [26, 27, 51, 52, 53], and Morin, Pomet, and Samson [37, 38, 39], can be given a new interpretation by examining them using the generalized averaging theory of [58].

Combining many of the ideas discussed above, Lucibello and Oriolo [30], and Vendittelli et al. [59] propose a robust iterative state steering approach for the stabilization of driftless nonholonomic control systems. The basic procedure begins with open loop inputs for steering a nonholonomic system, then build upon this by discretely updating the steering controls. The authors intentionally decompose the process into a series of disjoint computational steps, allowing for interchangeability of techniques within each step. The procedure found in this paper is conceptually similar, however, the entire process forms a continuous thread linked by averaging theory. The net result obtained from linking the control design to the closed loop analysis using averaging theory is beneficial, as exponential stabilization via continuous feedback is possible.

In a technique reminiscent of the approach found herein, Struemper and Krishnaprasad [50] utilize Floquet theory to prove linear stability of systems evolving on simple Lie groups. The control inputs, found through approximate inversion, are shown to be exponentially stabilizing. Additional methods exist, however, these solutions are largely found in specific application domains. Canudas de Wit and Sordalen [10, 13] give exponentially stabilizing controllers for mobile robots or underwater vehicles evolving on the Lie groups $SE(2)$ and $SE(3)$, respectively. Sorensen and Egeland discuss exponential stabilization of chained systems [49]. The focus of this paper is on a general and systematic methodology that is not limited to any specific class of underactuated driftless systems.

Contribution. General strategies for feedback stabilization of underactuated driftless nonlinear control systems using time-varying methods utilize many of the same ideas: homogeneity, averaging, and series expansions. This paper synthesizes some of the aforementioned ideas with a generalized averaging theory [58] to provide a practical, intuitive, and easily implementable control strategy for stabilization of underactuated driftless control systems. Importantly, the algorithm does not use nonlinear transformations of state. Neither are there *a priori* requirements, such as pre-existing controllers or Lyapunov functions, although such knowledge may be helpful. The technique culminates in a set of control functions with tunable gains and oscillatory parameters that can be well understood using concepts from linear control theory and the generalized averaging theory. One drawback to the technique, as is well known in the literature, is the rapid increase in frequency of the oscillatory controls as the number of Jacobi-Lie bracket required for stabilization increases. Nevertheless, the systematic nature of the calculations easily lends itself to algorithmic computation. The material here gives calculations for up to 3^{rd}-order averaging. A general algorithm is given to aid in the calculation of feedback control for higher levels of averaging. By following the procedure implicitly given, it is possible to construct a stabilizing controller without detailed understanding of the underlying mathematics of our averaging methods.

Organization. Section 2 summarizes the averaging theory that forms the basis for our method. The control of driftless systems is studied in Section 3, with a focus on using the results of averaging theory for control of underactuated driftless affine control systems. Section 4 demonstrates the application of this theory to a driftless model of the Hilare robot. Section 5 concludes with a synopsis of the current results and a vision of future work.

2 A Generalized Averaging Theory

The main goal of averaging theory is to find a time-independent (averaged) approximation of a time-dependent (usually time-periodic) vector field, and also determine what properties are invariant under the averaging process. The property most commonly sought is stability, which is important from a control theoretic perspective. Both Sanders and Verhulst [60], and Bogoliubov and Mitropolsky [3] provide a background of classical averaging theory. Guckenheimer and Holmes [14] discuss stability analysis of periodic systems via averaging and Poincaré maps.

The authors' prior work on a generalized averaging theory [58] extends the classical methods by demonstrating that averaging theory is the fusion of two distinct analysis techniques, nonlinear Floquet theory and perturbation methods. Nonlinear Floquet theory indicates that averaging methods and their related theorems may hold in the exact sense. For a nonlinear system too difficult to analyze in closed form, perturbation methods provide a useful means to approximate the system dynamics, resulting in a finite series approximation to the vector field or flow in question.

2.1 Nonlinear Floquet Theory

The flow of the differential equation

$$\dot{x} = X(x,t), \tag{1}$$

with X smooth in $x \in \mathbb{R}^n$, absolutely continuous as a function of t, and T-periodic, i.e., $X(x,t) = X(x,t+T)$, can be analyzed by a non-linear version of Floquet theory. This approach represents the flow as the composition of a periodic flow and the evolution of an autonomous averaged vector field.

The flow of a time-varying vector field $X(\cdot,t)$ will be denoted by $\Phi^X_{0,t}$, whereas the flow for an autonomous vector field $Z(\cdot)$ will be denoted by $\exp(Zt)$. The flow of a time varying vector field for a fixed time t may be related to the flow of an autonomous vector field for unit time via the *logarithm*,

$$Z = \ln\, \Phi^X_{0,t}. \tag{2}$$

The autonomous vector field Z satisfies the identity

$$\exp Z = \Phi^X_{0,t}$$

for fixed t [1, 58].

Definition 1 *The* monodromy map *corresponding to the flow $\Phi^X_{0,t}$ of the periodic system (1) is defined to be*

$$M(\cdot) = \Phi^X_{0,T}(\cdot),$$

where T is the period of X, i.e, $X(x,t) = X(x,t+T)$.

Theorem 1 (Nonlinear Floquet Theorem) *[58] Let $\Phi^X_{0,t}$ be the flow of the smooth time-periodic differential equation (1). If the monodromy map has a logarithm, then the flow $\Phi^X_{0,t}$ can be represented as a composition of flows*

$$\Phi^X_{0,t} = P(t) \circ \exp(Zt), \tag{3}$$

where P is T-periodic, and Z is an autonomous vector field.

The mapping $P(t)$ is called the *Floquet mapping*, and Z is called the *averaged vector field corresponding to X*. When the context is clear, Z will simply be called the *averaged vector field*.

Theorem 2 *[58] If the monodromy map has a fixed point, the actual flow has a periodic orbit whose stability is determined by the stability of the monodromy map.*

Corollary 1 *[58] If the flow of system (1) has a fixed point x^*, as does the monodromy map, then stability of the flow may be determined using the monodromy map. An asymptotically (exponentially) stable fixed point for the monodromy map implies an asymptotically (exponentially) stable fixed point for the flow.*

The monodromy map coincides with the flow of the autonomous averaged vector field, Z, at time T, e.g., $M = \exp(ZT)$. In the proof of the nonlinear Floquet theorem [58], the vector field Z is defined to be

$$Z \equiv \frac{1}{T}\ln(M). \tag{4}$$

The relationship between the mapping M and the vector field Z means that Z may be used to determine stability of (1).

Corollary 2 *[47] The stability properties of the logarithm of the monodromy map may be used to infer the stability properties of the monodromy map itself.*

2.2 Series Expansions and Perturbation Methods

Both $P(t)$ and Z from Equation (3) typically have infinite series expansions. If either $P(t)$ or Z cannot be found in closed form, then an approximation technique is required. By introducing a small paremeter ϵ, it is possible to obtain truncations of the series expansions. Consider the periodic system

$$\dot{x} = X(x,t;\epsilon) = \epsilon X(x,t), \tag{5}$$

where $X(\cdot, t+T;\epsilon) = X(\cdot,t;\epsilon)$.

The average of $X(\cdot,t;\epsilon)$ is the autonomous vector field Z from Theorem 1, obtained using Equation (4). The logarithm of the autonomous vector field Z has a power series expansion in ϵ denoted by

$$Z \equiv \sum_{\alpha=1}^{\infty} \epsilon^\alpha \Lambda^{(\alpha)}. \tag{6}$$

Due to the power series expansion for Z, we may define what it means to take a truncation of Z.

Definition 2 *If the function* F *can be given by a power series expansion in ϵ, then* $\mathrm{Trunc}_m(\mathrm{F})$ *is a truncation of the $(m+1)$ and higher-order terms in the series.*

Thus,

$$\mathrm{Trunc}_m(Z) = \sum_{\alpha=1}^{m} \epsilon^\alpha \Lambda^{(\alpha)}. \tag{7}$$

Calculation of the $\Lambda^{(k)}$ is an involved process [1, 58]. Truncations of Z relevant to this paper are given in Section 2.3.

Definition 3 *A truncated series expansion is considered to be a* stabilized expansion with respect to property $\mathcal{P}$ *if the inclusion of additional terms to the truncation does not affect the given property of the series expansion, i.e., if property $\mathcal{P}$ holds for all* $\mathrm{Trunc}_{m+k}(\mathrm{F})$*, $k > 0$, when property $\mathcal{P}$ holds for* $\mathrm{Trunc}_m(\mathrm{F})$.

In this treatise, the desired property is linear stability for vector fields.

Definition 4 *[47] A* stabilized truncated series expansion with respect to linear stability *for the vector field found in Equation (6) is a truncated vector field that has the same linear stability properties as any higher-order truncation of the vector field, and also the full series expansion of the vector field.*

At a stabilized truncation, linear stability can be assessed and used to determine the linear stability of the original system as per the previous propositions. The role of feedback will be to create a stabilized truncation with respect to linear stability for the truncated average (7).

Prior to implementing the truncations, it is useful to understand the error asymptotics due to truncations of the series expansions.

Theorem 3 *[58] The* m^{th}*-order truncation of the logarithm of the monodromy map gives an* m^{th}*-order approximation of the flow on a compact subset* $K \subset M$*,*

$$\exp(Zt) = \exp(\mathrm{Trunc}_m(Z)\, t) + O(\epsilon^m),$$

as $\epsilon \downarrow 0$ *on the time-scale* $\frac{1}{\epsilon}$.

Theorem 4 *[58] The* m^{th}*-order truncation of the time-periodic Floquet mapping is of order* $\epsilon^{(m+1)}$*-close to the time-periodic Floquet mapping,*

$$P(t) = \mathrm{Trunc}_m(P(t)) + O\left(\epsilon^{m+1}\right),$$

as $\epsilon \downarrow 0$ *on the time-scale* 1.

The removal of the higher-order terms may result in an aperiodic truncated Floquet mapping, however, it is often the case that $\mathrm{Trunc}_m(P(0)) = \mathrm{Trunc}_m(P(T))$. To define a periodic function $\mathrm{Trunc}_m(P(t))$, take the truncated Floquet mapping as defined on $t \in [0, T)$, then extend the domain of definition periodically for all time by defining $\mathrm{Trunc}_m(P(\tau + kT)) \equiv \mathrm{Trunc}_m(P(\tau))$, where $\tau \in [0, T)$ and $k \in \mathbb{Z}$. When the domain of definition for $\mathrm{Trunc}_m(P)$ is adjusted to be periodic, it is called the *amended truncation of the Floquet mapping* or, when the context is clear, the *amended truncation*.

