Distributed Control Subject to Delays Satisfying an $\mathcal{H}_\infty$ Norm Bound

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Abstract— This paper presents a characterization of distributed controllers subject to delay constraints induced by a strongly connected communication graph that achieve a prescribed closed loop $\mathcal{H}_\infty$ norm. Inspired by the solution to the $\mathcal{H}_2$ problem subject to delays, we exploit the fact that the communication graph is strongly connected to decompose the controller into a local finite impulse response component and a global but delayed infinite impulse response component. This allows us to reduce the control synthesis problem to a linear matrix inequality feasibility test.

I. INTRODUCTION

The identification of Quadratic Invariance[1](QI) [1] as an appropriate condition for the convexification of structured model matching problems has brought a renewed enthusiasm and excitement to optimal controller synthesis. In the following discussion, we survey recent results in this area, and in particular comment on three classes of quadratically invariant constraints: (1) sparsity constraints, in which we assume no delay in information sharing, but rather a restriction of what measurements each controller has access to, (2) delay constraints, in which we assume that controllers communicate with each other subject to delays induced by a strongly connected communication graph, and hence eventually have access to global, but delayed, information, and (3) delay-sparsity constraints, in which we allow both restrictions on measurement access and communication delay between controllers.

Before proceeding into a more detailed review of QI based results, it is worth mentioning that novel approaches to distributed control, not based on the QI framework, have begun to appear in the literature. Representative examples include: sparsity inducing control [5], [6], convex relaxations of rank constrained problems [7], [8], the minimization of convex surrogates to traditional performance metrics [9], [10], spatial truncation [11], [12], positive systems [13], [14], and localized distributed control [15], [16], [17].

Returning to QI constraint sets, in the $\mathcal{H}_2$ case, explicit state-space solutions exist for fixed and varying delay constrained [18], [19] , sparsity constrained [20], [4] and delay-sparsity constrained [21] state-feedback problems. When

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[1]QI [1] is closely related to funnel causality [2], partial nestedness [3] and poset causality [4].

moving to the output feedback case, specific sparsity constrained problems have been solved explicitly, such as the state-space solution for the two-player problem [22] and for lower-triangular systems [23]. The delay-sparsity-constrained case has earned considerable attention, with solutions via vectorization [1] and semi-definite programming [24], [25] existing – we note that although computationally tractable, in contrast with the sparsity constrained setting, none of these methods claim to yield a controller of minimal order. In the case of delay constraints without sparsity, the aforementioned results are applicable, but an additional method based on quadratic programming and spectral factorization [26] also exists.

The landscape of distributed $\mathcal{H}_\infty$ controller synthesis is comparably much sparser, so to speak. However, especially in the sparsity constrained case, there has recently been some progress. In particular, [27] provides a semi-definite programming solution for the structured optimal $\mathcal{H}_\infty$ output-feedback problem subject to nested sparsity constraints. In [28], an explicit state-space representation of the minimum-entropy solution to the two-player version of this problem is presented. A more general approach, applicable to all three classes of constraint types, is presented in [29]. It allows for a principled approximation of the problem via a semi-definite programming based solution that computes an optimal $\mathcal{H}_\infty$ controller within a fixed finite-dimensional subspace. By allowing this finite impulse response (FIR) approximation to be of large enough order, they are able to achieve near optimal performance in a computationally tractable manner.

This paper aims to provide a solution to the sub-optimal distributed $\mathcal{H}_\infty$ control problem subject to delay constraints – in particular, we seek a delay constrained controller that achieves a prescribed closed loop norm. Inspired by the results in [26], we exploit the fact that the controller can be written as a direct sum of a local FIR filter and a delayed, but global, infinite impulse response (IIR) element, and show that the synthesis problem can be reduced to a linear matrix inequality (LMI) feasibility test.

A caveat is that our method is based on the so-called "1984" approach to $\mathcal{H}_\infty$ control, and as such, suffers from the same computational burden that the centralized solution is subject to. We do not claim that our solution is computationally scalable, but provide it rather as evidence that in the case of delay constrained $\mathcal{H}_\infty$ synthesis, the problem admits a finite-dimensional formulation. Our hope is that this result, much as was the case for its centralized analogue, will be a stepping stone to more computationally scalable and explicit results.

This article is organized as follows: Section II establishes

notation, and formalizes the distributed $\mathcal{H}_\infty$ model matching problem subject to delay constraints. In Section III, we provide a refresher on the "1984" solution to the $\mathcal{H}_\infty$ problem, as described in [30]. Section IV provides the main result of the paper, and we demonstrate our algorithm on a three-player chain example in Section V. We end with a discussion and conclusions in Section VI, and the Appendix contains useful formulae for computing the transfer matrix factorizations and approximations required by our method.

II. PROBLEM FORMULATION

In all of the following, we work in discrete-time.

A. Notation and Operator Theoretic Preliminaries

Here we establish notation and remind the reader of some standard results from operator theory, taken from [30].

