A Projection Framework for Near-Potential Polynomial Games

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***Abstract*— It has been shown that in the case of finite games, games with a small Maximum Pairwise Difference (MPD) to a potential game share some of their favorable static and dynamic characteristics. In this paper, we extend these results to games in which strategy sets can be either finite, or closed intervals of the real line; and utility functions are polynomials in the players' actions. We define a notion of distance in the space of polynomial games in terms of the Maximum Differential Difference (MDD) between two games, and relate this concept to their MPD. We also show that a nearby polynomial potential game can be obtained from the solution of a semidefinite program.**

We then use polynomial potential games to study the static and dynamic properties of nearby polynomial games. In particular, we relate the approximate equilibria and approximate better response dynamics of a polynomial game to those of a nearby polynomial potential game in terms of their MDD.

I. Introduction and Motivation

Potential games are a well studied and useful class of games, as they have very appealing static and dynamic properties. Specifically, such games always possess pure-strategy Nash equilibria, and many simple user dynamics (e.g. best response) converge to a Nash equilibrium [2], [10]. As further motivation, potential games (and relaxations thereof) have been used successfully in modeling systems under cooperative control, allowing for the application of well established game theoretic methods to difficult distributed control problems [3], [4], [5].

Presumably, games that are "close" to potential games should share some of these appealing static and dynamic characteristics. Previous work [1], [6], [7] has formalized this idea, and shown that indeed in the case of games with finite strategy and player sets, a near-potential game's properties can be inferred from a nearby potential game approximation.

Although the finite player condition is not a restrictive one in a control setting, that of finite action sets can be – most systems of interest are governed by continuous (analog) dynamics, and therefore have continuous action sets. By restricting continuous systems to finite action sets, there is a tradeoff between the level of discretization, system performance, and problem complexity resulting from the potentially large cardinality of the discretized strategy set.

Although infinite games avoid these issues, they are inherently less tractable to study, as their analysis often requires solving infinite dimensional optimization problems – for example, only recently have useful results on the value and optimal strategies for certain classes of infinite games [8], [9] been developed.

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In this work, our goal is to extend the ideas introduced in [1], [6], [7] to a particular class of infinite games: namely strategic (noncooperative) games with (i) both finite actions sets, and continuous action sets that are closed intervals of the real line; and (ii) polynomial cost functions. We begin by studying continuous games, that is those in which all actions sets are closed intervals of the real line. We combine results from game theory, algebraic geometry, and convex optimization, to develop a computationally tractable projection framework for finding a nearby polynomial potential game approximation of an arbitrary polynomial game. Specifically, we introduce the *Maximum Differential Difference* (MDD) as a useful measure of "distance" between two continuous games, and show that the problem of projecting onto the space of continuous polynomial potential games with respect to the MDD can be cast as an optimization problem that is amenable to a sum of squares (SOS) relaxation.

We then extend this framework to games in which some or all of the action sets are finite by defining the continuous relaxation of such a game. We show how to use the MDD between a nearby polynomial potential game and the continuous relaxation of a game to bound the Maximum Pairwise Distance (MPD) (an analogous concept to the MDD, introduced in [6] for studying finite games) between the original game and the nearby polynomial potential game.

Next, we relate the static and dynamic properties of our initial game to the nearby polynomial game found by our projection framework. We show that the approximate equilibria, as well as the approximate better response dynamics in arbitrary polynomial strategic-form games can be analyzed using a nearby polynomial potential game. Not surprisingly, the closer a game is to being a continuous potential game, the tighter our characterization becomes.

This paper is organized as follows: in Section II we present the necessary game-theoretic and algebraic geometry background. Section III formally defines the MDD, MPD and continuous relaxation of a game, and introduces the projection framework for finding nearby polynomial potential games. In Section IV, we establish connections between an arbitrary polynomial game's static and dynamic properties and those of a nearby polynomial potential game. Section V presents an example demonstrating our results. Finally, Section VI provides conclusions and directions for future work.

II. Preliminaries

A. Game Theory and Potential Games

A (noncooperative) polynomial game in strategic form consists of:

- A finite set of players, denoted $\mathcal{N} = \{1, ..., N\}$
- Strategy spaces: each player $n \in \mathcal{N}$ has strategy set $E^n \subset \mathbb{R}$. We denote the space of strategy profiles as $E = \prod_{n \in \mathcal{N}} E^n$. Let $x^n \in E^n$ denote the strategy of player n.
- Utility functions: $u^n : E \to \mathbb{R},\ u^n \in \mathbb{R}[x]$, where $\mathbb{R}[x]$ denotes polynomials in $x^1, ..., x^N$ with real coefficients.