Corollary 3 *[55] If the (amended) truncation* $\mathrm{Trunc}_m(P(t))$ *is periodic with period* T*, and the period is on the time-scale* 1*, then the (amended) truncation is order* $\epsilon^{(m+1)}$*-close to* $P(t)$ *for all time.*[1]

2.3 Computing the Averaged Expansions

The generalized averaging theory in [58] provides a method to compute averaged expansions to arbitrary order. The algorithm results in a truncated Floquet mapping and a truncated autonomous averaged vector field,

$$x(t) = \mathrm{Trunc}_{m-1}(P(t))(z(t)) + O(\epsilon^m), \text{ and}$$
$$\dot{z} = \mathrm{Trunc}_m(Z) + O\left(\epsilon^{m+1}\right),$$

with the initial value $z(0) = x_0$. The first-order expansions of the Floquet mapping and the averaged vector field are

$$\mathrm{Trunc}_0(P(t)) = \mathrm{Id}, \text{ and}$$
$$\mathrm{Trunc}_1(Z) = \epsilon\overline{X},$$

where $\overline{X} = \frac{1}{T}\int_0^T X(\cdot, t)\,\mathrm{d}t$. The second-order expansions of the Floquet mapping and the averaged vector field are

$$\mathrm{Trunc}_1(P(t)) = \mathrm{Id} + \epsilon\int_0^t \left(X_\tau - \overline{X}\right)\mathrm{d}\tau, \text{ and}$$
$$\mathrm{Trunc}_2(Z) = \epsilon\overline{X(\cdot, t)} + \frac{1}{2}\epsilon^2\overline{\left[\int_0^t X_\tau\,\mathrm{d}\tau, X_t\right]}.$$

[1]The corollary can be modified to get a larger order of time at the sacrifice of the order of proximity.

By X_t, it is meant $X(\cdot, t)$. The third-order expansions of the Floquet mapping and the averaged vector field are

$$\begin{aligned}\mathrm{Trunc}_2\left(P(t)\right) = \mathrm{Id} + \epsilon \int_0^t \left(X_\tau - \overline{X}\right)\,\mathrm{d}\tau \,+ \frac{1}{2}\epsilon^2 \int_0^t \left(\left[\int_0^\tau X_s\,\mathrm{d}s\,, X_\tau\right] - \overline{\left[\int_0^t X_s\,\mathrm{d}s\,, X_t\right]}\right)\,\mathrm{d}\tau \\ + \frac{1}{2}\epsilon^2 \int_0^t X_\tau\,\mathrm{d}\tau \circ \int_0^t X_\tau\,\mathrm{d}\tau \,- \epsilon^2 \int_0^t X_\tau\,\mathrm{d}\circ \overline{X}\,t + \frac{1}{2}\epsilon^2 \overline{X} \circ \overline{X}\,t^2, \text{ and}\end{aligned}$$

$$\begin{aligned}\mathrm{Trunc}_3\left(Z\right) = \epsilon\overline{X} + \frac{1}{2}\epsilon^2 \overline{\left[\int_0^t X_\tau\,\mathrm{d}\tau\,, X_t\right]} + \frac{1}{4}T\epsilon^3\left[\overline{X}, \overline{\left[\int_0^t X_\tau\,\mathrm{d}\tau\,, X_t\right]}\right] \\ + \frac{1}{3}\epsilon^3 \overline{\left[\int_0^\tau X_{\tau_1}\,\mathrm{d}\tau_1\,, \left[\int_0^\tau X_{\tau_1}\,\mathrm{d}\tau_1\,, X_\tau\right]\right]}.\end{aligned}$$

The fourth order expansion can be found in [58].

3 Control of Driftless Systems

The standard form for an underactuated driftless affine control system is

$$\dot{q} = Y_a(q)u^a(q,t), \tag{12}$$

where $q \in Q$, with $Q \subset \mathbb{R}^n$ open, and $a = 1 \dots m < n \equiv \dim(Q)$ (full actuation occurs when $m = n$). The inputs u^a reside in $\mathcal{U}$, the set of functions piecewise analytic with respect to q, and absolutely continuous with respect to t. For driftless nonlinear affine control systems, small-time local controllability (see [48] for details) is based on the Lie Algebra Rank Condition (LARC),

$$\dim \overline{\Delta}(q) = \dim T_q D, \quad \forall q \in D \tag{13}$$

where $\overline{\Delta}$ is the involutive closure of the control vector field distribution, $\Delta \equiv \mathrm{span}\{\, Y_a \,\}$.

Theorem 5 (Chow's Theorem) *[48] The system (12) is small-time locally controllable if and only if the Lie Algebra Rank Condition holds.*

We assume that the LARC is satisfied and focus on relating the terms in $\overline{\Delta}$ to control system design.

3.1 Averaging Theory for Control

The control functions $u^a(q,t)$ from Equation (12) will be decomposed into state feedback terms and time-periodic terms: $u^a(q,t) = f^a(q) + v^a(t/\epsilon)$. The functions $f^a(q)$ are feedback terms for stabilization of the directly controlled states, and the functions $v^a(t)$ are T-periodic functions. In practice, the control functions $v^a(t)$ are also functions of state, however, by ensuring that the state dependence varies slowly in time compared to the period of oscillation, such a dependence can be ignored. Substitution into equation (12) gives

$$\dot{q} = Y_a(q)f^a(q) + Y_a(q)v^a(t/\epsilon) = X_S(q) + Y_a(q)v^a(t/\epsilon), \tag{14}$$

where $X_S(q) \equiv Y_a(q)f^a(q)$. A transformation of time, $t \mapsto \epsilon\tau$, converts (14) into a form compatible with averaging theory,

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = \epsilon X_S(q) + \epsilon Y_a(q)v^a(t). \tag{15}$$

The truncated average of (15) is

$$\begin{aligned} q(\tau) &= \mathrm{Trunc}_{m-1}\left(P(\tau)\right)\left(z(\tau)\right) + O\left(\epsilon^m\right), \text{ and} \\ \frac{\mathrm{d}z}{\mathrm{d}\tau} &= \mathrm{Trunc}_m\left(Z\right). \end{aligned}$$

Transforming back to time t, the truncated average of (14) is

$$q(t) = \mathrm{Trunc}_{m-1}\left(P(t/\epsilon)\right)\left(z(t/\epsilon)\right) + O\left(\epsilon^{m-1}\right), \text{ and}$$
$$\dot{z} = \frac{1}{\epsilon}\mathrm{Trunc}_{m}\left(Z\right).$$

Recall that $P(t)$ and Z can be expressed as power series in ϵ. The time transformation lowers the power series order by one. The Floquet mapping $P(t)$ is transformed from being $O\left(\epsilon^{m}\right)$-close to being $O\left(\epsilon^{m-1}\right)$-close. An additional order of approximation can be obtained by utilizing the *improved* m^{th}*-order average* with $\mathrm{Trunc}_{m}\left(P(t)\right)$,

$$q(t) = \mathrm{Trunc}_{m}\left(P(t/\epsilon)\right)\left(z(t/\epsilon)\right) + O\left(\epsilon^{m}\right), \text{ and}$$
$$\dot{z} = \frac{1}{\epsilon}\mathrm{Trunc}_{m}\left(Z\right).$$

3.1.1 Averaged Coefficients

The averaged vector field of (15) will contain vector fields that have combinations of time integrals and Jacobi-Lie brackets, see Section 2.3. Since the periodic inputs act as coefficients to the input vector fields, and iterated Jacobi-Lie brackets are multi-linear, the integrals can be factored. The integral terms represent the net effect of the inputs on the Jacobi-Lie bracket terms, and will be called *averaging coefficients*.

Define the following notation for the *averaging coefficients*:

$$\mathrm{V}_{(n)}^{(a)}(t) \equiv \int_{t_0}^{t}\int^{s_{n-1}}\dots\int^{s_2} v^a(s_1)ds_1\cdots ds_{n-1}. \tag{19}$$

For the purposes of this paper, the initial time will always be $t_0 = 0$. Cases of multiple upper and lower indices denote products of this type of integral. An example is $\mathrm{V}_{(1,1)}^{(a,b)}(t)$,

$$\mathrm{V}_{(1,1)}^{(a,b)}(t) = \mathrm{V}_{(1)}^{(a)}\,\mathrm{V}_{(1)}^{(b)} = \left(\int_0^t v^a(s_1)ds_1\right)\left(\int_0^t v^b(s_1)ds_1\right).$$

Time-averaged terms are called *averaged coefficients*. The single index averaged coefficients are

$$\overline{\mathrm{V}_{(n)}^{(a)}(\tau)} = \frac{1}{T}\int_0^T \mathrm{V}_{(n)}^{(a)}(\tau)\,\mathrm{d}\tau = \frac{1}{T}\mathrm{V}_{(n+1)}^{(a)}(T).$$

Additionally define $\widetilde{\mathrm{V}}_{(n)}^{(a)} \equiv \mathrm{V}_{(n)}^{(a)} - \overline{\mathrm{V}_{(n)}^{(a)}}$, whose multi-index version is defined to be $\widetilde{\mathrm{V}}_{(N)}^{(A)} \equiv \mathrm{V}_{(N)}^{(A)} - \overline{\mathrm{V}_{(N)}^{(A)}}$, where $(A) = (a_1, a_2, ..., a_{|A|})$ and $(N) = (n_1, n_2, ..., n_{|N|})$. The $\widehat{\ }$ symbol will denote integrals within the product structure. For example,

$$\mathrm{V}_{(\widehat{0,0},1)}^{(\widehat{a,b},c)}(t) = \left(\int_0^t \mathrm{V}_{(0,0)}^{(a,b)}(\tau)\,\mathrm{d}\tau\right)\left(\mathrm{V}_{(1)}^{(c)}(t)\right) = \left(\int_0^t \left(v^a(\tau)\,v^b(\tau)\right)\mathrm{d}\tau\right)\left(\int_0^t v^c(\tau)\,\mathrm{d}\tau\right).$$

The notation will simplify the expressions for the averaged expansions of (14).

3.1.2 Averaged Expansions

This section details how the averaged expansions of Section 2.3 fit within the control theoretic description of Equation (14). In particular the Jacobi-Lie brackets naturally appearing in the averaged expansions have direct relevance to control via Chow's Theorem.

First-order. The first-order averaged version of system (14) is

$$\dot{z} = X_S + \overline{\mathrm{V}_{(0)}^{(a)}(t)}Y_a(z), \text{ and} \tag{20a}$$
$$\mathrm{Trunc}_0\left(P(t/\epsilon)\right) = \mathrm{Id}. \tag{20b}$$

The improved average utilizes

$$\mathrm{Trunc}_1\left(P(t/\epsilon)\right) = \mathrm{Id} + \epsilon\int_0^{t/\epsilon}\widetilde{\mathrm{V}}_{(0)}^{(a)}(\tau)\,\mathrm{d}\tau\ \ Y_a(\cdot). \tag{20c}$$

Second-order. Second- and higher-order average approximations are usually sought when the first-order averages of the time-periodic inputs vanish, or when a higher order of approximation is desired.

Assumption 1 *For higher-order averaging, it is assumed that the time-averages of the oscillatory control inputs vanish, i.e.,* $\overline{\mathrm{V}_{(0)}^{(a)}(t)} = 0$.

A system with non-vanishing average is equivalent to a system with direct constant control through the function $f(q)$ from Eq. (14), together with zero-average time-periodic inputs $v^a(t)$. Consequently, Assumption 1 does not impose any limitations on the class of admissable inputs.