- $\mathcal{H}_2$ denotes the set of stable proper transfer matrices that are norm square integrable on the unit circle with vanishing negative Fourier coefficients; i.e. if $G \in \mathcal{H}_2$ then $H(z) = \sum_{i=0}^\infty H_i z^{-i}$ and $\|H\|_2^2 = \text{trace}\left(\sum_{i=0}^\infty H_i^* H_i\right)$.
- $\mathcal{H}_\infty$ denotes the set of stable proper transfer matrices. Note that $G \in \mathcal{H}_\infty$ implies $G \in \mathcal{H}_2$.
- $\mathcal{L}_\infty$ denotes the frequency domain Lesbesgue space of essentially bounded functions.
- The prefix $\mathcal{R}$ to a set $\mathcal{X}$ indicates the restriction to real-rational members of $\mathcal{X}$.
- $\|\cdot\|_\infty$ denotes the norm on $\mathcal{L}_\infty$.
- For $R \in \mathcal{L}_\infty$, let $\text{dist}(R, \mathcal{H}_\infty) := \inf\{\|R - X\|_\infty : X \in \mathcal{H}_\infty\}$.
- $\|\cdot\|$ denotes the spectral norm (maximum singular value).
- For a transfer matrix $G \in \mathcal{R}\mathcal{L}_\infty$, $G^\sim$ denotes its conjugate, i.e. $G^\sim(z) = G^*(z^{-1})$.
- For a transfer matrix $G \in \mathcal{R}\mathcal{L}_\infty$, $G^\dagger$ denotes its Moore-Penrose pseudo-inverse.
- $\oplus$, and $\perp$, denote the direct sum, and orthogonality, respectively, as defined with respect to the standard inner product on $\mathcal{H}_2$.
- Decompose $R \in \mathcal{R}\mathcal{L}_\infty$ as $R = R_1^\sim + R_2$, with R_1, $R_2 \in \mathcal{R}\mathcal{H}_\infty$, and R_1 strictly proper. We shall refer to (R_1, R_2) as an anti-stable/stable decomposition of R.
- Γ_F denotes the Hankel operator with symbol F, that is to say the Hankel mapping from $\mathcal{H}_2$ to $\mathcal{H}_2^\perp$. Note that if (F_1, F_2) is an anti-stable/stable decomposition of F, then $\Gamma_F = \Gamma_{F_1^\sim}$.
- $\tilde{\Gamma}_F$ denotes the adjoint Hankel operator with symbol F, that is to say the Hankel mapping from $\mathcal{H}_2^\perp$ to $\mathcal{H}_2$. The following useful fact then holds:
$$\|\Gamma_F\| = \|\Gamma_{F_1^\sim}\| = \|\tilde{\Gamma}_{F_1}\|. \tag{1}$$
- Δ_N denotes the N-delay operator, i.e. $\Delta_N G = \frac{1}{z^N} G$.

B. The model-matching problem subject to delay

We provide a brief overview of the distributed optimal control problem subject to delay, and refer the reader to [26] for a much more thorough and general exposition.
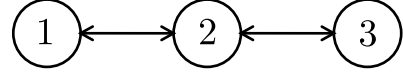

Fig. 1: The graph depicts the communication structure of the three-player chain problem. Edge weights (not shown) indicate the delay required to transmit information between nodes.

Let P be a stable discrete-time plant given by

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{2}$$

with inputs of dimension p_1, p_2 and outputs of dimension q_1, q_2. We restrict attention to stable plants for simplicity. These methods could also be applied to an unstable plant if a stable stabilizing nominal controller can be found, as in [1]. Future work will look to incorporate the results in [26], which are based on those in [31], into our procedure so as to have a general solution to the model matching problem.

Throughout, we assume that $D_{12}^T D_{12} > 0$, $D_{21} D_{21}^T > 0$, $C_1^T D_{12} = 0$, and $B_1 D_{21}^T = 0$, so as to ensure the existence of stabilizing solutions to the necessary discrete algebraic Riccati equations (DAREs).

For $N \geq 1$, define the space of $\mathcal{R}\mathcal{H}_\infty$ FIR transfer matrices by $\mathcal{X}_N = \oplus_{i=0}^{N-1} \frac{1}{z^i} \mathbb{C}^{p_2 \times q_2}$.

In this paper, we are concerned with controller constraints described by delay patterns that are imposed by *strongly connected communication graphs*. As such, let $\mathcal{S} \subset \mathcal{R}\mathcal{H}_\infty$ be a subspace of the form

$$\mathcal{S} = \mathcal{Y} \oplus \Delta_N \mathcal{R}\mathcal{H}_\infty \tag{3}$$

where

$$\mathcal{Y} = \oplus_{i=0}^{N-1} \frac{1}{z^i} \mathcal{Y}_i \subset \oplus_{i=0}^{N-1} \frac{1}{z^i} \mathbb{R}^{p_2 \times q_2} \subset \mathcal{X}_N. \tag{4}$$

Specifically, this implies that every decision-making agent has access to *all* measurements that are at least N time-steps old.

We can therefore partition the measured outputs y and control inputs u according to the dimension of the subsystems:

$$y = [\ y_1^T \quad \cdots \quad y_m^T]^T \quad u = [\ u_1^T \quad \cdots \quad u_n^T]^T$$

and then further partition each constraint set $\mathcal{Y}_i$ as

$$\mathcal{Y}_i = \begin{bmatrix} \mathcal{Y}_i^{11} & \cdots & \mathcal{Y}_i^{1m} \\ \vdots & \ddots & \vdots \\ \mathcal{Y}_i^{n1} & \cdots & \mathcal{Y}_i^{nm} \end{bmatrix}, \tag{5}$$

where

$$\mathcal{Y}_i^{jk} = \begin{cases} \mathbb{R}^{p_2^j \times q_2^k} & \text{if } u_j \text{ has access to } y_k \text{ at time } i \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and $\sum_{j=1}^n p_2^j = p_2$, $\sum_{k=1}^m q_2^k = q_2$.

Example 1: Consider the three player chain problem as illustrated in Figure 1, with communication delay τ_c between nodes. Then

$$\mathcal{S} = \begin{bmatrix} \mathcal{RH}_\infty & \frac{1}{z^{\tau_c}}\mathcal{RH}_\infty & \frac{1}{z^{2\tau_c}}\mathcal{RH}_\infty \\ \frac{1}{z^{\tau_c}}\mathcal{RH}_\infty & \mathcal{RH}_\infty & \frac{1}{z^{\tau_c}}\mathcal{RH}_\infty \\ \frac{1}{z^{2\tau_c}}\mathcal{RH}_\infty & \frac{1}{z^{\tau_c}}\mathcal{RH}_\infty & \mathcal{RH}_\infty \end{bmatrix} \quad (7)$$

$$= \oplus_{i=0}^{2\tau_c-1} \frac{1}{z^i}\mathcal{Y}_i \oplus \Delta_{2\tau_c}\mathcal{RH}_\infty$$

with

$$\mathcal{Y}_i = \begin{cases} \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} & \text{for } 0 \le i < \tau_c \\ \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} & \text{for } \tau_c \le i < 2\tau_c, \end{cases} \quad (8)$$

where, for compactness, * is used to denote a space of appropriately sized real matrices. In this setting, every decision maker then has access to all measurements that are at least $2\tau_c$ time-steps old.

