A Dual Problem in $\mathcal{H}_2$ Decentralized Control subject to Delays

Nikolai Matni and John C. Doyle

Abstract— It has been shown that the decentralized $\mathcal{H}_2$ model matching problem subject to delay can be solved by decomposing the controller into a centralized, but delayed, component and a decentralized FIR component, the latter of which can be solved for via a linearly constrained quadratic program. In this paper, we derive the dual to this optimization problem, show that strong duality holds, and exploit this to further analyze properties of the control problem. Namely, we determine *a priori* upper and lower bounds on the optimal $\mathcal{H}_2$ cost, and obtain further insight into the structure of the optimal FIR component. Furthermore, we show how the optimal dual variables can be used to inform communication graph augmentation, and illustrate this idea with a routing problem.

I. INTRODUCTION

Decentralized control problems arise when several decision makers, or controllers, need to determine their actions, or inputs, based only on a subset of the total information available about the system. These types of problems arise in areas as diverse as physiology, economics and the power grid. A particular class of decentralized control problems that has received a significant amount of attention over the past few decades is that of optimal $\mathcal{H}_2$ (or LQG) control subject to delay constraints. In this case, the information constraints can be interpreted as arising from a communication graph, in which edge weights between nodes correspond to the delay required to transmit information between them.

For the special case of the one-step delay information sharing pattern, the $\mathcal{H}_2$ problem was solved in the 1970s using dynamic programming [1], [2], [3]. For more complex delay patterns, the separation principle fails [4], [5], [6], making extensions beyond the state feedback case [7], [8] difficult, although recent work [9] provides two dynamic programming decompositions for the general delayed sharing model.

This paper focusses on the output feedback $\mathcal{H}_2$ problem with quadratically invariant [15] communication delays, which has been previously solved using vectorization [15] or linear matrix inequalities (LMI) [16], [17], and most recently using an extension of spectral factorization [18]. Additionally, methods and/or solutions exist for special instances of this problem, such as the two-player systems considered in [10], [11], and spatially invariant systems [12], [13], [14].

In [18], the problem is solved by decomposing the controller into a centralized, but delayed, component and a decentralized FIR component. It is shown that the optimal FIR component can be solved for via a linearly constrained

N. Matni and J.C. Doyle are with the Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, CA. nmatni@caltech.edu.

quadratic program. In this paper, we show how this problem can be converted to a semi-definite program (SDP) for which strong duality holds. Then, much in the spirit of [19], we exploit strong duality to further analyze the problem. Namely, we determine *a priori* upper and lower bounds on the optimal $\mathcal{H}_2$ cost, and obtain further insight into the structure of the optimal FIR component. Furthermore, most interestingly, and perhaps most practical, we show how the optimal dual variables can be used to inform communication graph augmentation, and illustrate this idea with a routing problem.

This paper is structured as follows. Section II introduces the general problem studied in this paper, and presents the primal SDP. In Section III, we derive the dual problem, show that strong duality holds, which we then exploit to further analyze the optimal control problem. Section IV presents a special type of routing problem for which we can determine the best action for the router to take based on the optimal dual variables, and Section V ends with conclusions and suggestions for future work.

II. PRELIMINARIES AND PRIMAL PROBLEM FORMULATION

1) $\mathcal{H}_2$ Preliminaries: Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of complex numbers. A function $G : (\mathbb{C} \bigcup \{\infty\}) \backslash \mathbb{D} \to \mathbb{C}^{p \times q}$ is in $\mathcal{H}_2$ if it can be expanded as

$$G(z) = \sum_{i=0}^{\infty} \frac{1}{z^i} G_i$$

where $G_i \in \mathbb{C}^{p \times q}$ and $\sum_{i=0}^{\infty} \text{Tr}(G_i G_i^*) < \infty$. Define the conjugate of G by

$$G(z)^{\sim} = \sum_{i=0}^{\infty} z^i G_i^*$$

$\mathcal{H}_2$ is a Hilbert space with inner product given by

$$
\begin{aligned}
< G, H > &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Tr}(G(e^{j\theta}) H(e^{j\theta})^{\sim}) d\theta \\
&= \sum_{i=0}^{\infty} \text{Tr}(G_i H_i^*),
\end{aligned}
$$

where the last equality follows from Parseval's identity.

Finally, if $\mathcal{M}$ is a subspace of $\mathcal{H}_2$, denote the orthogonal projection onto $\mathcal{M}$ by $\mathbb{P}_{\mathcal{M}}$.

2) Problem Formulation: Let P be a stable discrete-time plant given by

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with inputs of dimension p_1, p_2 and outputs of dimension q_1, q_2. We restrict attention to stable plants for simplicity. These methods could also be applied to an unstable plant if a stable stabilizing nominal controller can be found, as in [15]. We note that this task may be non-trivial, with strong guarantees existing only in the sparsity constrained setting [20].