The second-order average has the form,

$$\dot{z} = X_S(z) + \epsilon\overline{\mathrm{V}_{(1)}^{(a)}(t)}\,[Y_a(z), X_S(z)] + \frac{1}{2}\epsilon\overline{\mathrm{V}_{(1,0)}^{(a,b)}(t)}\,[Y_a(z), Y_b(z)]\,, \text{ and} \tag{21a}$$

$$\mathrm{Trunc}_1\left(P(t/\epsilon)\right) = \mathrm{Id} + \epsilon\mathrm{V}_{(1)}^{(a)}(t/\epsilon)Y_a(\cdot). \tag{21b}$$

Third-order If the LARC is satisfied via higher levels of iterated Jacobi-Lie brackets, then higher-order averaging is required. The averaged vector field to third order is

$$\begin{aligned}\dot{z} = X_S &+ \epsilon\overline{\mathrm{V}_{(1)}^{(a)}(t)}\,[Y_a, X_S] + \frac{1}{2}\epsilon\overline{\mathrm{V}_{(1,0)}^{(a,b)}(t)}\,[Y_a, Y_b] + \epsilon^2\left(\overline{\mathrm{V}_{(2)}^{(a)}(t)} - \frac{1}{2}T\overline{\mathrm{V}_{(1)}^{(a)}(t)}\right)[X_S, [X_S, Y_a]] \\ &- \frac{1}{3}\epsilon^2\left(\overline{\mathrm{V}_{(\widehat{1,0})}^{(\widehat{a,b})}(t)} + \frac{1}{2}T\overline{\mathrm{V}_{(1,0)}^{(a,b)}(t)}\right)[X_S, [Y_a, Y_b]] \\ &+ \frac{1}{3}\epsilon^2\left(\overline{\mathrm{V}_{(1,1)}^{(a,b)}(t)} + \overline{\mathrm{V}_{(\widehat{1,0})}^{(\widehat{a,b})}(t)} - T\overline{\mathrm{V}_{(1,0)}^{(a,b)}(t)}\right)[Y_a, [Y_b, X_S]] + \frac{1}{3}\epsilon^2\overline{\mathrm{V}_{(1,1,0)}^{(a,b,c)}(t)}\,[Y_a, [Y_b, Y_c]]\,.\end{aligned} \tag{22}$$

The Floquet mapping is

$$\begin{aligned}\mathrm{Trunc}_2\left(P(t/\epsilon)\right) = \mathrm{Id} + \epsilon\mathrm{V}_{(1)}^{(a)}(t/\epsilon)\,Y_a &+ \epsilon^2\int_0^{t/\epsilon}\widetilde{\mathrm{V}}_{(1)}^{(a)}(\tau)\,\mathrm{d}\tau\ [Y_a, X_S] \\ &+ \frac{1}{2}\epsilon^2\int_0^{t/\epsilon}\widetilde{\mathrm{V}}_{(1,0)}^{(a,b)}(\tau)\,\mathrm{d}\tau\ [Y_a, Y_b] + \frac{1}{2}\epsilon^2\mathrm{V}_{(1,1)}^{(a,b)}(t/\epsilon)\,Y_a\cdot Y_b\,.\end{aligned} \tag{23}$$

Higher-order. To compute fourth- and higher-order averages simply follow the averaged expansion procedure delineated in Chapter 2, taking into account the affine control decomposition. The reference [55] has explicit computations of the lower-order averaged expansions for guidance.

3.2 Sinusoidal Inputs for Indirect Actuation

The analysis of Section 3.1 demonstrates that although Jacobi-Lie brackets determine possible flow directions, the averaged coefficients dictate the degree of flow in those directions. Since the LARC predicts the controllable directions, one would like to have a similar procedure to determine when input functions contribute to flow in the critical Jacobi-Lie bracket directions. By approximating the flow using series expansions with averaged coefficients, it is possible to compute the amplitudes of the response to sinusoidal forcing in a given Jacobi-Lie bracket direction. This *approximate inversion* technique is successfully utilized by Bullo [7] and Martínez and Cortés [33] in deriving motion control algorithms for control of underactuated mechanical systems. Sussmann and Liu [26, 52, 53] give a strategy for finding the sinusoidal functions and frequencies to obtain individually excited averaged coefficients for driftless systems. This section examines the algorithm for select averaged coefficients with the goal of providing a practical approach to-, and interpretation of-, the Ω-sets found in the work of Sussman and Liu.

3.2.1 Second-order Averaged Coefficients

The process of determining the appropriate oscillatory control inputs for an arbitrary system to any given order of averaging is difficult. Sinusoidal input pairs in resonance will lead to Jacobi-Lie bracket motion. Each averaged coefficient has algebraic frequency requirements for resonance, which may conflict and cause coupling of motion across all directions whose Jacobi-Lie brackets contain the vector fields that are activated. Certain Jacobi-Lie brackets can never be actuated independently. Nevertheless, useful guidelines can be established by investigating simple, lower-order cases. Although difficult in practice, abstraction to higher-order follows naturally.

The oscillatory inputs for the function $v^a(t)$, $a = 1 \dots m$, will be sinusoidal inputs of the form $\alpha^a \sin(\omega_a t)$ or $\alpha^a \cos(\omega_a t)$, where $\omega_a \in \mathbb{Z}^+$. The value ω_a will be called the *carrier frequency* of the control input $v^a(t)$.

For second-order averaging, the averaged coefficient is

$$\overline{\mathrm{V}^{(a,b)}_{(0,1)}(t)} = \frac{1}{T}\int_0^T \int^t v^a(\tau)\,\mathrm{d}\tau \,.\, v^b(t)\,\mathrm{d}t\,.$$

Of the four possible permutations of desired input pairs, in-phase sinusoidal inputs do not work. In order to prevent coupling, the algebraic equality

$$\omega_a - \omega_b = 0 \tag{24}$$

should hold for the desired out-of-phase sinusoidal input pair and for no other oscillatory inputs. The net result of this analysis is a set of inputs that operate at unique carrier frequencies; a commmonly known fact.

Lemma 1 *Consider a control system of the form (14) and the second-order Jacobi-Lie bracket $[Y_a, Y_b]$ that arises in its averaged form (21). If the associated inputs*

$$v^a(t) = \alpha^a_{ab}\cos(\omega_{ab}t) \quad \textit{and} \quad v^b(t) = \alpha^b_{ab}\sin(\omega_{ab}t)$$

are chosen for some unique carrier frequency ω_{ab}, then only the bracket $[Y_a, Y_b]$ will be excited (i.e., have non-zero coefficient). The corresponding averaged coefficient will evaluate to

$$\overline{\mathrm{V}^{(a,b)}_{(1,0)}(t)} = \frac{\alpha_{ab}}{2\omega_{ab}}\,, \quad \textit{where} \;\; \alpha_{ab} = \alpha^a_{ab}\alpha^b_{ab}.$$

3.2.2 Third-Order Averaged Coefficients

The important factor for the selection of inputs in the previous analysis involved the algebraic equality (24). Higher-order expansions will have additional algebraic restrictions analogous to (24) in order to keep the effect of the inputs isolated, i.e., so that motion in specific Jacobi-Lie bracket directions can be commanded. These restrictions will also affect the construction from Lemma 1, if both second- and third-order effects are to be simultaneously included.

Consider the third order averaged vector field of Eq. (22) with averaged coefficient $\overline{\mathrm{V}^{(a,b,c)}_{(1,1,0)}(t)}$. The eight possible permutations of three inputs lead to the contributions found in Table 1, with potential coupling found in Table 2. Based on Table 1, the important algebraic equalities that lead to non-vanishing averaged coefficients are as follows,

$$\omega_a + \omega_b - \omega_c = 0,\; \omega_a - \omega_b - \omega_c = 0,\; \text{and}\; \omega_a - \omega_b + \omega_c = 0. \tag{25}$$

In order to avoid second-order coupling between terms when three distinct periodic controls are activated, the following inequality also needs to hold,

$$\omega_i - \omega_j \neq 0, \quad \text{for } i \neq j. \tag{26}$$

Failure to satisfy the inequality (26) will cause the control combinations aimed at exciting third order Jacobi-Lie brackets to excite a second order Jacobi-Lie bracket of the form $[Y_i, Y_j]$. With the above conditions met, the only useful combinations involve an odd number of cosines and an even number of sines. It is important to note that the algebraic inequality,

$$2\omega_i - \omega_j \neq 0, \quad \text{for } i \neq j, \tag{27}$$

may also need to hold according to the coupling of Table 2. Failure to satisfy the inequality (27) may excite the Jacobi-Lie brackets corresponding to the averaged coefficients found in Table 2. These Jacobi-Lie brackets take the form, $[Y_i, [Y_j, Y_i]]$.

Lemma 2 *Consider a control system of the form (14) and the third-order Jacobi-Lie bracket $[Y_a, [Y_b, Y_c]]$ that arises in its averaged form (22). For the case of three distinct vector fields entering into the third-order iterated Jacobi-Lie*

$v^a(t), v^b(t), v^c(t)$	$\overline{\mathrm{V}^{(a,b,c)}_{(1,1,0)}(t)}$
$\alpha^a \sin(\omega_a t),\ \alpha^b \sin(\omega_b t),\ \alpha^c \sin(\omega_c t)$	$= 0$
$\alpha^a \sin(\omega_a t),\ \alpha^b \sin(\omega_b t),\ \alpha^c \cos(\omega_c t)$	$= \begin{cases} \frac{\alpha^a\alpha^b\alpha^c}{4\omega_a\omega_b} & \text{if } \omega_a+\omega_b-\omega_c=0,\ \omega_a-\omega_b-\omega_c=0,\ \omega_a-\omega_b+\omega_c=0 \\ 0 & \text{otherwise} \end{cases}$
$\alpha^a \sin(\omega_a t),\ \alpha^b \cos(\omega_b t),\ \alpha^c \sin(\omega_c t)$	$= \begin{cases} \frac{\alpha^a\alpha^b\alpha^c}{4\omega_a\omega_b} & \text{if } \omega_a+\omega_b-\omega_c=0,\ \omega_a-\omega_b+\omega_c=0 \\ -\frac{\alpha^a\alpha^b\alpha^c}{4\omega_a\omega_b} & \text{if } \omega_a-\omega_b-\omega_c=0 \\ 0 & \text{otherwise} \end{cases}$
$\alpha^a \sin(\omega_a t),\ \alpha^b \cos(\omega_b t),\ \alpha^c \cos(\omega_c t)$	$= 0$
$\alpha^a \cos(\omega_a t),\ \alpha^b \sin(\omega_b t),\ \alpha^c \sin(\omega_c t)$	$= \begin{cases} -\frac{\alpha^a\alpha^b\alpha^c}{4\omega_a\omega_b} & \text{if } \omega_a-\omega_b-\omega_c=0,\ \omega_a-\omega_b+\omega_c=0 \\ \frac{\alpha^a\alpha^b\alpha^c}{4\omega_a\omega_b} & \text{if } \omega_a+\omega_b-\omega_c=0 \\ 0 & \text{otherwise} \end{cases}$
$\alpha^a \cos(\omega_a t),\ \alpha^b \sin(\omega_b t),\ \alpha^c \cos(\omega_c t)$	$= 0$
$\alpha^a \cos(\omega_a t),\ \alpha^b \cos(\omega_b t),\ \alpha^c \sin(\omega_c t)$	$= 0$
$\alpha^a \cos(\omega_a t),\ \alpha^b \cos(\omega_b t),\ \alpha^c \cos(\omega_c t)$	$= \begin{cases} \frac{\alpha^a\alpha^b\alpha^c}{4\omega_a\omega_b} & \text{if } \omega_a-\omega_b-\omega_c=0,\ \omega_a-\omega_b+\omega_c=0 \\ -\frac{\alpha^a\alpha^b\alpha^c}{4\omega_a\omega_b} & \text{if } \omega_a+\omega_b-\omega_c=0 \\ 0 & \text{otherwise} \end{cases}$