A polynomial game is said to be

i) *continuous* if for each $n \in \mathcal{N}$, E^n is a closed interval of the real line. In this case we say that the strategy set E^n is continuous.
ii) *finite* if for each $n \in \mathcal{N}$, $|E^n| < \infty$. In this case we say that the strategy set E^n is finite.
iii) *mixed* if there exist $c,\ f \in \mathcal{N}$, such that E^c is continuous and E^f is finite.

Throughout this paper, we assume without loss of generality that continuous action sets are given by $[-1, 1]$, and that finite actions sets lie within $[-1, 1]$.

Following the notation in [1], a game instance can then be defined by the tuple $(\mathcal{N}, \{u^n\}_{n\in\mathcal{N}}, E)$. For a strategy profile $x \in E$, we will often write $x = (x^n, x^{-n})$, where x^{-n} is a vector containing the strategies of all players except for the n^{th} one.

We now introduce a standard extension to the basic notion of a Nash Equlibrium (NE) – that of an (additive) ϵ-equilibrium.

Definition 1: A strategy profile $x \in E$ is an ϵ-equilibrium if $\forall n \in \mathcal{N}$

$$u^n(x^n, x^{-n}) \geq u^n(y^n, x^{-n}) - \epsilon \quad \forall y^n \in E^n \tag{1}$$

Note that this definition corresponds to a pure NE for $\epsilon = 0$.

We next define polynomial potential games (definitions modified from [2])

Definition 2: Consider a polynomial game $\mathcal{G} = (\mathcal{N}, \{u^n\}_{n\in\mathcal{N}}, E)$. If there exists a *potential function* $P : E \to \mathbb{R}$, $P \in \mathbb{R}[x]$ such that for every $n \in \mathcal{N}$, $x \in E$, $y^n \in E^n$

$$P(x^n, x^{-n}) - P(y^n, x^{-n}) = u^n(x^n, x^{-n}) - u^n(y^n, x^{-n}) \tag{2}$$

then $\mathcal{G}$ is a(n) (exact) *polynomial potential game*.

A useful characterization of continuous polynomial potential games in terms of algebraic constraints is given by the following (modified from [2])

Theorem 1: Let $\mathcal{G}$ be a continuous polynomial game as defined above. Then $\mathcal{G}$ is a potential game with potential function P if and only if P is continuously differentiable, and

$$\frac{\partial u^n}{\partial x^n} = \frac{\partial P}{\partial x^n} \quad \forall n \in \mathcal{N} \tag{3}$$

Clearly if we restrict P to be in $\mathbb{R}[x]$ the above smoothness conditions are satisfied, and Theorem 1 provides a full characterization of continuous polynomial potential games.

We conclude with a well known convergence result which will be fundamental in our analysis of dynamic properties of polynomial games [10]:

Theorem 2: (*Zangwill's Convergence Theorem*) Let $f : X \rightrightarrows X$ define an algorithm that, given $x_0 \in X$, generates a sequence $\{x_k\}$ via $x_{k+1} \in f(x_k)$. Let a compact solution set $Y^* \in X$ be given. Let

1) $\{x_k\}$ be a compact subset of X
2) $\exists$ a continuous function $\alpha : X \to \mathbb{R}$ such that
   a) if $x \notin Y^*$ then $\alpha(x') > \alpha(x)$ for all $x' \in f(x)$
   b) if $x \in Y^*$ then $\alpha(x') \geq \alpha(x)$ for all $x' \in f(x)$
3) f is closed at x if $x' \notin Y^*$ for all $x' \in f(x)$

Then either $\{x_k\}$ arrives at a solution in Y^* or every limit point of $\{x_k\}$ is in Y^*.

This theorem can be used to show, for example, convergence of potential games under best response dynamics to NE, by setting α to the potential function of the game, f to the best response dynamics and Y^* to the set of NE [10].

B. Sum of Squares and Non-Negativity of Polynomials

This subsection, adapted from [11], provides a brief summary of two key results from algebraic geometry that are necessary for our development.