To ensure the existence of stabilizing solutions to the appropriate Riccati equations (note that stabilizability and detectability of (A, B_2, C_2) is implied by the assumption of a stable plant), we assume

- $D_{12}^T D_{12} > 0$,
- $D_{21} D_{21}^T > 0$,
- $C_1^T D_{12} = 0$
- $B_1 D_{21}^T = 0$

For $N \geq 1$, define the space of strictly proper finite impulse response (FIR) transfer matrices by $\mathcal{X}_N = \oplus_{i=1}^{N} \frac{1}{z^i} \mathbb{C}^{p_2 \times q_2}$. Note that in the following, we sometimes suppress the subscript and write $\mathcal{X}_N = \mathcal{X}$ when N is clear from context. We can therefore decompose $\frac{1}{z}\mathcal{H}_2$ into orthogonal subspaces as

$$\frac{1}{z}\mathcal{H}_2 = \mathcal{X}_N \oplus \frac{1}{z^{N+1}}\mathcal{H}_2,$$

In this paper, we are concerned with controller constraints described by delay patterns that are imposed by *strongly connected communication graphs*. As such, let $\mathcal{R}_p$ be the space of proper real rational transfer matrices, and $\mathcal{S} \subset \frac{1}{z}\mathcal{R}_p$ be a subspace of the form

$$\mathcal{S} = \mathcal{Y} \oplus \frac{1}{z^{N+1}}\mathcal{R}_p \qquad (1)$$

where $\mathcal{Y} = \oplus_{i=1}^{N} \frac{1}{z^i}\mathcal{Y}_i \subset \oplus_{i=1}^{N} \frac{1}{z^i}\mathbb{R}^{p_2 \times q_2} \subset \mathcal{X}_N$. Specifically, this implies that every decision-making agent has access to *all* measurements that are at least $N+1$ time-steps old.

We can therefore partition the measured outputs y and control inputs u according to the dimension of the subsystems:

$$y = [\ y_1^T \ \cdots \ y_m^T]^T \quad u = [\ u_1^T \ \cdots \ u_n^T]^T$$

and then further partition each constraint set $\mathcal{Y}_i$ as

$$\mathcal{Y}_i = \begin{bmatrix} \mathcal{Y}_i^{11} & \cdots & \mathcal{Y}_i^{1m} \\ \vdots & \ddots & \vdots \\ \mathcal{Y}_i^{n1} & \cdots & \mathcal{Y}_i^{nm} \end{bmatrix}, \qquad (2)$$

where

$$\mathcal{Y}_i^{jk} = \begin{cases} \mathbb{R}^{p_2^j \times q_2^k} & \text{if } u_j \text{ has access to } y_k \text{ at time } i \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

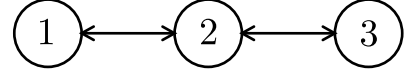and $\sum_{j=1}^{n} p_2^j = p_2$, $\sum_{k=1}^{m} q_2^k = m$.

Fig. 1: The graph depicts the communication structure of the three-player chain problem. Edge weights (not shown) indicate the delay required to transmit information between nodes.

Example 1: Consider the three player chain problem as illustrated in Figure 1, with communication delay τ_c between nodes. Then

$$\begin{aligned} \mathcal{S} &= \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^{1+\tau_c}}\mathcal{R}_p & \frac{1}{z^{1+2\tau_c}}\mathcal{R}_p \\ \frac{1}{z^{1+\tau_c}}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p & \frac{1}{z^{1+\tau_c}}\mathcal{R}_p \\ \frac{1}{z^{1+2\tau_c}}\mathcal{R}_p & \frac{1}{z^{1+\tau_c}}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix} \\ &= \oplus_{i=1}^{2\tau_c} \frac{1}{z^i}\mathcal{Y}_i \oplus \frac{1}{z^{2\tau_c+1}}\mathcal{R}_p \end{aligned}$$

with

$$\mathcal{Y}_i = \begin{cases} \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} & \text{for } i \leq \tau_c \\[20pt] \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} & \text{for } \tau_c < i \leq 2\tau_c \end{cases},$$

where, for compactness, * is used to denote a space of appropriately sized real matrices. In this setting, every decision maker then has access to all measurements that are at least $2\tau_c + 1$ time-steps old.

The decentralized control problem of interest is to design a controller $K \in \mathcal{S}$ so as to minimize the closed loop $\mathcal{H}_2$ norm of the system:

$$\begin{aligned} \underset{K}{\text{minimize}} \quad & \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|_{\mathcal{H}_2} \\ \text{s.t.} \, & K \in \mathcal{S} \end{aligned} \qquad (4)$$

In [15], it was shown that a necessary and sufficient condition to be able to pass to the Youla parameter $Q = K(I - P_{22}K)^{-1}$ in (4) is that the constraint set be *quadratically invariant*.

Definition 1: A set $\mathcal{S}$ is *quadratically invariant* under P_{22} if

$$KP_{22}K \in \mathcal{S} \text{ for all } K \in \mathcal{S}$$

Under this assumption on $\mathcal{S}$ and P_{22}, we can pass without loss to the Youla domain. Since K is strictly proper and stabilizing, Q must be strictly proper and stable; thus (4) can be reduced to the following model matching problem:

$$\begin{aligned} \underset{Q}{\text{minimize}} \quad & \|P_{11} + P_{12}QP_{21}\|_{\mathcal{H}_2} \\ \text{s.t.} \, & Q \in \mathcal{S} \cap \frac{1}{z}\mathcal{H}_2 \end{aligned} \qquad (5)$$

For technical simplicity, all controllers in this paper are assumed to be strictly proper – the results extend to non-strictly proper controllers, but the resulting formulas are more complicated. Although this problem admits several solutions [9], [15], [16], [17], we follow the one presented in [18], as it has structure that we exploit in the sequel.