Table 1: Averaged coefficients for third-order averaging of driftless systems.

bracket $[Y_a,[Y_b,Y_c]]$, ($a \neq b$, $a \neq c$, and $b \neq c$), no choice of associated inputs will result in the excitation of motion along only a single bracket direction. If the following associated inputs are chosen,

$$v^a(t) = \alpha^a_{abc}\cos(\omega_{abc}t), \quad v^b(t) = \alpha^b_{abc}\sin(3\omega_{abc}t), \quad \textit{and} \quad v^c(t) = \alpha^c_{abc}\sin(2\omega_{abc}t),$$

for some principle carrier frequency ω_{abc}, then the bracket $[Y_a,[Y_b,Y_c]]$ and a cyclicly related bracket, $[Y_c,[Y_a,Y_b]]$ or $[Y_b,[Y_c,Y_a]]$, will be excited. The corresponding averaged coefficients will be,

$$\overline{\mathrm{V}^{(a,b,c)}_{(1,1,0)}(t)} = \frac{3\alpha_{abc}}{8\omega^2_{abc}} \quad \textit{and} \quad \overline{\mathrm{V}^{(c,a,b)}_{(1,1,0)}(t)} = \frac{\alpha_{abc}}{8\omega^2_{abc}}$$

or

$$\overline{\mathrm{V}^{(a,b,c)}_{(1,1,0)}(t)} = \frac{\alpha_{abc}}{4\omega^2_{abc}} \quad \textit{and} \quad \overline{\mathrm{V}^{(b,c,a)}_{(1,1,0)}(t)} = -\frac{\alpha_{abc}}{8\omega^2_{abc}},$$

where $\alpha_{abc} = \alpha^a_{abc}\alpha^b_{abc}\alpha^c_{abc}$.

proof

Assume for now that, v^a, v^b, and v^c, are the only nonzero inputs to the system. Without loss of generality, let $a = 1$, $b = 2$, and $c = 3$. Let the input functions be

$$v^1(t) = \alpha^1\cos(\omega_1 t),\ v^2(t) = \alpha^2\sin(\omega_2 t),\ v^3(t) = \alpha^3\sin(\omega_3 t).$$

$v^e(t), v^f(t)$		$\overline{\mathrm{V}^{(i,j,k)}_{(1,1,0)}(t)},\ i,j,k \in \{\, e, f \,\}$
$\alpha^e \sin(\omega_e t),\ \alpha^f \sin(\omega_f t)$	$=$	0
$\alpha^e \sin(\omega_e t),\ \alpha^f \cos(\omega_f t),$	$=$	$\begin{cases} \overline{\mathrm{V}^{(e,e,f)}_{(1,1,0)}(t)} = \frac{(\alpha^e)^2 \alpha^f}{4\omega_e^2} & \text{if } 2\omega_e = \omega_f \\ \overline{\mathrm{V}^{(e,f,e)}_{(1,1,0)}(t)} = -\frac{(\alpha^e)^2 \alpha^f}{4\omega_e \omega_f} & \text{if } 2\omega_e = \omega_f \\ \overline{\mathrm{V}^{(f,e,e)}_{(1,1,0)}(t)} = -\frac{(\alpha^e)^2 \alpha^f}{4\omega_e \omega_f} & \text{if } 2\omega_e = \omega_f \\ 0 & \text{otherwise} \end{cases}$
$\alpha^e \cos(\omega_e t),\ \alpha^f \sin(\omega_f t),$	$=$	$\begin{cases} \overline{\mathrm{V}^{(e,e,f)}_{(1,1,0)}(t)} = \frac{(\alpha^f)^2 \alpha^e}{4\omega_f^2} & \text{if } 2\omega_f = \omega_e \\ \overline{\mathrm{V}^{(f,e,f)}_{(1,1,0)}(t)} = -\frac{(\alpha^f)^2 \alpha^e}{4\omega_e \omega_f} & \text{if } 2\omega_f = \omega_e \\ \overline{\mathrm{V}^{(e,f,f)}_{(1,1,0)}(t)} = -\frac{(\alpha^f)^2 \alpha^e}{4\omega_e \omega_f} & \text{if } 2\omega_f = \omega_e \\ 0 & \text{otherwise} \end{cases}$
$\alpha^e \cos(\omega_e t),\ \alpha^f \cos(\omega_f t),$	$=$	$\begin{cases} \overline{\mathrm{V}^{(e,e,f)}_{(1,1,0)}(t)} = -\frac{(\alpha^e)^2 \alpha^f}{4\omega_e^2} & \text{if } 2\omega_e = \omega_f \\ \overline{\mathrm{V}^{(e,f,e)}_{(1,1,0)}(t)} = \frac{(\alpha^e)^2 \alpha^f}{4\omega_e \omega_f} & \text{if } 2\omega_e = \omega_f \\ \overline{\mathrm{V}^{(f,e,e)}_{(1,1,0)}(t)} = \frac{(\alpha^e)^2 \alpha^f}{4\omega_e \omega_f} & \text{if } 2\omega_e = \omega_f \\ \overline{\mathrm{V}^{(f,f,e)}_{(1,1,0)}(t)} = -\frac{(\alpha^f)^2 \alpha^e}{4\omega_f^2} & \text{if } 2\omega_f = \omega_e \\ \overline{\mathrm{V}^{(f,e,f)}_{(1,1,0)}(t)} = \frac{(\alpha^f)^2 \alpha^e}{4\omega_e \omega_f} & \text{if } 2\omega_f = \omega_e \\ \overline{\mathrm{V}^{(e,f,f)}_{(1,1,0)}(t)} = \frac{(\alpha^f)^2 \alpha^e}{4\omega_e \omega_f} & \text{if } 2\omega_f = \omega_e \\ 0 & \text{otherwise} \end{cases}$

Table 2: Coupling of averaged coefficients for third-order averaging of driftless systems.

Assume that the algebraic inequality in Equation (27) for the coupling found in Table 2 is satisfied, and no coupling occurs. The averaged coefficients are

$$\overline{\mathrm{V}^{(i,j,k)}_{(1,1,0)}(t)} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha^1\alpha^2\alpha^3}{4\omega_1\omega_2} \end{bmatrix} & \begin{bmatrix} 0 \\ -\frac{\alpha^1\alpha^2\alpha^3}{4\omega_1\omega_3} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ \frac{\alpha^1\alpha^2\alpha^3}{4\omega_1\omega_2} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{\alpha^1\alpha^2\alpha^3}{4\omega_2\omega_3} \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -\frac{\alpha^1\alpha^2\alpha^3}{4\omega_1\omega_3} \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{\alpha^1\alpha^2\alpha^3}{4\omega_2\omega_3} \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}, \tag{28}$$

where $\omega_2 > \omega_3 > \omega_1$ and $i, j, k \in \{\, 1, 2, 3 \,\}$. The matrix representing the averaged coefficient in Eq. (28) is read as follows: i gives the outer row, j gives the column, and k gives the inner row. The contributions to the averaged vector field are

$$\overline{\mathrm{V}^{(i,j,k)}_{(1,1,0)}(t)}\, [Y_i, [Y_j, Y_k]] = \frac{\alpha^1\alpha^2\alpha^3}{4} \left(\hat{\omega}_1 \,[Y_1, [Y_2, Y_3]] + \hat{\omega}_2 \,[Y_2, [Y_3, Y_1]] - \hat{\omega}_3 \,[Y_3, [Y_1, Y_2]] \right), \tag{29}$$

where

$$\hat{\omega}_1 \equiv \frac{1}{\omega_1}\left(\frac{1}{\omega_2} + \frac{1}{\omega_3}\right),\ \hat{\omega}_2 \equiv \frac{1}{\omega_2}\left(\frac{1}{\omega_3} - \frac{1}{\omega_1}\right),\ \text{and } \hat{\omega}_3 \equiv \frac{1}{\omega_3}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right).$$

With these definitions, $\hat{\omega}_1 > 0,\ \hat{\omega}_2 < 0,\ \hat{\omega}_3 > 0$.

The Jacobi identity may be used to remove one of the Jacobi-Lie brackets, $[Y_2,[Y_3,Y_1]]$ or $[Y_3,[Y_1,Y_2]]$. Select the last Jacobi-Lie bracket from Equation (29) to be removed using the Jacobi identity,

$$\overline{\mathrm{V}_{(1,1,0)}^{(i,j,k)}(t)}\,[Y_i,[Y_j,Y_k]] = \frac{\alpha^1\alpha^2\alpha^3}{4}\left((\hat{\omega}_1+\hat{\omega}_3)\,[Y_1,[Y_2,Y_3]] + (\hat{\omega}_2+\hat{\omega}_3)\,[Y_2,[Y_3,Y_1]]\right). \tag{30}$$

The second Jacobi-Lie bracket in Equation (30) can be cancelled only if there exists a selection of ω_i satisfying one of the equalities (25), such that

$$\hat{\omega}_2+\hat{\omega}_3 = 0. \tag{31}$$

In other words, ω_1 and ω_3 must satisfy

$$\frac{2}{(\omega_1+\omega_3)\omega_3} - \frac{1}{(\omega_1+\omega_3)\omega_1} + \frac{1}{\omega_1\omega_3} = 0.$$

The equivalent problem

$$2\omega_1 - \omega_3 + (\omega_1+\omega_3) = 3\omega_1 = 0$$

does not have a valid solution, therefore there does not exist a selection of $\omega_2 > \omega_3 > \omega_1$ such that only one Jacobi-Lie bracket contribution is achieved.

Selecting any $\omega_2 > \omega_3 > \omega_1$ such that the equality $\omega_2 = \omega_1 + \omega_3$ holds will excite the equivalent of two Jacobi-Lie brackets. Choosing $\omega_3 = 2\omega_1 = 2\omega$ will avoid the coupling described in Table 2 according to the algebraic inequality in Equation (27). The cosine carrier frequency is chosen to be the smallest to avoid coupling. The final contribution is

$$\overline{\mathrm{V}_{(1,1,0)}^{(i,j,k)}(t)}\,[Y_i,[Y_j,Y_k]] = \frac{\alpha^1\alpha^2\alpha^3}{8\omega^2}\left(3\,[Y_1,[Y_2,Y_3]] + [Y_2,[Y_3,Y_1]]\right).$$

A similar procedure can be used to cancel the second Jacobi-Lie bracket from Equation (29) instead of the third, leading to the contribution

$$\overline{\mathrm{V}_{(1,1,0)}^{(i,j,k)}(t)}\,[Y_i,[Y_j,Y_k]] = \frac{\alpha^1\alpha^2\alpha^3}{8\omega^2}\left(2\,[Y_1,[Y_2,Y_3]] - [Y_2,[Y_3,Y_1]]\right),$$

where $\omega_3 = 2\omega_1 = 2\omega$.