Definition 3: A polynomial $p \in \mathbb{R}[x]$ admits a *sum of squares* (SOS) decomposition if there exists a set of polynomials f_k, $k = 1, ..., K$ such that

$$p(x) = \sum_{k=1}^{K} f_k^2(x) \tag{4}$$

Let $\mathcal{P}_{N,m}$ denote the set of nonzero forms (i.e. polynomials of homogeneous degree) in N variables of degree m, with coefficients in $\mathbb{R}$ that are non-negative on $\mathbb{R}^N$, and let $\Sigma_{N,m}$ denote the set of SOS polynomials in N variables of degree m (note that m is necessarily even in both cases). Clearly, $\Sigma_{N,m} \subset \mathcal{P}_{N,m}$, with the containment being strict in general, although exceptions do exists (c.f. Chapter 4.2, [11]). However, in contrast with the $\mathcal{NP}$-hardness of determining the non-negativity of a general polynomial, SOS polynomials can be characterized through a semidefinite program (SDP).

Finally, the following lemma provides algebraic constraints that are sufficient (and in certain cases, necessary) conditions for a polynomial to be non-negative over the hybercube $[-1, 1]^N$.

Lemma 1: Let $p \in \mathbb{R}[x]$. If there exist $\{\lambda_0, \lambda_1, ..., \lambda_N\} \in \bigcup_{m\geq 0} \Sigma_{N,m}$ such that

$$p(x) = \lambda_0 + \sum_{i=1}^{N} \lambda_i (1 - x_i^2) \tag{5}$$

then $p(x) \geq 0$ on $[-1, 1]^N$.

Proof: Invoke the Positivstellensatz as in [11], [12] ■

III. Projection Framework

A. The Maximum Differential Difference

We begin by defining the *Maximum Differential Difference* between two continuous games:

Definition 4: Let $\mathcal{G}$ and $\hat{\mathcal{G}}$ be two continuous polynomial games with set of players $\mathcal{N}$ and collections of utility functions $\{u^n\}_{n\in\mathcal{N}}$ and $\{\hat{u}^n\}_{n\in\mathcal{N}}$, respectively. The *Maximum Differential Difference* (MDD) between these games is defined as

$$\mathrm{MDD}(\mathcal{G},\hat{\mathcal{G}}) := \sup_{n\in\mathcal{N},x\in E}\left|\frac{\partial u^n}{\partial x^n}(x) - \frac{\partial \hat{u}^n}{\partial x^n}(x)\right| \tag{6}$$

The MDD captures how different two games are in terms of the utility improvements due to infinitesimal unilateral deviations.

The MDD can be seen as an extension to continuous games of the MPD [6] between two games, defined as

$$\begin{aligned}&\mathrm{MPD}(\mathcal{G},\hat{\mathcal{G}}) := \\ &\sup_{n\in\mathcal{N},y^n\in E^n,x\in E} |(u^n(y^n,x^{-n}) - u^n(x^n,x^{-n}))- \\ &\quad(\hat{u}^n(y^n,x^{-n}) - \hat{u}^n(x^n,x^{-n}))|\end{aligned} \tag{7}$$

Although originally used to study the properties of finite games, the MPD is equally applicable to continuous games, although in general, computing the MPD of two continuous games is intractable as it results in an infinite dimensional optimization problem. We can however bound the MPD of two continuous games in terms of their MDD.

Lemma 2: Let $\mathcal{G}$ and $\hat{\mathcal{G}}$ be two continuous polynomial games satisfying $\mathrm{MDD}(\mathcal{G},\hat{\mathcal{G}}) \leq \gamma$. Then $\mathrm{MPD}(\mathcal{G},\hat{\mathcal{G}}) \leq 2\gamma$.

Proof: Let $n\in\mathcal{N}, y^n\in E^n, x\in E$ be arbitrary. Then

$$\begin{aligned}&|(u^n(y^n,x^{-n}) - u^n(x^n,x^{-n}))- \\ &\quad(\hat{u}^n(y^n,x^{-n}) - \hat{u}^n(x^n,x^{-n}))| \\ &= \left|\int_{y^n}^{x^n} \frac{\partial u^n}{\partial \tau^n}(\tau^n,x^{-n}) - \frac{\partial \hat{u}^n}{\partial \tau^n}(\tau^n,x^{-n})\mathrm{d}\tau^n\right| \\ &\leq \int_{-1}^{1}\left|\frac{\partial u^n}{\partial \tau^n}(\tau^n,x^{-n}) - \frac{\partial \hat{u}^n}{\partial \tau^n}(\tau^n,x^{-n})\right|\mathrm{d}\tau^n \\ &\leq \gamma\int_{-1}^{1}\mathrm{d}\tau^n = 2\gamma\end{aligned} \tag{8}$$