3) Reduction to a Quadratic Program: Let X, Y be the stabilizing solutions to the following Riccati Equations

$$
\begin{aligned}
X &= C_1^T C_1 + A^T X A - (A^T X B_2 + C_1^T D_{12}) \times \\
&\quad \Omega^{-1}(A^T X B_2 + C_1^T D_{12})^T \\
Y &= B_1^T B_1 + A Y A^T - (A Y C_2^T + B_1 D_{21}^T) \times \\
&\quad \Psi^{-1}(A Y C_2^T + B_1 D_{21}^T)^T
\end{aligned}
$$

where $\Omega := D_{12}^T D_{12} + B_2^T X B_2$, and $\Psi := D_{21} D_{21}^T + C_2 Y C_2^T$. Define the regulator and filter gains, respectively, as

$$
\begin{aligned}
K &= -\Omega^{-1}(B_2^T X A + D_{12}^T C_1) \\
L &= -(A Y C_2^T + B_1 D_{21}^T)\Psi^{-1}
\end{aligned}
$$

and the auxiliary matrix T by

$$
T = \Omega^{1/2} \left[\begin{array}{c|c} A & L \\ \hline K & 0 \end{array} \right] \Psi^{1/2}. \tag{6}
$$

Finally, let W_L and W_R be left and right spectral factors for $P_{12}^\sim P_{12}$ and $P_{21} P_{21}^\sim$ such that

$$
\begin{aligned}
P_{12}^\sim P_{12} &= W_L^{-\sim} W_L^{-1} \\
P_{21} P_{21}^\sim &= W_R^{-1} W_R^{-\sim}.
\end{aligned}
$$

We first present the classical solution to the delayed model matching problem, from which the decentralized solution is then constructed.

Theorem 1: The optimal solution to the delayed model matching problem

$$
\begin{aligned}
\text{minimize}_Q \quad & \|P_{11} + P_{12} Q P_{21}\|_{\mathcal{H}_2} \\
\text{s.t.} \, & Q \in \frac{1}{z^{N+1}}\mathcal{H}_2
\end{aligned}
$$

is given by $Q_N = -W_L \mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2}(T) W_R$.

Theorem 2: (From [18]) The optimal solution to (5) is given by

$$
Q^* = U^* + V^*
$$

where $V^* \in \mathcal{Y}$ is the unique minimizer of

$$
\|G(V)\|_{\mathcal{H}_2}^2 + 2 < G(V), T > \tag{7}
$$

with $G(V) = \mathbb{P}_\mathcal{X}(W_L^{-1} V W_R^{-1})$, and

$$
U^* = Q_N - W_L \mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2}(W_L^{-1} V^* W_R^{-1}) W_R \in \frac{1}{z^{N+1}}\mathcal{H}_2. \tag{8}
$$

The optimal cost is then given by

$$
\|P_{11} + P_{12} Q_N P_{21}\|_{\mathcal{H}_2}^2 + \|G(V^*)\|_{\mathcal{H}_2}^2 + 2 < G(V^*), T > \tag{9}
$$

The assumption of a strongly connected graph is key in the above, as it allows for the optimal controller Q^* to be decomposed as the direct sum of a FIR filter V^*, and a delayed, but centralized, component U^* that depends only on *globally available* information.

We now present the primal optimization problem that is solved to obtain the FIR component V^* of the optimal decentralized controller. For ease of notation, let $G_i(V) = G_i$, and $H = W_L^{-1}$, $J = W_R^{-1}$. Note that H and J can be expanded as $H = \sum_{i=0}^\infty \frac{1}{z^i} H_i$ and $J = \sum_{i=0}^\infty \frac{1}{z^i} J_i$.

Similarly, T and V admit the expansions $T = \sum_{i=1}^\infty \frac{1}{z^i} T_i$, and $V = \sum_{i=1}^N \frac{1}{z^i} V_i \in \mathcal{Y}$, with each $V_i \in \mathcal{Y}_i$.[1]

Lemma 1: (Primal Problem): The FIR transfer matrix $G(V)$ can be written as

$$
G(V) = \sum_{i=1}^N \frac{1}{z^i} G_i, \text{ with } G_i = \sum_{\substack{j,l \geq 0, k \geq 1 \\ j+k+l=i}} H_j V_k J_l
$$

and, applying Parseval's identity to (7), we can formulate the optimization problem as

$$
\begin{aligned}
\text{minimize}_V \quad & \sum_{i=1}^N \text{Tr} G_i G_i^* + 2 \sum_{i=1}^N \text{Tr} G_i T_i^* \\
\text{s.t.} \, & V_i \in \mathcal{Y}_i
\end{aligned} \tag{10}
$$

Remark 1: It was shown in [18] that (10) is a convex quadratic program with a unique solution.