Suppose now that a, b, and c are arbitrary, and the carrier frequency ω_{abc} is chosen to be unique in the sense that the algebraic inequalities of Equation (26), and Equation (27) according to Table 2, hold. Furthermore, the algebraic equalities in Equation (25) hold with no undesired coupling. The analysis implies either,

$$\overline{\mathrm{V}_{(1,1,0)}^{(d,e,f)}(t)}\,[Y_d,[Y_e,Y_f]] = \frac{\alpha_{abc}}{8\omega_{abc}^2}\left(3\,[Y_a,[Y_b,Y_c]] + [Y_b,[Y_c,Y_a]]\right),$$

or,

$$\overline{\mathrm{V}_{(1,1,0)}^{(d,e,f)}(t)}\,[Y_d,[Y_e,Y_f]] = \frac{\alpha_{abc}}{8\omega_{abc}^2}\left(2\,[Y_a,[Y_b,Y_c]] - [Y_b,[Y_c,Y_a]]\right),$$

for summation over all indices $d,e,f \in \{1\ldots m\}$, and where $\alpha_{abc} = \alpha_{abc}^a\alpha_{abc}^b\alpha_{abc}^c$.

■

There is nothing that can be done to resolve the problem of exciting a second Jacobi-Lie bracket in this case. In the two-input, three-bracket case, it is possible to obtain a unique contribution.

Lemma 3 *Consider a control system of the form (14) and the third-order Jacobi-Lie bracket $[Y_a,[Y_b,Y_c]]$ that arises in its averaged form (22). For the case of two distinct vector fields entering into the third-order iterated Jacobi-Lie bracket $[Y_b,[Y_a,Y_b]]$, if the following associated inputs are chosen,*

$$v^a(t) = \alpha_{bab}^a\cos(2\omega_{bab}t),\ v^b(t) = \alpha_{bab}^b\sin(\omega_{bab}t),$$

for some unique principle carrier frequency, ω_{bab}, then only the bracket $[Y_b,[Y_a,Y_b]]$ will be excited. The corresponding averaged coefficient is

$$\overline{\mathrm{V}_{(1,1,0)}^{(b,a,b)}(t)} = -\frac{3\alpha_{bab}}{8\omega_{bab}^2},$$

where $\alpha_{bab} = \alpha_{bab}^a\left(\alpha_{bab}^b\right)^2$.

proof
Follows the proof of Lemma 2, with a simplification due to the vanishing Jacobi-Lie bracket $[Y_a, [Y_b, Y_b]]$.
■

3.2.3 Sinusoidal Inputs for Higher-Order Expansions

For higher-order expansions, myriad algebraic identities must hold in order to find input combinations that isolate motions related to specific Jacobi-Lie brackets. Each averaged coefficient must be examined to determine its contribution, and the limitations arising from the chosen set of input functions. Once a calculation is done for a particular Jacobi-Lie bracket structure, it need not be repeated for another problem with the same Jacobi-Lie bracket structure. The general strategy proven in [26, 52, 53] also allows for the frequency choices to be positive rational numbers instead of only positive integers.

Definition 5 *Let B be an iterated Jacobi-Lie bracket. Define $\delta_a(B)$ to be the number of times that the control vector field Y_a appears in the iterated Jacobi-Lie bracket B.*

In [26], it was shown that for any Jacobi-Lie bracket B such that $\delta_i(B) = \delta_j(B)$ for all $i, j = 1 \dots m$, there is no way to obtain isolated activation of the Jacobi-Lie bracket B. The cyclicly related Jacobi-Lie brackets will also be excited, as was seen in Lemma 2. Furthermore, as the degree of the Jacobi-Lie bracket increases, the frequencies of the oscillatory inputs required for control increase at a rapid rate. A similar consequence holds as the quantity of Jacobi-Lie brackets requiring unique activation increases.

We have chosen time-periodic inputs whose averaged coefficients scale linearly with the parameters α_J. For purposes of feedback, the α_J should be defined as functions of state so as to achieve stabilization for the averaged equations of motion. There is no unique method for picking the coefficients α^I_J to obtain the desired product $\alpha_J = \Pi_{i\in I}\alpha^i_J$. Different methods of decomposing the product α_J will result in different controllers, although the averaged evolution will be the same [55]. The theorems and analysis of Section 3.3 below touch upon the implications of this freedom.

3.3 Stabilization Using Sinusoids

To summarize, we have obtained the system response to a set of oscillatory control inputs for an arbitrary order of averaging. We have also analyzed the effects of the oscillatory control inputs on the averaged expansion, leading to an α-parametrized control form. Now, we must determine a stabilizing feedback strategy.

For convenience, order the Jacobi-Lie brackets as they appear in the series expansion of the averaged vector field. Let $\left\{ \widehat{\mathrm{Y}}_j \right\}$ denote the Jacobi-Lie brackets, and let $\left\{ \mathrm{T}^j \right\}$ be their corresponding averaged coefficients. With this ordering, the averaged equations of motion can be put into the form:

$$\dot{z} = X_S + \mathrm{T}^i(\alpha)\widehat{\mathrm{Y}}_i = X_S + B(z)H(\alpha), \tag{32}$$

where the matrices B and H are

$$B(z) = [\widehat{\mathrm{Y}}_1 \dots \widehat{\mathrm{Y}}_N] \quad \text{and} \quad H(\alpha) = [\mathrm{T}^1 \dots \mathrm{T}^N]^{\mathrm{T}}, \tag{33}$$

and X_S corresponds to the directly controlled states. The α parameters will be used to obtain control authority over the indirectly controllable states. The remainder of this section describes different techniques to select the α parameters, and concludes with a discussion and comparison of the stabilization techniques.

3.3.1 Oscillatory Control via Discretized Feedback

We first consider feedback based on time discretization, while a continuous time version is developed below. State error will be used as feedback to modulate the α-parameters, converting the problem to periodic discrete feedback. It is similar to the motion control algorithms, based on motion primitives, that use approximate inversion for open loop stabilization and trajectory tracking [6, 33]. In the language of Floquet theory, our goal is to utilize feedback so as to create stable Floquet multipliers.

Theorem 6 *Consider a system of the form (14) which satisfies the LARC at $q^* \in Q$. Let $u^a(t)$ be the corresponding set of α-parametrized, T-periodic input functions where $a = 1 \dots m$ and $\alpha \in \mathbb{R}^{n-m}$. Let $z(t)$ be the averaged system response to the inputs. Given the averaged system (32), assuming that the m directly controlled states have been linearly stabilized and that the linearization of H from Equation (33) with respect to α at $\alpha = 0$ and $z = q^*$ is invertible on the $(n-m)$ dimensional subspace to control, then there exists a $K \in \mathbb{R}^{(n-m)\times n}$ such that for*

$$\alpha = -\Lambda K z(T\lfloor t/T \rfloor),$$

the average system response is stabilized. Here, $\Lambda^{(n-m)\times(n-m)}$ is invertible and $\lfloor \cdot \rfloor$ denotes the floor function,

proof
Given the assumptions on the system, the averaged system (32) is stabilizable. Linearization of the system with respect to z and α, followed by translation of the fixed point to the origin, yields

$$\dot{z} = Az + B\frac{\partial H}{\partial \alpha}\alpha \equiv Az + B\Upsilon\alpha. \tag{34}$$

Choosing α constant over a control period and integrating the linearized system yields the discrete time system:

$$z(k+1) = \hat{A}z(k) + \hat{B}\alpha,$$

where

$$\hat{A} = e^{AT}, \quad \text{and} \quad \hat{B} = e^{AT}\int_0^T e^{-At}\mathrm{d}t B\Upsilon.$$

The control assumptions on the system imply that the matrix $\hat{B}$ has a pseudo-inverse on the $(n-m)$ dimensional subspace to be stabilized. Denote this pseudo-inverse by Λ and choose the matrix K such that the eigenvalues of $\hat{A} - B\Lambda K$ lie within the unit circle. With this choice of Λ and K, the discrete system has been stabilized.
■
The directly controlled subspace is continuously stabilized, whereas the indirectly controlled subspace is stabilized with time-discretized feedback. The state is sampled at the end of every period, then the error is computed and used to update the α-parameters, which remain fixed for the duration of a period. Although the α-parameters are state dependent, by being held constant over the period, the requirements of averaging theory are met.

Using the analysis of Sussman and Liu [53, 27], Morin and Samson [38] derived a discretized feedback strategy which functions the same as the one proven in Theorem 6. Morin and Samson considered an extended system, consisting of an extra copy of the configuration space evolving discretely in time. The discretely evolving configuration space is used for feedback modulation of sinusoidal inputs. We have more explicitly linked the discretely evolving system to the average of the actual oscillatory system. Additional distinctions are that, for the method presented here, the directly controllable subsystem is still continuously stabilized, a homogeneous approximation is not needed to construct the controller, and, by appealing to the generalized averaging theory, it is possible to make modifications and improvements to the control strategy [55].

3.3.2 Oscillatory Control via Continuous Feedback

The control design procedure outlined above required integration of the linearized dynamical model over a period of actuation and a time discretized feedback strategy. Corollary 2 implies that stability can also be determined from the Floquet exponents, i.e., the stability of the continuous autonomous averaged vector field. A version of Theorem 6 can be proven without discretizing the closed-loop system (see [55] for the proof details).

Theorem 7 *Consider a system of the form (14) which satisfies the LARC at $q^* \in Q$. Let $u^a(t)$ be the corresponding set of α-parametrized, T-periodic input functions where $a = 1 \dots m$ and $\alpha \in \mathbb{R}^{n-m}$. Lastly, denote by $z(t)$ the averaged system response to the inputs. Given the averaged system (32), assuming that the m directly controlled states have been linearly stabilized and that the linearization of H with respect to α at $\alpha = 0$ and $z = q^*$ is invertible on the $(n-m)$ dimensional subspace to control, then there exists a $K \in \mathbb{R}^{(n-m)\times n}$ such that for*

$$\alpha = -\Lambda K z(t),$$

the average system response is stabilized. Here, $\Lambda^{(n-m)\times(n-m)}$ is invertible.

3.3.3 Oscillatory Control via Lyapunov Functions

In Section 2, linearization of the autonomous Floquet vector field Z was emphasized as a means to prove stability of the original oscillatory flow. In reality, any method to demonstrate stability of the average can be applied. This section demonstrates how geometric homogeneity [34, 35] and Lyapunov analysis can lead to exponential or ρ-exponential stability. The methods provide a way to improve the controllers found in [34, 35, 45] for two reasons: (1) the Lyapunov function for the averaged system will take a simpler form as it will not be time-varying and (2) the modification introduces adjustable controller gains for tuning the closed-loop system response.