The distributed control problem of interest is to design a controller $K \in \mathcal{S}$ so as to achieve a pre-defined closed loop $\mathcal{H}_\infty$ norm. Specifically, the problem is to find an internally stabilizing $K \in \mathcal{S}$ such that

$$\|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|_\infty \le \gamma \quad (9)$$

for some pre-defined $\gamma > \gamma_{\inf}$, where $\gamma_{\inf}$ is the optimal achievable closed loop $\mathcal{H}_\infty$ norm.

In order to reformulate this problem as a *convex* model matching problem, we require the notion of QI.

Definition 1: A set $\mathcal{S}$ is *quadratically invariant* under P_{22} if

$$KP_{22}K \in \mathcal{S} \text{ for all } K \in \mathcal{S}$$

In [1], it was shown that if $\mathcal{S}$ is quadratically invariant under P_{22}, then $K \in \mathcal{S}$ if and only if $Q = K(I - P_{22}K)^{-1} \in \mathcal{S}$. In the case of delay-constraints imposed by a communication graph, intuitive and easily verifiable conditions for QI can be stated [32]. Essentially these conditions say that in order to have QI, controllers must be able to communicate with each other faster than their control actions propagate through the plant – this is closely related to funnel causality [2], partial nestedness [3] and poset causality [4].

Thus, if quadratic invariance holds, the feasibility problem (9) can be reduced, via the Youla parameterization, to the following equivalent model matching problem:

Problem 1: Find $Q \in \mathcal{S} \bigcap \mathcal{RH}_\infty$ such that

$$\|\mathcal{T}_1 - \mathcal{T}_2 Q \mathcal{T}_3\|_\infty \le \gamma \quad (10)$$

for some $\gamma > \gamma_{\inf}$, with $\mathcal{T}_1 = P_{11}$, $\mathcal{T}_2 = P_{12}$ and $\mathcal{T}_3 = P_{21}$.

III. A Review of "1984" $\mathcal{H}_\infty$ Control

As our solution is based on the so-called "1984" approach to $\mathcal{H}_\infty$ control, we review it in this section. The following is based on material found in chapter 8 of [30].

A. $\mathcal{T}_3 = I$ Case

We begin with the solution to the sub-optimal model matching problem with $\mathcal{T}_3 = I$ first, as the general case follows from a nearly identical derivation. Specifically, we consider the problem:

Problem 2: Find $Q \in \mathcal{RH}_\infty$ such that $\|\mathcal{T}_1 - \mathcal{T}_2 Q\|_\infty \le \gamma$ for some $\gamma > \gamma_{\inf} \ge 0$, where $\gamma_{\inf}$ is the optimal achievable closed loop $\mathcal{H}_\infty$ norm.

In order to state the main result, we first define the following transfer matrices:

1) Let U_i, U_o be an inner-outer factorization of $\mathcal{T}_2$ such that $\mathcal{T}_2 = U_i U_o$, with $U_i^\sim U_i = I$, and U_i, U_o, $U_o^\dagger \in \mathcal{RH}_\infty$.
2) Let $Y := (I - U_i U_i^\sim)\mathcal{T}_1$.
3) For $\gamma > \|Y\|_\infty$, let Y_o be a bi-stable spectral factor of $\gamma^2 I - Y^\sim Y$ such that $\gamma^2 I - Y^\sim Y = Y_o^\sim Y_o$, with Y_o, $Y_o^{-1} \in \mathcal{RH}_\infty$.
4) Define the $\mathcal{RL}_\infty$ matrix $R := U_i^\sim \mathcal{T}_1 Y_o^{-1}$.

Theorem 1: Let $\alpha := \inf\{\|\mathcal{T}_1 - \mathcal{T}_2 Q\|_\infty : Q \in \mathcal{RH}_\infty\}$. Then

1) $\alpha = \inf\{\gamma : \|Y\|_\infty < \gamma, \text{ dist}(R, \mathcal{RH}_\infty) < 1\}$, and
2) For $\gamma > \alpha$ and $Q, X \in \mathcal{RH}_\infty$ such that
   - $\|R - X\|_\infty \le 1$, and
   - $X = U_o Q Y_o^{-1}$,

   we have that $\|\mathcal{T}_1 - \mathcal{T}_2 Q\|_\infty \le \gamma$.

Before proving this result, we need the following two preliminary lemmas:

Lemma 1: Let U be inner and $E \in \mathcal{RL}_\infty$ be given by

$$E := \begin{bmatrix} U^\sim \\ I - UU^\sim \end{bmatrix}.$$

Then for all $G \in \mathcal{RL}_\infty$, we have that $\|EG\|_\infty = \|G\|_\infty$.

Lemma 2: For $F, G \in \mathcal{RL}_\infty$ with the same number of columns, if

$$\left\| \begin{bmatrix} F \\ G \end{bmatrix} \right\|_\infty < \gamma \quad (11)$$

then $\|G\|_\infty < \gamma$ and $\|FG_o^{-1}\|_\infty < 1$, where G_o is a bi-stable spectral factor of $\gamma^2 I - G^\sim G$.

Conversely, if $\|G\|_\infty < \gamma$ and $\|FG_o^{-1}\|_\infty \le 1$, then (11) holds.

Lemma 3 (Nehari's Theorem): For any $R \in \mathcal{RL}_\infty$, we have that

$$\text{dist}(R, \mathcal{RH}_\infty) = \text{dist}(R, \mathcal{H}_\infty) = \|\Gamma_R\|,$$

and that there exists $X \in \mathcal{RH}_\infty$ such that $\|R - X\|_\infty = \text{dist}(R, \mathcal{RH}_\infty)$.

We may now prove Theorem 1.

Proof:

1) Let $\gamma_{\inf} := \inf\{\gamma : \|Y\|_\infty < \gamma, \text{ dist}(R, \mathcal{RH}_\infty) < 1\}$.