We now apply standard techniques from convex analysis and optimization to reformulate (10) as a semi-definite program (SDP) that is more amenable to dualization.

Lemma 2: The quadratic program (10) can be recast as the following SDP

$$
\begin{aligned}
\underset{\{V_i\}_{i=1}^N, \{W_i\}_{i=1}^N}{\text{minimize}} \quad & \sum_{i=1}^N \text{Tr} W_i + 2 \sum_{i=1}^N \text{Tr} G_i T_i^* \\
\text{s.t.} \, & \begin{bmatrix} I & G_i^* \\ G_i & W_i \end{bmatrix} > 0, \text{ for } i = 1, ..., N \\
& e_j^* V_i e_k = 0, \forall i, j, k \text{ s.t. } \mathcal{Y}_i^{jk} = 0
\end{aligned} \tag{11}
$$

where $e_i = [0, ..., I, ..., 0]^T$, with the identity matrix I in the ith position taken to be of appropriate dimension based on context.

Proof: We first note that $V_i \in \mathcal{Y}_i \iff e_j^* V_i e_k = 0 \, \forall j, k$ s.t. $\mathcal{Y}_i^{jk} = 0$. Introducing slack variables $\{W_i\}_{i=1}^N$ we rewrite (10) as

$$
\begin{aligned}
\underset{\{V_i\}_{i=1}^N, \{W_i\}_{i=1}^N}{\text{minimize}} \quad & \sum_{i=1}^N \text{Tr} W_i + 2 \sum_{i=1}^N \text{Tr} G_i T_i^* \\
\text{s.t.} \, & W_i - G_i G_i^* > 0, \text{ for } i = 1, ..., N \\
& e_j^* V_i e_k = 0, \forall i, j, k \text{ s.t. } \mathcal{Y}_i(j, k) = 0
\end{aligned} \tag{12}
$$

We complete the proof by applying a Schur-complement transformation to the inequality constraints. ∎

III. DUAL PROBLEM FORMULATION

Before proceeding to the derivation of the dual problem, we present the following useful lemma:

Lemma 3: Let G_i be defined as above, and Z_i be any matrix of compatible dimension. Then

$$
\sum_{i=1}^N \text{Tr} G_i Z_i = \sum_{i=1}^N \sum_{\substack{j,l \geq 0, k \geq 1 \\ k-j-l=i}} \text{Tr} V_i J_l Z_k H_j
$$

Proof: Easily verified using the cyclical property of the trace operator and the definition of G_i. ∎

We now present the main result of the paper, the dual formulation of (11).

Theorem 3: The dual problem to (11) is given by

$$\underset{\nu_{jk}^i, X_i}{\text{maximize}} \quad -\sum_{i=1}^{N} \text{Tr} X_i X_i^*$$
$$\text{s.t.} \quad \sum_{(j,k)\in\mathcal{I}_i} e_k \nu_{jk}^i e_j^* - 2 \sum_{\substack{j,l\geq 0, k\geq 1 \\ k-j-l=i}} J_l Z_k H_j = 0$$
$$\text{for } i = 1, ..., N$$
$$\tag{13}$$

where $\mathcal{I}_i := \{(j,k) \; : \; \mathcal{Y}_i^{jk} = 0\}$ and $Z_k = (T_k^* - X_k)$. Furthermore, strong duality holds, the dual optimum d^* is achieved, and is bounded by $0 \geq d^* \geq -||\mathbb{P}_\mathcal{X}(T)||_{\mathcal{H}_2}^2$.

Proof: Introduce Lagrange multipliers $\{v_{jk}^i\}_{(j,k)\in\mathcal{I}_i}^{i=1,...,N}$ and $\{\chi_i\}_{i=1,...N}$ with

$$\chi_i = \begin{bmatrix} X_{11}^i & X_{12}^i \\ X_{12}^{i*} & X_{22}^i \end{bmatrix} \geq 0.$$

The Lagrangian of (11) can then be written as

$$L(\{W_i\}, \{V_i\}; \{\chi_i\}, \{\nu_{jk}^i\}) = \sum_{i=1}^{N} \Big(\text{Tr}W_i + 2\text{Tr}G_i T_i^*$$
$$-\text{Tr}\begin{bmatrix} X_{11}^i & X_{12}^i \\ X_{21}^{i*} & X_{22}^i \end{bmatrix}\begin{bmatrix} I & G_i^* \\ G_i & W_i \end{bmatrix} + \sum_{(j,k)\in\mathcal{I}_i} \text{Tr}\nu_{jk}^i e_j^* V_i e_k \Big)$$
$$= \sum_{i=1}^{N} \Big(\text{Tr}W_i(I - X_{22}^i) + 2\text{Tr}G_i(T_i^* - X_{12}^i)$$
$$-\text{Tr}X_{11}^i + \sum_{(j,k)\in\mathcal{I}_i} \text{Tr}V_i e_k \nu_{jk}^i e_j^* \Big)$$