Definition 6 *A* stabilized truncated series expansion with respect to the Lyapunov function V *for the vector field (6) is a truncated vector field that has the same stability property with respect to the Lyapunov function* V *as any higher-order truncation of the vector field, and also the full series expansion of the vector field.*

Choose a parametrization $\alpha = p(z)$. That is, let α be a function of the averaged system states. The averaged system response (32) may be expressed as

$$\dot{z} = Z(z) \equiv X_S(z) + B(z)H \circ p(z). \tag{35}$$

Theorem 8 *Consider a system of the form (14) which satisfies the LARC at $q^* \in Q$. Let $u^a(t)$ be the corresponding set of α-parametrized, T-periodic input functions where $a = 1 \ldots m$ and $\alpha \in \mathbb{R}^{n-m}$. Lastly,denote by $z(t)$ the averaged system response to the inputs. Given the averaged system (32), assume that there exists an α-parametrization $p(z)$ and a Lyapunov function V such that (35) is a stabilized truncated series expansion with respect to the Lyapunov function, V. If the system in Equation (35) is shown to be asymptotically (exponentially) stable using the Lyapunov function, then the average system response is asymptotically (exponentially) stable.*

proof
Follows the proof of Theorem 7. Instead of linearizing the system, use the Lyapunov function V to demonstrate stability.
■
Finding a suitable Lyapunov function for the nonlinear system (35) may be difficult. In a local neighborhood of the equilibrium, some of the nonlinear terms do not affect stability and may be disregarded, facilitating the Lyapunov analysis. A general strategy for extracting the dominant stabilizing terms and removing the unnecessary nonlinear terms from the truncated average is needed for the Lyapunov technique to be useful. Below, we discuss an extension of the work found in [47] and its synthesis with [34, 35, 45].

Homogeneous Systems. Geometric homogeneity will be used to identify the higher order terms that can be removed. Although what follows can be described more generally, for brevity we follow the material from [16, 34]. Without loss of generality, the equilibrium point will be translated to the origin of $\mathbb{R}^n$.

Definition 7 *Let $x = (x^1, \ldots, x^n)$ denote a set of coordinates for $\mathbb{R}^n$. A* dilation *is a mapping $\Delta^r_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$\Delta^r_\lambda(x) = (\lambda^{r_1} x^1, \ldots, \lambda^{r_n} x^n),$$

where $\lambda \in \mathbb{R}^+$ and $r = (r_1, \ldots r_n)$ are n positive rationals such that $r_1 = 1 \le r_i \le r_j \le r_n$ for $1 < i < j < n$.

Typically r, called the *scaling*, will be given and the notation Δ_λ will be used in place of Δ^r_λ.

Definition 8 *A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $l \ge 0$ with respect to Δ_λ if $f(\Delta_\lambda(x)) = \lambda^l f(x)$.*

Definition 9 *The homogeneous space, $\mathcal{H}_k$, is the set of all continuous functions, $f : \mathbb{R}^n \to \mathbb{R}$, with homogeneous degree k.*

Definition 10 *A continuous vector field $X(x,t) = X^i(x,t)\frac{\partial}{\partial x^i}$ is homogeneous of degree $m \le r_n$ with respect to Δ_λ, if $X^i(\cdot, t) \in \mathcal{H}_{r_i - m}$.*

Definition 11 *A continuous map from $\rho : \mathbb{R}^n \to \mathbb{R}$, is called a* homogeneous norm *with respect to the dilation Δ_λ, when*

1. $\rho(x) \geq 0, \;\; \rho(x) = 0 \iff x = 0$.

2. $\rho(\Delta_\lambda(x)) = \lambda \rho(x), \;\; \forall \, \lambda > 0$.

The notions of homogeneous degree and homogeneous norms are used to characterize vector fields and truncatable components of vector fields.

Definition 12 *The* m^{th}-degree homogeneous truncation of a vector field, *denoted* $\mathrm{Trunc}_m^\Delta(\cdot)$, *is the trunction obtained by removing all terms of homogeneous degree greater than* m.

The m-degree homogeneous truncation with respect to the dilation Δ results in the differential equation,

$$\dot{z} = Z_m = \mathrm{Trunc}_m^\Delta \left(X_S(z) + B(z) H \circ p(z) \right). \tag{36}$$

If there exists a Lyapunov function such that the system in Equation (36) is stable, then the averaged system is locally stable.

Theorem 9 *Consider a system of the form (14) which satisfies the LARC at* $0 \in \mathbb{R}^n$. *Let* $u^a(t)$ *be the corresponding set of* α*-parametrized,* T*-periodic input functions where* $a = 1 \ldots m$ *and* $\alpha \in \mathbb{R}^{n-m}$. *Lastly, denote by* $z(t)$ *the averaged system response to the inputs. Given the averaged system (32), assuming that there exists an* α*-parametrization* $p(z)$ *and a Lyapunov function* V *such that the homogeneous truncation in Equation (36) is asymptotically (*ρ*-exponentially) stable with respect to the Lyapunov function, then the average system response is locally asymptotically (*ρ*-exponentially) stable.*

proof
Rosier [46, Theorem 3] has proven that the homogeneous truncation of Equation (36) will preserve the dominant terms vis-a-vis stability. If the Lyapunov function can be used to demonstrate stability of the homogeneous truncation (36), then the Lyapunov function proves local stability with respect to the averaged expansion of Equation (32). Invocation of Theorem 8 completes the proof.
■
To achieve ρ-exponential stability for the dynamical system in Equation (35), the homogeneous truncation of order 0 must be stabilizing. The 0-degree homogeneous truncation with respect to the dilation Δ results in the system

$$\dot{z} = Z_0 = \mathrm{Trunc}_0^\Delta \left(X_S(z) + B(z) H \circ p(z) \right). \tag{37}$$

If there exists a Lyapunov function such that the system in Equation (37) is stable, then the averaged system is locally ρ-exponentially stable.

Corollary 4 *Consider a system of the form (14) which satisfies the LARC at* $0 \in \mathbb{R}^n$. *Let* $u^a(t)$ *be the corresponding set of* α *parametrized,* T*-periodic input functions where* $a = 1 \ldots m$ *and* $\alpha \in \mathbb{R}^{n-m}$. *Lastly, denote by* $z(t)$ *the averaged system response to the inputs. Given the averaged system (32), assuming that there exists an* α*-parametrization* $p(z)$ *and a Lyapunov function* V *such that the* 0*-degree homogeneous truncation in Equation (37) is asymptotically stable with respect to the Lyapunov function, then the averaged system response is locally* ρ*-exponentially stable.*

If the parametrization is contructed so that the averaged coefficients are of homogeneous order 0 and are stabilizing, then the existence of a Lyapunov function for the homogeneous truncation of the averaged vector field is guaranteed by Rosier [46].

Corollary 5 *Consider a system of the form (14) which satisfies the LARC at* $0 \in \mathbb{R}^n$. *Let* $u^a(t)$ *be the corresponding set of* α *parametrized,* T*-periodic input functions where* $a = 1 \ldots m$ *and* $\alpha \in \mathbb{R}^{n-m}$. *Lastly, denote by* $z(t)$ *the averaged system response to the inputs. Given the averaged system (32), assuming that there exists an* α*-parametrization* $p(z)$ *such that the* 0*-degree homogeneous truncation in Equation (37) is asymptotically stable, then the averaged system response is locally* ρ*-exponentially stable.*

In M'Closkey [34], dilation is used in the construction of the stabilizing controller. Consequently, the dilation is adapted to the Lie algebra of the underactuated driftless control system[2], which is why ρ-norms must be used. As a consequence, only ρ-exponential stability can be proven. We have provided an alternative method for constructing the time-varying controllers, which only uses the dilation and homogeneous norm to define a homogeneous truncation. Consequently, the dilation need not be adapted to the Lie algebra, but may be the standard dilation where $r_i = 1$. The linear stabilization in Theorem 7 is a special case of a homogeneous order 0 stable system with the standard dilation.

[2] In a dilation adapted to the Lie algebra, the scalings r will not satisfy $r^i = r^j$ for all $i, j \in [1, n]$. See M'Closkey [34] for more details.

3.3.4 Comments on the Stabilizing Controllers

Theorems 6-9 and Corollaries 4-5 stabilize an equilibrium point of the averaged system. If the α-parametrized control input functions do not vanish at the equilibrium q^*, then by Theorem 2, the flow of the actual system stabilizes to an orbit (of size $O(\epsilon)$) around the fixed point. If, on the other hand, the input functions do vanish at the equilibrium, then Corollary 1 implies that the flow of the actual system stabilizes to the fixed point (i.e., the orbit collapses to the fixed point).

In order to effectively implement a feedback strategy to stabilize the control system (14) via the average system (32), the appropriate states must be used in the feedback loop; the instantaneous values of the averaged system should be used rather than the instantaneous values of the actual (oscillatory) system. Trajectories of the actual flow are related to the average flow by the Floquet mapping,

$$x(t) = P(t)(z(t)). \tag{38}$$

We may solve for the average $z(t)$ using the current state $x(t)$ and the inverse mapping $P^{-1}(t)$. Since $P(t)$ is given by a series expansion, it's inverse can also be given by a series expansion. For the discretized feedback strategy, the difference in the averaged and instantaneous indirectly controlled states is not a critical factor to consider due to the fact that $P(t)$ is periodic, i.e., $P(kT) = P(0) = \mathrm{Id},\ k \in \mathbb{Z}^+$. Note, however, that the directly stabilized states will require the average for feedback.

Performance of a stabilizing feedback will degrade if the inverse to the Floquet mapping $P(t)$ from Eq. (38) is not used to recover the average. In this case, it is as though Equation (38) were ignored and the equation $x(t) = z(t)$ were used instead. The oscillatory nature of the instantaneous states $x(t)$ will place an upper bound on the feedback gains. The oscillatory inputs should be faster than the natural stabilizing dynamics of the directly stabilized subsytem, otherwise there will be attenuation of the oscillatory signal (and consequently the feedback signal to the indirectly controlled states). With the exception of [38], this effect has not been discussed in prior presentations of feedback strategies that utilize averaging techniques [34, 45].

In practice, one may utilize averages computed in realtime as continuous feedback, $\bar{x}(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,\mathrm{d}\tau$. The benefit of this latter approach is that the averaging process may serve to filter out noise in the sensor signals. It will also attenuate the feedback of external disturbances. As the continually computed average is not equal to $z(t) = P^{-1}(t)\,(x(t))$, the resulting trajectories may not be the same. When performing averaging of sensed measurements, it is important to analyze $P(t)$ to determine which states require averaging. Recall that $P(t)$ is a periodic function with a power series representation in ϵ. The states with dominant oscillatory terms will require averaging to filter out these dominant oscillatory dynamics, whereas the states that have no oscillatory terms, or ignorable, perturbative oscillatory terms, in $P(t)$ will not require averaging for feedback.

Trajectory Tracking. To track a trajectory, replace $x(t)$ with $x(t) - x_d(t)$; the system must be locally controllable along the trajectory. As the trajectory evolves, the Jacobi-Lie brackets contributing to the Lie algebra rank condition may vary. The trajectory tracking feedback strategy becomes more complicated if multiple α-parametrizations are needed to preserve controllability along the trajectory. Lastly, for the discretized feedback, the Nyquist criteria is a limiting factor in tracking a trajectory for the indirectly controlled states.