Choose $\epsilon > 0$ such that $\alpha < \gamma < \alpha + \epsilon$, implying that there exists $Q \in \mathcal{RH}_\infty$ such that $\|\mathcal{T}_1 - \mathcal{T}_2 Q\|_\infty < \gamma$. Then, by Lemma 1, we have that

$$\left\| \begin{bmatrix} U_i^\sim \\ I - U_i U_i^\sim \end{bmatrix} (\mathcal{T}_1 - \mathcal{T}_2 Q) \right\|_\infty < \gamma. \quad (12)$$

Now, notice that

$$\begin{bmatrix} U_i^\sim \\ I - U_i U_i^\sim \end{bmatrix} \mathcal{T}_2 = \begin{bmatrix} U_o \\ 0 \end{bmatrix}, \tag{13}$$

making (12) equivalent to

$$\left\| \begin{bmatrix} U_i^\sim \mathcal{T}_1 - U_o Q \\ Y \end{bmatrix} \right\|_\infty < \gamma. \tag{14}$$

Applying Lemma 2, this then implies that

$$\|Y\|_\infty < \gamma, \tag{15}$$

and

$$\|U_i^\sim T_1 Y_o^{-1} - U_o Q Y_o^{-1}\|_\infty < 1 \tag{16}$$

By Lemma 3, this in turn implies that $\mathrm{dist}\left(R, U_o(\mathcal{RH}_\infty) Y_o^{-1}\right) < 1$, which, noting that U_o is right invertible in $\mathcal{RH}_\infty$ and that Y_o is invertible in $\mathcal{RH}_\infty$, is equivalent to

$$\mathrm{dist}\left(R, \mathcal{RH}_\infty\right) < 1 \tag{17}$$

Then, from (15) and (17), and the definition of $\gamma_{\inf}$ we conclude that $\gamma_{\inf} \leq \gamma$, and thus that $\gamma < \alpha + \epsilon$. Since ϵ was arbitrary, we then have that $\gamma_{\inf} \leq \alpha$.

To prove the reverse inequality, again choose $\epsilon > 0$ and γ such that $\gamma_{\inf} < \gamma < \gamma_{\inf} + \epsilon$. Then (15) and (17) hold, so (16) holds for some $Q \in \mathcal{RH}_\infty$. Applying the converse of Lemma 2, this in turn implies that

$$\left\| \begin{bmatrix} U_i^\sim \mathcal{T}_1 - U_o Q \\ Y \end{bmatrix} \right\|_\infty \leq \gamma. \tag{18}$$

Finally, reversing the above steps, this leads to $\|\mathcal{T}_1 - \mathcal{T}_2 Q\|_\infty \leq \gamma$. Thus $\alpha \leq \gamma < \gamma_{\inf} + \epsilon$, and hence $\alpha \leq \gamma_{\inf}$.

2) This follows immediately from the previous derivation. ∎

Thus, a high level outline for computing an $\mathcal{H}_\infty$ controller satisfying a γ bound in closed loop is

1) Compute Y and $\|Y\|_\infty$.
2) Select a trial value $\gamma > \|Y\|_\infty$.
3) Compute R and $\|\Gamma_R\|$. Then $\|\Gamma_R\| < 1$ if and only if $\alpha < \gamma$, so increase or decrease γ accordingly, and return to step 2 until a sufficiently accurate upper bound for α is obtained.
4) Find a matrix $X \in \mathcal{RH}_\infty$ such that $\|R - X\|_\infty \leq 1$.
5) Solve $X = U_o Q Y_o^{-1}$ for a $Q \in \mathcal{RH}_\infty$ satisfying $\|\mathcal{T}_1 - \mathcal{T}_2 Q\|_\infty \leq \gamma$.

B. General $\mathcal{T}_3$

We now state the result for general $\mathcal{T}_3$. First, define the following matrices

1) Let U_i, U_o be an inner-outer factorization of $\mathcal{T}_2$ such that $\mathcal{T}_2 = U_i U_o$, with $U_i^\sim U_i = I$, and U_i, U_o, $U_o^\dagger \in \mathcal{RH}_\infty$.
2) Let $Y := (I - U_i U_i^\sim) \mathcal{T}_1$.
3) For $\gamma > \|Y\|_\infty$, let Y_o be a bi-stable spectral factor of $\gamma^2 I - Y^\sim Y$ such that $\gamma^2 I - Y^\sim Y = Y_o^\sim Y_o$, with Y_o, $Y_o^{-1} \in \mathcal{RH}_\infty$.

4) Let V_{co}, V_{ci} be a co-inner-outer factorization of $\mathcal{T}_3 Y_o^{-1}$ such that $\mathcal{T}_3 Y_o^{-1} = V_{co} V_{ci}$ and V_{ci}, V_{co}, $V_{co}^\dagger \in \mathcal{RH}_\infty$.
5) Let $Z := U_i^\sim \mathcal{T}_1 Y_o^{-1} \left(I - V_{ci}^\sim V_{ci} \right)$.
6) If $\|Z\|_\infty < 1$, let Z_{co} be a bi-stable co-spectral factor of $I - ZZ^\sim$ such that $I - ZZ^\sim = Z_{co} Z_{co}^\sim$, with Z_{co}, $Z_{co}^{-1} \in \mathcal{RH}_\infty$.
7) Let $R := Z_{co}^{-1} U_i^\sim \mathcal{T}_1 Y_o^{-1} V_{ci}^\sim$.

Theorem 2: Let $\alpha := \inf\{\|\mathcal{T}_1 - \mathcal{T}_2 Q \mathcal{T}_3\|_\infty : Q \in \mathcal{RH}_\infty\}$. Then

1) $\alpha = \inf\{\gamma : \|Y\|_\infty < \gamma, \|Z\|_\infty < 1, \mathrm{dist}\,(R, \mathcal{RH}_\infty) < 1\}$, and
2) For $\gamma > \alpha$ and Q, $X \in \mathcal{RH}_\infty$ such that
   - $\|R - X\|_\infty \leq 1$, and
   - $X = Z_{co}^{-1} U_o Q V_{co}$,

   we have that $\|\mathcal{T}_1 - \mathcal{T}_2 Q \mathcal{T}_3\|_\infty \leq \gamma$.