Applying Lemma 3 to $\sum_{i=1}^{N} \text{Tr}G_i Z_i$ and grouping like terms, the above can be rewritten as

$$L(\{W_i\}, \{V_i\}; \{\chi_i\}, \{\nu_{jk}^i\}) =$$
$$\sum_{i=1}^{N} \big(\text{Tr}W_i(I - X_{22}^i) - \text{Tr}X_{11}^i \big)$$
$$+ \sum \text{Tr}V_i \left(\sum_{(j,k)\in\mathcal{I}_i} e_k \nu_{jk}^i e_j^* - 2 \sum_{\substack{j,l\geq 0, k\geq 1 \\ k-j-l=i}} J_l Z_k H_j \right)$$

Infimizing the above expression over $\{W_i\}, \{V_i\}$, we see that it is bounded below if and only if

$$X_{22}^i = I$$
$$\sum_{(j,k)\in\mathcal{I}_i} e_k \nu_{jk}^i e_j^* - 2 \sum_{\substack{j,l\geq 0, k\geq 1 \\ k-j-l=i}} J_l Z_k H_j = 0$$

for all $i = 1, ..., N$. The dual problem then becomes

$$\underset{\nu_{jk}^i, X_i}{\text{maximize}} \; -\sum_{i=1}^{N} \text{Tr} X_{11}^i \; \text{s.t.}$$
$$\sum_{(j,k)\in\mathcal{I}_i} e_k \nu_{jk}^i e_j^* - 2 \sum_{\substack{j,l\geq 0, k\geq 1 \\ k-j-l=i}} J_l Z_k H_j = 0$$
$$\begin{bmatrix} X_{11}^i & X_{12}^i \\ X_{12}^{i*} & I \end{bmatrix} \geq 0, \quad \forall i = 1, ..., N$$

Applying a Schur-complement argument to the inequality constraints, we have $X_{11}^i - X_{12}^i X_{12}^{i*} \geq 0$, which can be taken to be equality without loss as the objective is to minimize $\text{Tr}X_{11}^i$. Relabeling X_{12}^i as X_i, we obtain (13).

To show that strong duality holds, and that the dual optimum is achieved, by Slater's condition, it suffices to show that (11) admits a strictly feasible point. Indeed, it is trivially verified that $V_i = 0$, $W_i = \frac{\epsilon I}{N p_2 q_2}$, $\epsilon > 0$, $\forall i = 1, ..., N$ is such a point. We therefore have that the primal optimal value $p^* \leq \epsilon$ for arbitrary $\epsilon > 0$.

Finally, setting $X_i = T_i$ and $\nu_{jk}^i \equiv 0$, we see that this is a dual feasible point, with dual objective $-\sum_{i=1}^{N} \text{Tr}T_i T_i^* = -||\mathbb{P}_\mathcal{X}(T)||_{\mathcal{H}_2}^2$. Therefore, by strong duality, $0 \geq p^* = d^* \geq -||\mathbb{P}_\mathcal{X}(T)||_{\mathcal{H}_2}^2$, where d^* is the dual optimal value ∎

An immediate consequence of the above formulation are the intuitive inequalities among the different optimization problems:

Corollary 1: Let C_0 be the optimal centralized cost of (7) with $N = 0$, C_d the optimal decentralized cost of (5) and C_N be the delayed centralized optimal cost of (7). Then

$$C_0 \leq C_d \leq C_N$$

Proof: Note that, from (9) $C_d = C_N + p^*$, where $-||\mathbb{P}_\mathcal{X}(T)||_{\mathcal{H}_2}^2 \leq p^* \leq 0$ is the optimal value of (11) and (13). Therefore the inequalilty $C_d \leq C_N$ follows immediately. It therefore suffices to show that $C_0 = C_N - ||\mathbb{P}_\mathcal{X}(T)||_{\mathcal{H}_2}^2$, but

$$
\begin{aligned}
C_0 &= ||P_{11}||_{\mathcal{H}_2}^2 - ||T||_{\mathcal{H}_2}^2 \\
&= ||P_{11}||_{\mathcal{H}_2}^2 - ||\mathbb{P}_{\frac{1}{z^{N+1}}}(T)||_{\mathcal{H}_2}^2 - ||\mathbb{P}_\mathcal{X}(T)||_{\mathcal{H}_2}^2 \\
&= C_N - ||\mathbb{P}_\mathcal{X}(T)||_{\mathcal{H}_2}^2
\end{aligned}
$$

∎

A. Further analysis

In this subsection, we explore further properties that can be inferred from the primal/dual optimization problems.

1) Refined upper and lower bounds: We refine the *a priori* upper and lower bounds on the optimal closed loop norm by finding primal and dual feasible points, respectively.