Existence and Continuity of Solutions. M'Closkey [34] has shown that any exponentially stabilizing feedback-law is necessarily non-Lipschitz. The feedback laws created by the algorithm in [34] are smooth on $Q \backslash \{ q^* \}$, and continuous at $q^* \in Q$. The stabilization is (ρ-exponentially) asymptotic; although the control system may stabilize to a neighborhood of the origin in finite time, it will not stabilize to the origin in finite time. The stabilizing feedback strategies using the generalized averaging theory are also smooth on $Q \backslash \{ q^* \}$, and continuous at $q^* \in Q$ (some are piecewise smooth and piecewise continuous, respectively). According to [34], solutions to the series expansions exist and are unique. On a related note, conditions for the continuity of solution trajectories with respect to the control inputs may be desired. Liu and Sussman [28, 29] have published work regarding the continuous dependence of trajectories with respect to the input functions.

4 Example

Here, the Hilare robot is examined. Additional examples requiring higher-order Jacobi-Lie bracket systems and/or detailing comparisons to existing stabilizing controllers for driftless systems can be found in [55]. Because the Hilare robot evolves on the matrix Lie group $SE(2)$, the Lie group exponential is used instead. It is possible to provide a stabilizing controller without the use of coordinate transformations. The essential information needed for stability is found in the averaged coefficients and Jacobi-Lie brackets. For more details regarding the evolution of systems on Lie groups, see [32, 40].

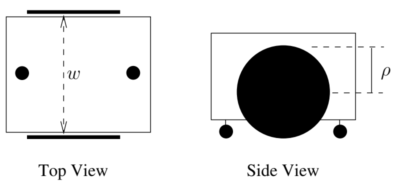


Figure 1: Hilare robot.

Hilare Robot. For the Hilare robot, the control vector fields of Equation (12) are

$$Y_1(g) = L(g)\,\xi_1 \quad \text{and} \quad Y_2(g) = L(g)\,\xi_2,$$

with $g = (x, y, \theta) \in SE(2)$,

$$L(g) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \xi_1 = \begin{bmatrix} -\frac{\rho}{2} \\ 0 \\ \frac{\rho}{2w} \end{bmatrix}, \quad \text{and} \quad \xi_2 = \begin{bmatrix} -\frac{\rho}{2} \\ 0 \\ -\frac{\rho}{2w} \end{bmatrix},$$

where ξ_1 and ξ_2 are elements of the Lie algebra $\mathfrak{se}(2)$, and ρ and w are the radius of the wheels and the distance separating the two wheels, respectively. The Jacobi-Lie brackets of the two control vector fields are

$$[Y_1, Y_1] = 0, \quad [Y_2, Y_2] = 0, \quad \text{and} \quad [Y_1, Y_2] = -[Y_2, Y_1] = L(g)\ [\xi_1, \xi_2] = L(g)\,\widehat{\xi} \equiv L(g) \begin{bmatrix} 0 \\ -\frac{\rho^2}{2w} \\ 0 \end{bmatrix}.$$

Together, the control vector fields Y_i and the Jacobi-Lie bracket $[Y_1, Y_2]$ span the tangent space to the Lie group at any given point $g \in SE(2)$, therefore the system is (small-time) locally controllable and only second order averaging will be required. The oscillatory controls and α-parametrization should be such that the average system results in a fully stabilized system evolving on $SE(2)$. Following Bullo and Murray [9], define the error on $SE(2)$ to be

$$g_{err} = (x_{err}, y_{err}, \theta_{err}) \equiv L^{-1}(g)\, g_{des},$$

where g_{des} is the desired $SE(2)$ position. According to (14), the controls take the form

$$u^a(t) = f^a(g; g_{des}) + v^a(t/\epsilon), \tag{39}$$

where

$$\begin{aligned} f^1(g; g_{des}) &= k_1 x_{err} + k_3 \theta_{err} \quad \text{and} \\ f^2(g; g_{des}) &= k_1 x_{err} - k_3 \theta_{err} \end{aligned} \tag{40}$$

are feedback terms for stabilization of the directly controlled states. Following Lemma 1, the oscillatory inputs are

$$v^1(t) = \alpha_{12}^1 \sin(t) \ \text{ and } \ v^2(t) = \alpha_{12}^2 \cos(t). \tag{41}$$

From (32), the resulting averaged equations of motion in the Lie group are

$$\dot{h} = -L(h)\left(X_S + \epsilon \overline{\mathrm{V}_{(1,0)}^{(1,2)}(t)\widehat{\xi}}\right) = -L(h)\left(\xi_1 f^1 + \xi_2 f^2 + \epsilon \alpha_{12} \widehat{\xi}\right), \quad \text{where } \alpha_{12} = \alpha_{12}^1 \alpha_{12}^2.$$

The corresponding truncated Floquet mapping is

$$\mathrm{Trunc}_1\left(P(t/\epsilon)\right)(g) = \mathrm{Id} + L(g)\,\widetilde{\xi},$$

where

$$\widetilde{\xi} = \epsilon \alpha_{12}^1 (1 - \cos(t/\epsilon))\xi_1 + \epsilon \alpha_{12}^2 \sin(t/\epsilon)\xi_2.$$

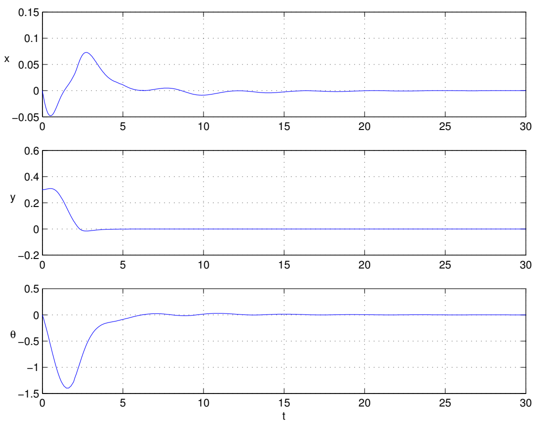
The average trajectory, used for feedback of the averaged state, is obtained from the actual trajectory via the approximation

$$\operatorname{Trunc}_1\left(P^{-1}(t)\right)(g) = g - L(g)\,\widetilde{\xi}. \tag{42}$$

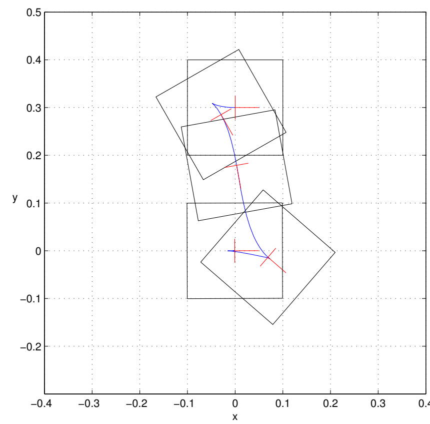
To stabilize the system according to Theorem 7, using the matrix Lie group exponential instead of the Euclidean exponential, the term α_{12} should be equal to $k_2 y_{err}$. One factoring of α_{12} into α_{12}^1 and α_{12}^2 achieving the equality is

$$\alpha_{12}^1 = \operatorname{sign}(k_2 y_{err})\sqrt{|k_2 y_{err}|} \text{ and } \alpha_{12}^2 = \sqrt{|k_2 y_{err}|}. \tag{43}$$

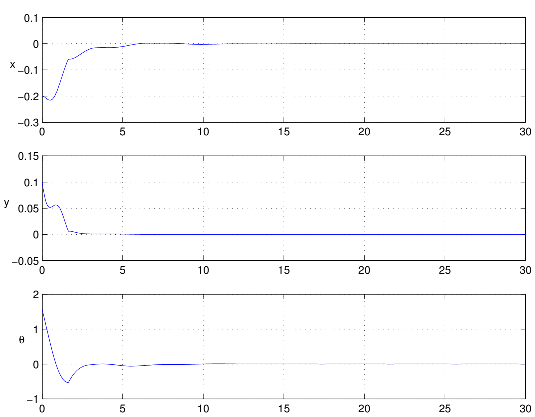
Figure 2 shows the resulting controller, obtained from Equations (39)-(43), stabilizing to the origin for two different initial conditions. The simulation parameters are $k_1 = 6$, $k_2 = 4\sqrt{5}$, $k_3 = \frac{3}{2}$, and $\epsilon = \frac{2}{3}$. A saturation function was included to prevent the wheels from rotating at excessive speeds. The saturation value of 4.5 radians per second was obtained from M'Closkey [34]. The dimensions of the Hilare robot are $\rho = 0.05$, and $w = 0.2$. The snapshots of the vehicle have a forward-pointing dagger locating the position and orientation of the Hilare robot.



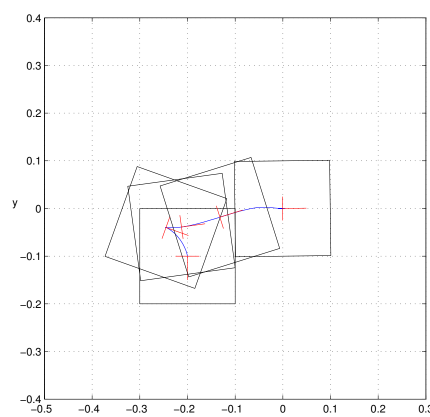
(a) Evolution of state coordinate from initial condition $g_0 = (0, 0.3, 0)$.



(b) Vehicle trajectory with snapshots of vehicle position at $t = 0$, 1.0, 1.6, 2.5, and 7.6 seconds.



(c) Evolution of state coordinates from initial condition $g_0 = (-0.2, -0.1, \frac{\pi}{2})$.



(d) Vehicle trajectory with snapshots of vehicle position at $t = 0$, 1.0, 2.0, 3.0, and 6.6 seconds.

Figure 2: Hilare robot point stabilization with continuous feedback of average state.

5 Conclusion

A recently developed "generalized averaging theory" was applied to underactuated driftless affine control systems for the construction of an exponentially stabilizing control strategy. The algorithm itself is decomposed into distinct controller synthesis steps: (1) averaging, (2) oscillatory actuation design, and (3) feedback parametrization. The only knowledge utilized is the Lie algebra structure of the underactuated driftless control system, which avoids the use of nonlinear transformations, Lyapunov functions, and pre-existing controllers. Although no special knowledge of the system is required, any additional information or structure can be used to facilitate and optimize the process [55].

The averaging method takes a time-varying nonlinear driftless control system and determines an autonomous approximation to the control system. It was shown that Lyapunov techniques could be applied to the averaged autonomous system to determine stability. By using the averaged autonomous system, the use of time-varying Lyapunov analysis was avoided. Recently, Peuteman and Aeyels have investigated the use of averaging methods to determine stability of nonlinear systems via Lyapunov techniques [42, 43, 44]. The generalized averaging theory can be used not only to recover many of their results, but also to extend them.