Proof: Analogous from that of Theorem 1, and therefore omitted. ∎

Similarly, we may outline a general high level algorithm for computing a controller using Theorem 2:

1) Compute Y and $\|Y\|_\infty$.
2) Select a trial value $\gamma > \|Y\|_\infty$.
3) Compute Z and $\|Z\|_\infty$.
4) If $\|Z\|_\infty < 1$, continue; if not, increase γ and return to step 3.
5) Compute R and $\|\Gamma_R\|$. Then $\|\Gamma_R\| < 1$ if and only if $\alpha < \gamma$, so increase or decrease γ accordingly, and return to step 3 until a sufficiently accurate upper bound for α is obtained.
6) Find a matrix $X \in \mathcal{RH}_\infty$ such that $\|R - X\|_\infty \leq 1$.
7) Solve $X = Z_{co}^{-1} U_o Q V_{co}$ for a $Q \in \mathcal{RH}_\infty$ satisfying $\|\mathcal{T}_1 - \mathcal{T}_2 Q \mathcal{T}_3\|_\infty \leq \gamma$.

IV. DISTRIBUTED $\mathcal{H}_\infty$ CONTROL SUBJECT TO DELAYS

As in [26], we exploit the fact that the communication graph is strongly connected to decompose Q into a local distributed FIR filter $V \in \mathcal{Y}$ and a global, but delayed, IIR component $\Delta_N D \in \frac{1}{z^N} \mathcal{RH}_\infty$, where in particular, $D \in \mathcal{RH}_\infty$ is unconstrained:

$$Q = V + \Delta D \in \mathcal{S}, \text{ with } V \in \mathcal{Y}, D \in \mathcal{RH}_\infty \tag{19}$$

We will show that when Q admits such a decomposition, the norm bound test of Theorem 1 reduces to verifying the existence of a FIR filter $V \in \mathcal{Y}$ such that $\|\Gamma_{\hat{R}(V)}\| < 1$, where $\hat{R}(V)$ is a transfer matrix to be defined that depends affinely on V. Further we will show that verifying the existence of such a V, and constructing it if it exists, can be done by solving a LMI.

A. $\mathcal{T}_3 = I$ Case

We begin with a solution to the $\mathcal{T}_3 = I$ case to simplify the exposition, as the general case, much as in the centralized problem, follows from an analogous argument.

Let

- $\hat{\mathcal{T}}_1(V) := \mathcal{T}_1 - \mathcal{T}_2 V,$
- $\hat{\mathcal{T}}_2 := \mathcal{T}_2 \Delta_N,$

- $\hat{U}_i := U_i \Delta_N$, $\hat{U}_o = U_o \in \mathcal{RH}_\infty$ be inner and outer, respectively, such that $\hat{\mathcal{T}}_2 = \hat{U}_i \hat{U}_o$, and $\hat{U}_o^{-1} \in \mathcal{RH}_\infty$.
- $\hat{R}(V) := \Delta_N^\sim R - \hat{U}_o(\Delta_N^\sim V) Y_o^{-1}$,

with Y_o^{-1} and R defined as in Section III-A. We then have that

Theorem 3: Let $\alpha := \inf\{\|\hat{\mathcal{T}}_1(V) - \hat{\mathcal{T}}_2 D\|_\infty \; : \ldots$
$D \in \mathcal{RH}_\infty, V \in \mathcal{Y}\}$.

Then

1) $\alpha = \inf\{\gamma \; : \; \|Y\|_\infty < \gamma, \exists V \in \mathcal{Y}$ s.t.
   $\mathrm{dist}\left(\hat{R}(V), \mathcal{RH}_\infty\right) < 1\}$, and
2) For $\gamma > \alpha$ and $D, X \in \mathcal{RH}_\infty$ such that
   - $\|\hat{R}(V) - X\|_\infty \leq 1$, and
   - $X = \hat{U}_o D Y_o^{-1}$,

   we have that $\|\hat{\mathcal{T}}_1(V) - \hat{\mathcal{T}}_2 D\|_\infty \leq \gamma$.

Before proving this result, we will need the following lemma:

Lemma 4: For $\hat{Y}(V) := (I - \hat{U}_i \hat{U}_i^\sim)\hat{\mathcal{T}}_1(V)$, we have that $\hat{Y}(V) = Y$, where Y is as defined in Section III-A.

Proof: Straightforward, and thus omitted. ∎

We may now prove Theorem 3.

Proof: 1) Choose $\epsilon > 0$ such that $\alpha < \gamma < \alpha + \epsilon$, implying that there exists $V \in \mathcal{Y}$ and $D \in \mathcal{RH}_\infty$ such that $\|\hat{\mathcal{T}}_1(V) - \hat{\mathcal{T}}_2 D\|_\infty < \gamma$.

We now proceed as in the proof of Theorem 1, and premultiply by

$$\begin{bmatrix} \hat{U}_i^\sim \\ (I - \hat{U}_i \hat{U}_i^\sim) \end{bmatrix}, \tag{20}$$

and apply Lemma 2 to obtain the equivalence between $\|\hat{\mathcal{T}}_1(V) - \hat{\mathcal{T}}_2 D\|_\infty \leq \gamma$ and

$$\left\| \begin{bmatrix} (\hat{U}_i^\sim \hat{\mathcal{T}}_1(V) - \hat{U}_o D) \\ \hat{Y}(V) \end{bmatrix} \right\|_\infty < \gamma. \tag{21}$$

By Lemma 2 and Lemma 4, (21) is equivalent to

$$\|Y\|_\infty < \gamma \tag{22}$$

and

$$\|\hat{U}_i^\sim \hat{\mathcal{T}}_1(V) Y_o^{-1} - \hat{U}_o D Y_o^{-1}\|_\infty < 1. \tag{23}$$

Noting that

$$\begin{aligned}
\hat{U}_i^\sim \hat{\mathcal{T}}_1(V) Y_o^{-1} &= \hat{U}_i^\sim \mathcal{T}_1 Y_o^{-1} - \hat{U}_i^\sim(\mathcal{T}_2 \Delta_N)\Delta_N^\sim V Y_o^{-1} \\
&= \Delta_N^\sim R - \hat{U}_o \Delta_N^\sim V Y_o^{-1} \\
&= \hat{R}(V)
\end{aligned} \tag{24}$$

this is then equivalent to

$$\|\hat{R}(V) - \hat{U}_o D Y_o^{-1}\|_\infty < 1, \tag{25}$$

which by the arguments of the proof of Theorem 1, is equivalent to $\|\Gamma_{\hat{R}(V)}\| < 1$.