Lemma 4: Let $k \in \{1, ..., N\}$ and $S_k \in \mathcal{Y}_k$ be fixed. Then a primal feasible point is given by

$$
\begin{aligned}
W_i &= \epsilon I + G_i(V)G_i(V)^*, \; \epsilon > 0 \\
V_k &= v_k S_k \\
V_i &= 0 \; \forall i \neq k \\
v_k &= -\frac{\beta_k}{\alpha_k}
\end{aligned}
$$

with primal objective value (as $\epsilon \downarrow 0$)

$$p_k = -\frac{\beta_k^2}{\alpha_k} \leq 0 \tag{14}$$

where

$$
\begin{aligned}
\beta_k &= \sum_{i=1}^{N} \sum_{\substack{j,l\geq 0 \\ j+l+k=i}} \text{Tr}H_j S_k J_l T_i^* \\
\alpha_k &= \sum_{i=1}^{N} \sum_{\substack{j,l\geq 0 \\ j+l+k=i}} \text{Tr}H_j S_k J_l J_l^* S_k^* H_j^* > 0.
\end{aligned}
$$

Proof: From (12) it is clear that $\{W_i\}$ and $\{V_i\}$ are feasible points. Letting $\epsilon \downarrow 0$, the optimization then reduces to an unconstrained one over v_k:

$$\underset{v_k}{\text{minimize}} \; v_k^2 \sum_{i=1}^{N} \sum_{\substack{j,l\geq 0 \\ j+l+k=i}} \text{Tr}H_j S_k J_l J_l^* S_k^* H_j^*$$
$$+ 2v_k \sum_{i=1}^{N} \sum_{\substack{j,l\geq 0 \\ j+l+k=i}} \text{Tr}H_j S_k J_l T_i^*$$

which can be solved as $\alpha_k > 0$ (it is the sum of the traces of positive semi-definite matrices, that are not identically zero). The optimizer is easily found to be $v_k = -\frac{\beta_k}{\alpha_k}$, resulting in the primal objective value p_k as given in (14). ∎

Lemma 5: Let $(j,k) \in \mathcal{I}_N$. Then a dual feasible point is given by

$$X_i = T_i^{*,} \quad i = 1, ..., N-1 \tag{15a}$$

$$X_N = T_N^* + \Delta \tag{15b}$$

$$\Delta = -\frac{1}{2}J_0^{-1}e_k\nu_{jk}^N e_j^* H_0^{-1} \tag{15c}$$

$$\nu_{jk}^N = \frac{\text{Tr}e_j^* H_0^{-1}J_0^{-1}e_k}{\text{Tr}e_k^* J_0^{-2}e_k e_j^* H_0^{-2}e_j}I \tag{15d}$$

$$\nu_{mn}^i = 0, \quad \forall(i,m,n) \neq (N,j,k) \tag{15e}$$

with dual objective

$$d_{jk} = -||\mathbb{P}_{\mathcal{X}}(T)||_{\mathcal{H}_2}^2 + \frac{\left(\text{Tr}e_j^* H_0^{-1}J_0^{-1}e_k\right)^2}{\text{Tr}e_k^* J_0^{-2}e_k e_j^* H_0^{-2}e_j} \tag{16}$$

Proof: Defining $\{X_i\}$ by (15a, 15b), the dual problem then reduces to

$$\underset{\nu_{jk}^N, \Delta}{\text{maximize}} \quad -\left(\sum_{i=1}^N \text{Tr}T_i T_i^*\right) + 2\text{Tr}T_N\Delta^* - \text{Tr}\Delta\Delta^*$$
$$\text{s.t.} \quad e_k\nu_{jk}^N e_j^* + 2J_0\Delta H_0 = 0$$

Noting that $H_0 = \Omega^{\frac{1}{2}} > 0$, $J_0 = \Psi^{\frac{1}{2}} > 0$ by assumption (c.f. [18] for details), we can solve for Δ as in (15c), which reduces the optimization problem to an unconstrained one in ν_{jk}^N. Proceeding in a similar manner as the previous proof, we set $\nu_{jk}^N = vI$ and obtain the minimizer ν_{jk}^N as in (15d), resulting in a dual objective of d_{jk} as in (16). ∎

With these two lemmas, we can refine the bounds in Corollary 1 to

$$C_0 + \max_{(j,k)\in\mathcal{I}_N} \gamma_{jk} \leq C_d \leq C_N + \min_{k\in\{1,...,N\}} p_k$$

where $\gamma_{jk} := d_{jk} + ||\mathbb{P}_{\mathcal{X}}(T)||_{\mathcal{H}_2}^2$.

2) Dual variable interpretation and application to communication network design: By applying the shadow price interpretation to the dual variables $\{\nu_{jk}^i\}$, we are able to identify the most active equality constraint indices $(i,j,k) = \arg\max_{i,j,k} \text{Tr}(\nu_{jk}^i \nu_{jk}^{i*})$ (corresponding to the constraint $e_j^* V_i e_k = 0$) imposed by our controller constraint set $\mathcal{S}$. Eliminating this constraint will therefore result in the greatest incremental improvement in the optimal closed loop norm.

In this same spirit, suppose that we are interested in identifying the delay which, if reduced, would yield the best improvement in the closed loop norm. Using the same interpretation, we can solve for $(N,j,k) = \arg\max_{j,k} \text{Tr}(\nu_{jk}^N \nu_{jk}^{N*})$. This information can be used to inform certain types of routing decisions, as will be illustrated in Section IV.