The control strategy based on averaging has many other attractive features. Since integration of time dependent portions of the inputs smoothes out discontinuities, the approach can be used for legged locomotion control design. Recent research has studied this idea for a simple bipedal robot model [56]. Although there are unique problems inherent to systems with drift, the general strategy set forth in this paper still holds in this case. For example, averaging methods have recently been applied to stabilization of a carangiform fish [36] and the snakeboard [56]. After averaging, both of these systems with drift have remarkably similar control laws.

The linear (or homogeneous degree 0) control model approximating the actual nonlinear control system may be used to determine stability margins under uncertainty. Morin and Samson [38] have demonstrated robustness of a discretized feedback method for stabilization of driftless systems. The analysis is based on the work of Sussmann and Liu [53, 27], which utilizes averaging methods to obtain a fully controllable representation of the original underactuated system. The discretized feedback approach found herein also conforms to the strategy found in the work of Lucibello and Oriolo [30] and Vendittelli et al. [59], who proved robustness of the method for the stabilization of driftless nonholonomic control systems. Consequently, the successes demonstrated by [30, 38, 59] validate this research avenue.

References

[1] A. Agračhev and R. Gamkrelidze. The exponential representation of flows and the chronological calculus. *USSR Sbornik*, 35(6):727–785, 1978.

[2] A. Agračhev and R. Gamkrelidze. Chronological algebras and nonstationary vector fields. *Journal of Soviet Mathematics*, 17:1650–1675, 1979.

[3] N. M. Bogoliubov and Y. A. Mitropolsky. *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Hindustan Publishing Co., 1961.

[4] R. W. Brockett. Asymptotic stability and feedback stabilization. In *Differential Geometric Control Theory*, pages 181–91. Birkhauser, 1993.

[5] R. W. Brockett and L. Dai. Nonholonomic kinematics and the role of elliptic functions in constructive controllability. In Z. X. Li and J. F. Canny, editors, *Progress in Nonholonomic Motion Planning*, pages 1–22. Kluwer, 1992.

[6] F. Bullo. *Nonlinear Control of Mechanical Systems: A Riemannian Geometry Approach*. PhD thesis, California Institute of Technology, 1999.

[7] F. Bullo. Series expansions for the evolution of mechanical control systems. *SIAM Journal of Control and Optimization*, 40(1):166–190, 2001.

[8] F. Bullo, N. Leonard, and A.D. Lewis. Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups. *IEEE Transactions on Automation and Control*, 45(8):1437–1454, 2000.

[9] F. Bullo and R. M. Murray. Tracking for fully actuated mechanical systems: A geometric framework. *Automatica*, 35(1):17–34, 1999.

[10] C. Canudas de Wit and O. J. Sordalen. Exponential stabilization of mobile robots with nonholonomic constraints. *IEEE Transactions on Automation and Control*, 37(11):1791–7, 1992.

[11] K. Chen. Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula. *Annals of Mathematics*, 65(1):163–178, 1957.

[12] J. M. Coron. Global asymptotic stabilization for controllable systems without drift. *Mathematics, Control, Signals, and Systems*, 5(3):295–312, 1992.

[13] O. Egeland, M. Dalsmo, and O. J. Sordalen. Feedback control of a nonholonomic underwater vehicle with a constant desired configuration. *International Journal of Robotics Research*, 15(1):24–35, 1996.

[14] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, volume 42 of *AMS*. Springer Verlag, N.Y., 1990.

[15] M. Kawski. Chronological algebras and nonlinear control. In *Proceedings of the Asian Control Conference*, 1994.

[16] M. Kawski. Geometric homogeneity and applications to stabilization. In A. Krener and D. Mayne, editors, *Proceedings IFAC Symposium on Nonlinear Control Systems Design*, pages 147–152, Tahoe City, CA, 1995.

[17] M. Kawski. Nonlinear control and combinatorics of words. In B. Jakubczyk and W. Respondek, editors, *Geometry of Feedback and Optimal Control*, pages 305–346. Dekker, 1998.

[18] M. Kawski. Controllability via chronological calculus. In *Proceedings of IEEE Conference on Decision and Control*, pages 2920–2925, 1999.

[19] M. Kawski and H. Sussman. Noncommutative power series and formal Lie-algebraic techniques in nonlinear control theory. In U. Helmke, D. Pratzel-Wolters, and Zerz E., editors, *Operators, Systems, and Linear Algebra*, pages 111–128. Teubner, 1997.

[20] I. Kolmanovsky and N. H. McClamroch. Developments in nonholonomic control problems. *IEEE Control Systems Magazine*, 15(6):20–36, 1995.

[21] J. Kurzweil and J. Jarník. Limit processes in ordinary differential equations. *Journal of Applied Mathematics and Physics*, 38, 1987.

[22] J. Kurzweil and J. Jarník. Iterated Lie brackets in limit processes in ordinary differential equations. *Results in Math*, 14, 1988.

[23] G. Lafferriere and H. J. Sussmann. Motion planning for controllable systems without drift. *Proceedings of IEEE International Conference on Robotics and Automation*, pages 1148–1153, 1991.

[24] G. Lafferriere and H. J. Sussmann. A differential geometric approach to motion planning. In Z. X. Li and J. F. Canny, editors, *Progress in Nonholonomic Motion Planning*, pages 235–270. Kluwer, 1992.

[25] N. E. Leonard and P. S. Krishnaprasad. High-order averaging on Lie groups and control of an autonomous underwater vehicle. Technical Report T.R. 93-70, UMD, 1993.

[26] W. S. Liu. An approximation algorithm for nonholonomic systems. *SIAM Journal of Control and Optimization*, 35(4):1328–1365, 1997.

[27] W. S. Liu. Averaging theorems for highly oscillatory differential equations and iterated Lie brackets. *SIAM Journal of Control and Optimization*, 35(6):1989–2020, 1997.

[28] W. S. Liu and H. J. Sussmann. Continuous dependence of trajectories with respect to the input. In *Proceedings of IEEE Conference on Decision and Control*, Pheonix, AZ, 1999.

[29] W. S. Liu and H. J. Sussmann. Continuous dependence with respect to the input of trajectories of control-affine systems. *SIAM Journal of Control and Optimization*, 37(3):777–803, 1999.

[30] P. Lucibello and G. Oriolo. Robust stabilization by iterative state steering with an application to chained-form systems. *Automatica*, 37(1):71–79, 2001.

[31] W. Magnus. On the exponential solutions of differential equations for a linear operator. *Communications in Pure and Applied Mathematics*, VII:649–673, 1954.

[32] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry*, volume 17 of *TAM*. Springer-Verlag, 1994.

[33] S. Martínez and J. Cortés. Motion control algorithms for simple mechanical systems with symmetry. *Submitted to Acta Applicandae Mathetmaticae*, 2001.

[34] R. T. M'Closkey. *Exponential Stabilization of Driftless Nonlinear Control Systems*. PhD thesis, California Institute of Technology, 1994.

[35] R. T. M'Closkey and R. M. Murray. Exponential stabilization of driftless nonlinear control systems using homogeneous feedback. *IEEE Transactions on Automation and Control*, 42(5):614–28, 1997.

[36] K. A. Morgansen, P. A. Vela, and J. W. Burdick. Trajectory stabilization for a planar carangiform robot fish. In *Proceedings of IEEE International Conference on Robotics and Automation*, Washinton D.C., 2002.

[37] P. Morin, J.-B. Pomet, and C. Samson. Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of Lie brackets in closed loop. *SIAM Journal of Control and Optimization*, 38(1):22–49, 1999.

[38] P. Morin and C. Samson. *Robust Point-Stabilization of Nonlinear Affine Control Systems*, volume 216 of *Lecture Notes in Control and Information Sciences*, pages 215–237. Springer-Verlag, 1999.

[39] P. Morin and C. Samson. A characterization of the Lie algebra rank condition by transverse periodic functions. *SIAM Journal of Control and Optimization*, 40(4):1227–1249, 2001.

[40] R. M. Murray, Z. X. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.

[41] R. M. Murray and S. S. Sastry. Nonholonommic motion planning: Steering using sinusoids. *IEEE Transactions on Automation and Control*, 38(5):700–716, 1993.

[42] J. Peuteman and D. Aeyels. Averaging results and the study of uniform asymptotic stability of homogeneous differential equations that are not fast time-varying. *SIAM Journal of Control and Optimization*, 37(4):997–1010, 1999.

[43] J. Peuteman and D. Aeyels. Exponential stability of nonlinear time-varying differential equations and partial averaging. *Mathematics, Control, Signals, and Systems*, 15(1):42–70, 2002.

[44] J. Peuteman and D. Aeyels. Exponential stability of slowly time-varying nonlinear systems. *Mathematics, Control, Signals, and Systems*, 15(3):202–228, 2002.

[45] J. B. Pomet. Explicit design of time-varying stabilization control laws for a class of controllable systems without drift. *Systems and Control Letters*, 18(2):147–158, 1992.

[46] L. Rosier. Homogeneous Lyapunov functions for homogeneous continuous vector field. *Systems and Control Letters*, 19(6):467–473, 1992.

[47] A. V. Sarychev. Lie- and chronologico-algebraic tools for studying stability of time-varying systems. *Systems and Control Letters*, 43(1):59–76, 2001.

[48] S. S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control*. Springer, 1999.

[49] O. J. Sordalen and O. Egeland. Exponential stabilization of nonholonomic chained systems. *IEEE Transactions on Automation and Control*, 40(1):35–49, 1995.

[50] H. Struemper and P. S. Krishnaprasad. Approximate inversion and feedback stabilization for systems on matrix Lie groups. In *Proceedings of the American Control Conference*, Albuquerque, NM, 1997.

[51] H. J. Sussmann and W. S. Liu. Limiting behavior of trajectories for highly oscillatory controls. Technical Report 02, Rutgers SYCON, 1991.

[52] H. J. Sussmann and W. S. Liu. Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories. In *Proceedings of IEEE Conference on Decision and Control*, pages 437–442, 1991.

[53] H. J. Sussmann and W. S. Liu. Lie bracket extensions and averaging: The single bracket case. In Z. X. Li and J. F. Canny, editors, *Progress in Nonholonomic Motion Planning*, pages 109–148. Kluwer, 1992.

[54] A. R. Teel, R. M. Murray, and G. C. Walsh. Non-holonomic control systems: From steering to stabilization with sinusoids. *International Journal of Control*, 62(4):849–870, 1995.

[55] P. A. Vela. *Averaging and Control of Nonlinear Systems*. PhD thesis, California Institute of Technology, 2003.

[56] P. A. Vela and J. W. Burdick. Control of underactuated mechanical systems via averaging theory. In *Proceedings of IEEE International Conference on Robotics and Automation*, 2002.

[57] P. A. Vela and J. W. Burdick. Averaging methods for control part II: Systems with drift. *preprint*, 2003.

[58] P. A. Vela and J. W. Burdick. A general averaging theory via series expansions. *preprint*, 2003.

[59] M. Vendittelli, G. Oriolo, F. Jean, and J.-P. Laumond. Nonhomogeneous nilpotent approximations for nonholonomic systems with singularities. *to appear IEEE Transactions on Automation and Control*.

[60] F. Verhulst and J. A. Sanders. *Averaging Methods in Nonlinear Dynamical Systems*. Springer-Verlag, N. Y., 1985.