The rest of the proof proceeds as that of Theorem 1. ∎

Thus, for a fixed γ, we have reduced the problem to a feasibility test: does there exist a FIR filter $V \in \mathcal{Y}$ such that $\|\Gamma_{\hat{R}(V)}\| < 1$. As per identity (1), this is equivalent to $\|\tilde{\Gamma}_{\hat{R}_1(V)}\| < 1$, with $(\hat{R}_1(V), \hat{R}_2(V))$ an anti-stable/stable decomposition of $\hat{R}(V)$.

Reduction to a LMI: Let R_1 and R_2 be an anti-stable/stable decomposition of $\Delta_N^\sim R$. Now, define $G(V) \in \mathcal{RH}_\infty$ as

$$\begin{aligned}
G(V) &:= \hat{U}_o V Y_o^{-1} \\
&= \textstyle\sum_{i=0}^\infty \frac{1}{z^i} G_i(V).
\end{aligned} \tag{26}$$

where the terms $G_i(V)$ are the impulse response elements of G. It is easily verified that these terms are *affine* in $\{V_i\}$, the impulse response elements of V (i.e. $V = \sum_{i=0}^{N-1} \frac{1}{z^i} V_i$). Note that $G(V) \in \mathcal{RH}_\infty$ follows from $U_o, V, Y_o^{-1} \in \mathcal{RH}_\infty$. As such, let

$$G(V) := \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array} \right]$$

be a minimal stable realization of G.

We then have that

$$\begin{aligned}
\hat{U}_o \Delta_N^\sim V Y_o^{-1} &= \Delta_N^\sim G \\
&= z^N \textstyle\sum_{i=0}^\infty \frac{1}{z^i} G_i(V) \\
&= \textstyle\sum_{k=1}^N z^k G_{N-k}(V) + \ldots \\
&\qquad \textstyle\sum_{j=0}^\infty \frac{1}{z^j} G_{j+N}(V) \\
&=: q(V)^\sim + N_G(V).
\end{aligned} \tag{27}$$

with $q(V) = \sum_{k=1}^N \frac{1}{z^k} G_{N-k}^\top(V) \in \mathcal{RH}_\infty$ and strictly proper.

Also note that $N_G(V)$ has the following state space representation

$$N_G(V) = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G A_G^N & C_G A_G^{N-1} B_G \end{array} \right], \tag{28}$$

and is therefore also clearly in $\mathcal{RH}_\infty$.

The following lemma is an immediate consequence of the previous discussion.

Lemma 5: Let $\hat{R}(V)$ be as defined. Then an anti-stable/stable decomposition of $\hat{R}(V)$ is given by

$$\begin{aligned}
\hat{R}_1(V) &= R_1 - q(V) \\
\hat{R}_2(V) &= R_2 - N_G(V)
\end{aligned} \tag{29}$$

From our previous discussion, we have thus reduced the problem to finding an FIR filter V such that $\|\tilde{\Gamma}_{\hat{R}_1(V)}\| < 1$, for $\hat{R}_1(V)$ given as in (29).

We begin by deriving a state space representation for $\hat{R}_1(V)$, and then use this representation to formulate the Hankel norm bound test as a LMI.

First note that $q(V)$ is simply a strictly causal FIR filter, and thus has a state space representation given by

$$q(V) = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q(V) & 0 \end{array} \right], \tag{30}$$

where A_q is the down-shift operator (i.e. a block matrix with appropriately dimensioned Identity matrices along the first sub block diagonal, and zeros elsewhere), $B_q = [I, 0 \ldots, 0]^\top$, and $C_q(V) = [G_{N-1}(V)^\top, \ldots, G_0(V)^\top]$. Note that only $C_q(V)$ is a function of our design variable V.

Letting the strictly proper $R_1 \in \mathcal{RH}_\infty$ have a minimal stable realization

$$R_1 = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & 0 \end{array} \right] \tag{31}$$

we then have the following realization for $\hat{R}_1(V) \in \mathcal{RH}_\infty$:

$$\hat{R}_1(V) = \left[\begin{array}{cc|c} A_r & 0 & B_r \\ 0 & A_q & B_q \\ \hline C_r & -C_q(V) & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_R & B_R \\ \hline C_R(V) & 0 \end{array}\right]. \tag{32}$$

We emphasize again that our design variable V appears only in $C_R(V)$.

We now recall the variational formulation for the Hankel norm of a strictly proper transfer matrix $F \in \mathcal{RH}_\infty$.

Proposition 1: For a system

$$F = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \mathcal{RH}_\infty,$$

we have that $\|\tilde{\Gamma}_F\| < 1$ if and only if there exist matrices P, $Q \geq 0$ and scalar $\lambda \geq 0$ such that

$$\begin{aligned} \begin{bmatrix} A^\top Q A - Q & C^\top \\ C & -\lambda I \end{bmatrix} &\leq 0 \\[2mm] \begin{bmatrix} -P & PB & PA \\ B^\top P & -I & 0 \\ A^\top P & 0 & -P \end{bmatrix} &\leq 0 \\[2mm] P - Q &\geq 0 \\ \lambda &< 1 \end{aligned} \tag{33}$$

Proof: This is the discrete-time analog of the variational formulation found in Section 6.3.1 of [33]. ∎

Substituting our realization (32) into (33), we see that this is an LMI in the variables $\{V_i\}_{i=0}^{N-1}$, P, Q, and λ, and is feasible if and only if there exists an FIR filter $V \in \mathcal{Y}$ such that $\|\tilde{\Gamma}_{\hat{R}_1(V)}\| < 1$.

Thus, a high level outline for computing a distributed controller satisfying an $\mathcal{H}_\infty$ norm bound of γ in closed loop is

1) Compute Y and $\|Y\|_\infty$.
2) Select a trial value $\gamma > \|Y\|_\infty$.
3) Construct $\hat{R}_1(V)$ and check if the LMI in variables $\{V_i\}_{i=0}^{N-1}$, Q, P and λ

$$\begin{aligned} \begin{bmatrix} A_R^\top Q A_R - Q & C_R(V)^\top \\ C_R(V) & -\lambda I \end{bmatrix} &\leq 0 \\[2mm] \begin{bmatrix} -P & PB_R & PA_R \\ B_R^\top P & -I & 0 \\ A_R^\top P & 0 & -P \end{bmatrix} &\leq 0 \\[2mm] P - Q &\geq 0 \\ \lambda &< 1 \end{aligned} \tag{34}$$

is feasible for $V = \sum_{i=0}^{N-1} \frac{1}{z^i} V_i \in \mathcal{Y}$. This LMI is feasible if and only if $\|\tilde{\Gamma}_{\hat{R}_1(V)}\| < 1$, which in turn occurs if and only if $\alpha < \gamma$, so increase or decrease γ accordingly. This feasibility test will additionally yield an FIR filter $V \in \mathcal{Y}$ that satisfies this bound.