3) FIR reconstruction from dual optimal variables: By strong duality, complementary slackness applies, and we can therefore relate the dual and primal optimal variables in the

following manner

$$\begin{bmatrix} X_i X_i^* & X_i \\ X_i^* & I \end{bmatrix} \begin{bmatrix} I & G_i^* \\ G_i & W_i \end{bmatrix} = 0 \implies G_i = -X_i^*$$

This relation can be used to solve for the optimal V_i from the dual optimal variables X_i in a recursive manner. For the case $i = 1$, we have

$$G_1 = H_0 V_1 J_0 = -X_1^* \implies V_1 = -H_0^{-1}X_1^* J_0^{-1}.$$

Similarly, for $i = 2$,

$$G_2 = H_0 V_2 J_0 + \sum_{\substack{j,l\geq 0,k=1 \\ j+1+l=i}} H_j V_1 J_l = -X_2^*$$
$$\implies V_2 = -H_0^{-1}(X_2^* + \sum_{\substack{j,l\geq 0,k=1 \\ j+1+l=i}} H_j V_1 J_l)J_0^{-1}.$$

Continuing in this manner, we can solve for any V_i, $i \in 1, ..., N$ as

$$V_i = -H_0^{-1}(X_i^* + \sum_{\substack{j,l\geq 0,1\leq k\leq i-1 \\ j+k+l=i}} H_j V_k J_l)J_0^{-1}$$

This form for the components of the FIR component offers additional insight into the structure of the controller, making explicit the interdependence between the $\{V_i\}$.

IV. EXAMPLE – 5 SYSTEM RING COMMUNICATION GRAPH AUGMENTATION

In this section, we consider augmenting the communication graph of the 5 scalar plant undirected ring system shown in Figure 2 so as to improve the closed loop norm as much as possible. This system is described by

$$\tilde{A} = \begin{bmatrix} 10 & 1 & 0 & 0 & 10 \\ 1 & 10 & 1 & 0 & 0 \\ 0 & 1 & 10 & 1 & 0 \\ 0 & 0 & 1 & 10 & 1 \\ 1 & 0 & 0 & 10 & 10 \end{bmatrix}, \quad A = \frac{.99\tilde{A}}{|\lambda_{\max}(\tilde{A})|}$$
$$B_1 = \begin{bmatrix} I_5 + 99e_4 e_4^* & 0_{5\times 5} \end{bmatrix}$$
$$B_2 = C_2 = I_5$$
$$C_1 = \begin{bmatrix} 2I_5 \\ 0_{5\times 5} \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0_{5\times 5} \\ I_5 \end{bmatrix}$$
$$D_{21} = \begin{bmatrix} 0_{5\times 5} & I_5 \end{bmatrix}, \quad D_{11} = 0_{10\times 10}, \quad D_{22} = 0_{5\times 5}$$

We assume that the communication structure mimics the dynamic structure of the plant, such that the controller constraint set is given by

$$\mathcal{S} = \frac{1}{z}\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \oplus \frac{1}{z^2}\begin{bmatrix} * & * & 0 & 0 & * \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ * & 0 & 0 & * & * \end{bmatrix} \oplus \frac{1}{z^3}\mathcal{R}_p$$

The structure of the system is such that it takes one time step for a node to access its own local measurements (due to the assumption of strict properness), two time steps to access an immediate neighbor's measurements, and three time steps to access any other node's measurements. Suppose now that there is a central router that can provide a directed link between any two nodes X and Y, such that node X has access to node Y's measurements after two time steps. In
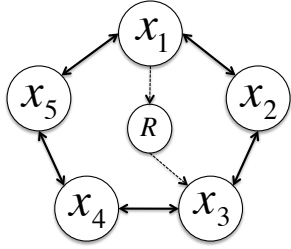
Fig. 2: The graph depicts the the communication structure of the five-player ring problem considered in Section IV. The routing/control problem consists of determining which link the router should establish, and to compute the optimal $\mathcal{H}_2$ decentralized controller. This problem is solved using the primal/dual methods described in Section III.

this case, it is obvious that the only links to consider are between nodes that are not immediate neighbors. Using the previously described shadow price interpretation, we identify the most active equality constraint on V_2 as $(2,1,4)$ (i.e. $e_1^* V_2 e_4 = 0$): $|\nu_{14}^2| \gg |\nu_{jk}^2|$ for all other $(j,k) \in \mathcal{I}_2$. We therefore eliminate this constraint to yield the new controller constraint set

$$\mathcal{S}_{14}^2 = \frac{1}{z}\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \oplus \frac{1}{z^2}\begin{bmatrix} * & * & 0 & * & * \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ * & 0 & 0 & * & * \end{bmatrix} \oplus \frac{\mathcal{R}_p}{z^3}.$$

This is intuitively a reasonable choice, as a larger disturbance enters at node 4 (due to the form of B_1) and is barely attenuated as it propagates around the ring towards node 1. The optimal $\mathcal{H}_2$ closed loop norm with respect to the original constraint set is 401.7, but drops significantly to 362 if the aforementioned constraint is removed. For comparison, if we remove the equality constraint $e_2 V_2 e_4^* = 0$ (another reasonable guess based on the structure of the plant), the optimal cost barely changes, decreasing to 401.6.