4) Find a matrix $X \in \mathcal{RH}_\infty$, implicitly dependent on V, such that $\|\hat{R}(V) - X\|_\infty \leq 1$ (such a matrix is guaranteed to exist by the same arguments as those used in the centralized case).

5) Solve $X = \hat{U}_o D Y_o^{-1}$ for $D \in \mathcal{RH}_\infty$ satisfying $\|\hat{\mathcal{T}}_1(V) - \hat{\mathcal{T}}_2 D\|_\infty \leq \gamma$.
6) Set $Q = V + \Delta_N D \in \mathcal{S} \bigcap \mathcal{RH}_\infty$

B. General $\mathcal{T}_3$

Define the following transfer matrices
1) $\hat{Z} = \hat{U}_i^\sim \mathcal{T}_1 Y_o^{-1}(I - V_{ci} V_{ci}^\sim)$,
2) $\hat{R}(V) := \Delta_N^\sim R - Z_{co}^{-1} \hat{U}_o (\Delta_N^\sim V) V_{co}$,

and let Y_o^{-1}, R, V_{co} and Z_{co}^{-1} be as defined in Section III-B, and $\hat{\mathcal{T}}_1(V)$, $\hat{\mathcal{T}}_2$, $\hat{U}_i$ and $\hat{U}_o$ be as defined in Section IV-A. We note that just as $\hat{Y}(V)$ was independent of V, so too would be the analogous $\hat{Z}(V)$ – as such we simply define $\hat{Z}$ and not $\hat{Z}(V)$.

Theorem 4: Let $\alpha := \inf\{\|\hat{\mathcal{T}}_1(V) - \hat{\mathcal{T}}_2 D \mathcal{T}_3\|_\infty : D \in \mathcal{RH}_\infty, V \in \mathcal{Y}\}$.

Then

1) $\alpha = \inf\{\gamma : \|Y\|_\infty < \gamma, \|Z\|_\infty < 1, \exists V \in \mathcal{Y} \text{ s.t. } \mathrm{dist}\left(\hat{R}(V), \mathcal{RH}_\infty\right) < 1\}$, and
2) For $\gamma > \alpha$ and D, $X \in \mathcal{RH}_\infty$ such that
   - $\|\hat{R}(V) - X\|_\infty \leq 1$, and
   - $X = Z_{co}^{-1} \hat{U}_o D V_{co}$,

   we have that $\|\hat{\mathcal{T}}_1(V) - \hat{\mathcal{T}}_2 D \mathcal{T}_3\|_\infty \leq \gamma$.

Proof: Analogous to that of Theorem 3, and therefore omitted. ∎

Just as in the $\mathcal{T}_3 = I$ case, this problem has now been reduced to finding an FIR filter $V \in \mathcal{Y}$ such that $\|\tilde{\Gamma}_{\hat{R}(V)}\| < 1$. The arguments of the preceding section apply nearly verbatim, with the exception of replacing equation (26) with

$$G(V) := Z_{co}^{-1} \hat{U}_o V V_{co} \tag{35}$$

Therefore, a high level outline for computing a distributed controller satisfying an $\mathcal{H}_\infty$ norm bound of γ in closed loop is

1) Compute Y and $\|Y\|_\infty$.
2) Select a trial value $\gamma > \|Y\|_\infty$.
3) Compute $\hat{Z}$ and $\|\hat{Z}\|_\infty$.
4) If $\|\hat{Z}\|_\infty < 1$, continue; if not, increase γ and return to step 3.
5) Construct $\hat{R}_1(V)$ according to (29), with $G(V)$ defined as in (35), and check if there exists $V \in \mathcal{Y}$ such that LMI (34) is feasible. This LMI is feasible if and only if $\|\tilde{\Gamma}_{\hat{R}_1(V)}\| < 1$, which in turn occurs if and only if $\alpha < \gamma$, so increase or decrease γ accordingly. This feasibility test will additionally yield an FIR filter $V \in \mathcal{Y}$ that satisfies this bound.
6) Find a matrix $X \in \mathcal{RH}_\infty$, implicilty dependent on V, such that $\|\hat{R}(V) - X\|_\infty \leq 1$.
7) Solve $X = Z_{co}^{-1} \hat{U}_o D V_{co}$ for $D \in \mathcal{RH}_\infty$ satisfying $\|\hat{\mathcal{T}}_1(V) - \hat{\mathcal{T}}_2 D \mathcal{T}_3\|_\infty \leq \gamma$.
8) Set $Q = V + \Delta_N D \in \mathcal{S} \bigcap \mathcal{RH}_\infty$

V. EXAMPLE

For the convenience of the reader, we provide explicit state-space formulae for the factorizations and approximations required to implement our algorithm in the Appendix.

We consider a three-player chain with communication delay of $\tau_c = 1$ – the sparsity constraint $\mathcal{Y}$ on the FIR filter is as given in equation (8). We first consider the simplified case of $P_{21} = I$ – the remaining dynamics of P_{11}, P_{12} and P_{22} are given by

$$
A = \begin{bmatrix} .5 & .2 & 0 \\ .2 & .5 & .2 \\ 0 & .2 & .5 \end{bmatrix}, \quad B_1 = [I_{3\times 3}\; 0_{3\times 3}] \quad B_2 = I_{3\times 3},
$$
$$
C_1 = \begin{bmatrix} I_{3\times 3} \\ 0_{3\times 3} \end{bmatrix}, \qquad D_{11} = 0_{6\times 6}, \qquad D_{12} = \begin{bmatrix} 0_{3\times 3} \\ I_{3\times 3} \end{bmatrix}
$$
$$
C_2 = I_{3\times 3}, \qquad D_{21} = [0_{3\times 3}\; I_{3\times 3}], \quad D_{22} = 0_{3\times 3}, \tag{36}
$$

Note that this is a suitably modified version of the output feedback problem considered in [26].