V. CONCLUSION

In [18], the decentralized $\mathcal{H}_2$ model matching problem subject to QI delays is solved by decomposing the controller into a centralized, but delayed, component and a decentralized FIR component, with the latter being solved for via a linearly constrained quadratic program. We showed how this problem can be converted to a semi-definite program (SDP) for which strong duality holds, and derived the dual problem. We then found primal and dual feasible points so as to identify *a priori* upper and lower bounds on the optimal $\mathcal{H}_2$ cost. Exploiting complementary slackness, we are also able to obtain further insight into the structure of the optimal FIR component. Finally, we showed how the optimal dual variables can be used to inform communication graph augmentation, and illustrated this idea with a routing problem.

There are many interesting avenues for future work, such as attempting to identify further refined upper and lower bounds, as well as delving more into the structure of the FIR component of the controller. Furthermore, although the optimal dual variables only provide information about incremental changes to the communication structure (i.e.

guidance as to which *single* constraint/delay is most detrimental to closed loop performance), it will be interesting to see whether they can be used to formulate an effective heuristic for communication graph design.

REFERENCES

[1] J. Sandell, N. and M. Athans, "Solution of some nonclassical lqg stochastic decision problems," *Automatic Control, IEEE Transactions on*, vol. 19, no. 2, pp. 108 – 116, apr 1974.

[2] B.-Z. Kurtaran and R. Sivan, "Linear-quadratic-gaussian control with one-step-delay sharing pattern," *Automatic Control, IEEE Transactions on*, vol. 19, no. 5, pp. 571 – 574, oct 1974.

[3] T. Yoshikawa, "Dynamic programming approach to decentralized stochastic control problems," *Automatic Control, IEEE Transactions on*, vol. 20, no. 6, pp. 796 – 797, dec 1975.

[4] P. Varaiya and J. Walrand, "On delayed sharing patterns," *Automatic Control, IEEE Transactions on*, vol. 23, no. 3, pp. 443 – 445, jun 1978.

[5] T. Yoshikawa and H. Kobayashi, "Separation of estimation and control for decentralized stochastic control systems," *Automatica*, vol. 14, no. 6, pp. 623 – 628, 1978. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0005109878900523

[6] B. Kurtaran, "Corrections and extensions to "decentralized stochastic control with delayed sharing information pattern"," *Automatic Control, IEEE Transactions on*, vol. 24, no. 4, pp. 656 – 657, aug 1979.

[7] A. Lamperski and J. C. Doyle, "On the structure of state-feedback lqg controllers for distributed systems with communication delays," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, dec. 2011, pp. 6901 –6906.

[8] ——, "Dynamic programming solutions for decentralized state-feedback lqg problems with comminication delays," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, jun. 2012.

[9] A. Nayyar, A. Mahajan, and D. Teneketzis, "Optimal control strategies in delayed sharing information structures," *Automatic Control, IEEE Transactions on*, vol. 56, no. 7, pp. 1606 –1620, july 2011.

[10] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *Automatic Control, IEEE Transactions on*, vol. 50, no. 12, pp. 1984 –1996, dec. 2005.

[11] A. Rantzer, "A separation principle for distributed control," in *Decision and Control, 2006 45th IEEE Conference on*, dec. 2006, pp. 3609 – 3613.

[12] A. Gattami, "Generalized linear quadratic control theory," in *Decision and Control, 2006 45th IEEE Conference on*, dec. 2006, pp. 1510 –1514.

[13] A. Lamperski and J. C. Doyle, "Output feedback $\mathcal{H}_2$ model matching for decentralized systems with delays," in *2013 IEEE American Control Conference, To appear.*, sept. 2012.

[14] J. Swigart and S. Lall, "An explicit state-space solution for a decentralized two-player optimal linear-quadratic regulator," in *American Control Conference (ACC), 2010*. IEEE, 2010, pp. 6385–6390.

[15] L. Lessard and S. Lall, "A state-space solution to the two-player decentralized optimal control problem," in *49th Annual Allerton Conference on Communication, Control, and Computing*. IEEE, 2011, pp. 1559–1564.

[16] B. Bamieh and P. G. Voulgaris, "A convex characterization of distributed control problems in spatially invariant systems with communication constraints," *Systems & Control Letters*, vol. 54, no. 6, pp. 575–583, 2005.

[17] P. G. Voulgaris, G. Bianchini, and B. Bamieh, "Optimal h2 controllers for spatially invariant systems with delayed communication requirements," *Systems & control letters*, vol. 50, no. 5, pp. 347–361, 2003.

[18] M. Fardad and M. R. Jovanović, "Design of optimal controllers for spatially invariant systems with finite communication speed," *Automatica*, vol. 47, no. 5, pp. 880–889, 2011.

[19] V. Balakrishnan and L. Vandenberghe, "Semidefinite programming duality and linear time-invariant systems," *Automatic Control, IEEE Transactions on*, vol. 48, no. 1, pp. 30 – 41, jan 2003.

[20] S. Sabau and N. C. Martins, "Stabilizability and norm-optimal control design subject to sparsity constraints," *arXiv preprint arXiv:1209.1123*, 2012.