We first computed the optimal centralized norm of the system using classical results [34], and obtained a centralized closed loop norm of .9772. We note that this is the theoretical lower bound as given by $\|Y\|_\infty$ from the algorithms we described above. To verify the consistency of our algorithm, we used our LMI formulation to compute a centralized controller as well. This was done by allowing the elements of the FIR filter V_0 and V_1 to be unconstrained, and not suprisingly, we were also able to achieve a closed loop norm of .9772 in this manner. We then constrained V to lie in the subspace $\mathcal{Y}$ as given by (8), and surprisingly, we were still able to achieve a closed loop norm of .9772. This is a significant improvement over the delayed system (i.e. V_0 and V_1 constrained to be zero), for which we were only able to achieve a closed loop norm of 1.6856.

We then considered the general output-feedback problem, with P_{21} given by the parameters in (36) as well. The centralized and LMI computed centralized closed loop norms were both found to be 1.502, with the best distributed norm found to be 1.515. Once again, we see near identical performance from the centralized and distributed solutions, whereas the delayed controller was only able to achieve a closed loop norm of 2.213.

VI. Conclusion

This paper presented an LMI based characterization of the sub-optimal delay-constrained distributed $\mathcal{H}_\infty$ control problem. By exploiting the strongly connected nature of the communication graph, we were able to reduce the problem to a feasibility test in terms of the Hankel norm of a certain transfer matrix that is a function of the localized FIR component of the controller. We note that much as in the $\mathcal{H}_2$ case, by reducing the control synthesis problem to one that is convex in the FIR filter, communication delay co-design [35], [36] and augmentation [37] methods are applicable. However, although finite dimensional, this method is based on the "1984" approach to $\mathcal{H}_\infty$ control – as such, the computational burden is quite high, limiting the scalability of the approach.

Future work will therefore focus on the following three aspects: (1) adapting the parameterization used in [26] so as to relax the assumption of a stable plant, (2) formally integrating communication delay co-design methods into the controller synthesis procedure, and most pressingly (3) seeking more direct and computationally scalable means of identifying appropriate FIR filters.

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APPENDIX

In all of the following, we assume that the conditions needed for the existence of the required stabilizing solution of the corresponding Discrete Algebraic Riccati Equations (DARE) are met – the reader is referred to [34] and [38] for more details. All "co-X" factorizations, where "X" may be either inner-outer or bi-stable spectral, can be obtained by transposing the "X" factorization of the transpose system.

A. Inner-Outer Factorizations

Let
$$G := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathcal{RH}_\infty.$$

From [34], an inner-outer factorization $G = U_i U_o$ of G, with U_i inner and U_o outer, is given by

$$U_i = \left[\begin{array}{c|c} A + BF & BH^{-1} \\ \hline C + DF & DH^{-1} \end{array} \right] \tag{37}$$

$$U_o = \left[\begin{array}{c|c} A & B \\ \hline -HF & H \end{array} \right] \tag{38}$$

with $H = (D^\top D + B^\top X B)^{\frac{1}{2}}$, and X the stabilizing solution of the following DARE

$$\begin{array}{rcl} X & = & A^\top X A + C^\top C + A^\top X B F, \\ F & = & -(D^\top D + B^\top X B)^{-1} B^\top X A. \end{array} \tag{39}$$

B. Bi-stable Spectral Factorizations

Let $Y \in \mathcal{RH}_\infty$ be strictly proper, and let

$$G_Y = \left[\begin{array}{c|c} A_Y & B_Y \\ \hline C_Y & 0 \end{array} \right]$$

be a state-space realization of the strictly proper $\mathcal{RH}_\infty$ component of $Y^\sim Y$.

If A_Y is invertible, then it holds that

$$\gamma^2 I - Y^\sim Y = G_Y + G_Y^\sim + D_Y + D_Y^\top$$

where $D_Y = \frac{1}{2} \left(\gamma^2 I + B_Y^\top A_Y^{-\top} C_Y^\top \right)$.

A bi-stable spectral factorization $\gamma^2 I - Y^\sim Y = M^\sim M$, with M, $M^{-1} \in \mathcal{RH}_\infty$ is then given by

$$M = \left[\begin{array}{c|c} A_Y & B_Y \\ \hline H^{-1}(C_Y + B_Y^\top X A_Y) & H \end{array} \right] \tag{40}$$

with $H = (D_Y + D_Y^\top + B_Y^\top X B_Y)^{\frac{1}{2}}$, and X the stabilizing solution of the following DARE

$$\begin{array}{rcl} X & = & A_Y^\top X A_Y + (A_Y^\top X B_Y + C_Y^\top)F, \\ F & = & -(D_Y^\top + D_Y + B_Y^\top X B_Y)^{-1}(B_Y^\top X A_Y + C_Y). \end{array} \tag{41}$$

This result follows directly from standard results on spectral factors and positive real systems [34]

C. Stable Approximations

The following is taken from [38]. Let

$$G := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathcal{RH}_\infty$$

be a minimal state-space representation, and assume that $\rho = \|\tilde{\Gamma}_G\| < \gamma$. Let X and Y be the controllability and observability Gramians of G, respectively.

Let $Q \in \mathcal{RH}_\infty$ have the state-space representation

$$Q := \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array} \right]$$

with

$$\begin{array}{ll} A_Q = A - BC_Q, & B_Q = AXC^\top + BE^\top, \\ C_Q = (E^\top C + B^\top Y A)N, & D_Q = D^\top - E^\top, \end{array}$$

where $N = (\gamma^2 I - XY)^{-1}$, and for any unitary matrix U,

$$E = -(I + CNXC^\top)^{-1} CNXA^\top Y B$$
$$+ \gamma(I + CNXC^\top)^{-\frac{1}{2}} U(I + B^\top Y N B)^{-\frac{1}{2}}.$$

Then $\|G - Q^\sim\|_\infty = \gamma$ and $(G - Q^\sim)^\sim(G - Q^\sim) = \gamma^2 I$.