As $n\in\mathcal{N}, y^n\in E^n, x\in E$ were arbitrary, the claim follows. ∎

B. Projection Framework for Continuous Polynomial Games

The problem to be addressed in this section can be formally defined as

Problem 1: Given a continuous polynomial game $\mathcal{G}$, find the continuous polynomial potential game $\hat{\mathcal{G}}$ which minimizes $\mathrm{MDD}(\mathcal{G},\hat{\mathcal{G}})$. This can be formulated as the following optimization problem:

$$\begin{aligned}&\inf_{P,\{\hat{u}^n\}_{n\in\mathcal{N}}\in\mathbb{R}[x]}\ \sup_{n\in\mathcal{N},x\in E}\left|\frac{\partial u^n}{\partial x^n}(x) - \frac{\partial \hat{u}^n}{\partial x^n}(x)\right| \quad s.t. \\ &P(x^n,x^{-n}) - P(y^n,x^{-n}) = \hat{u}^n(x^n,x^{-n}) - \hat{u}^n(y^n,x^{-n}) \\ &\qquad\qquad\forall n\in\mathcal{N},\ x\in E,\ y^n\in E^n\end{aligned} \tag{9}$$

We now present the main result of this section, which provides a tractable semidefinite relaxation of (9).

Theorem 3: Let $\mathcal{G} = (\mathcal{N},\{u^n\}_{n\in\mathcal{N}}, E=[-1,1]^N)$ be a continuous polynomial game, and let $(\gamma,\ P,\ \{\hat{u}^n\}_{n\in\mathcal{N}})$ be solutions to the following SOS program

$$\begin{aligned}&\inf_{P,\{\hat{u}^n\}_{n\in\mathcal{N}}\in\mathbb{R}[x],\{\lambda^n_{\pm i}\}^{i=0,\dots,N}_{n\in\mathcal{N}}\in\Sigma_N,\gamma\in\mathbb{R}} \gamma \quad \text{s.t.} \\ &\tfrac{\partial \hat{u}^n}{\partial x^n}(x) = \tfrac{\partial P}{\partial x^n}(x)\ \forall n\in\mathcal{N},\ x\in E \\ &\lambda^n_{\pm 0} + \textstyle\sum_{i=1}^N \lambda^n_{\pm i}(1-x_i^2) = \gamma \pm (\tfrac{\partial u^n}{\partial x^n}(x) - \tfrac{\partial \hat{u}^n}{\partial x^n}(x)) \\ &\qquad\qquad\forall n\in\mathcal{N},\ x\in E\end{aligned} \tag{10}$$

Then $\mathrm{MDD}(\mathcal{G},\hat{\mathcal{G}}) \leq \gamma$, where $\hat{\mathcal{G}} = (\mathcal{N},\{\hat{u}^n\}_{n\in\mathcal{N}},E)$ is a polynomial potential game with potential function P.

Proof: We begin with (9) and apply Theorem 1 to rewrite it as

$$\begin{aligned}&\inf_{P,\{\hat{u}^n\}_{n\in\mathcal{N}}\in\mathbb{R}[x]}\ \sup_{n\in\mathcal{N},x\in E}\left|\frac{\partial u^n}{\partial x^n}(x) - \frac{\partial \hat{u}^n}{\partial x^n}(x)\right| \quad s.t. \\ &\tfrac{\partial \hat{u}^n}{\partial x^n}(x) = \tfrac{\partial P}{\partial x^n}(x)\ \forall n\in\mathcal{N},\ x\in E\end{aligned} \tag{11}$$

The two problems are equivalent, in that both constraints guarantee that P is a potential function for the game generated by $\{\hat{u}^n\}$. We now introduce a slack variable γ to rewrite (11) as

$$\begin{aligned}&\inf_{P,\{\hat{u}^n\}_{n\in\mathcal{N}}\in\mathbb{R}[x],\gamma\in\mathbb{R}} \gamma \quad s.t. \\ &\tfrac{\partial \hat{u}^n}{\partial x^n}(x) = \tfrac{\partial P}{\partial x^n}(x)\ \forall n\in\mathcal{N},\ x\in E \\ &\gamma \pm (\tfrac{\partial u^n}{\partial x^n}(x) - \tfrac{\partial \hat{u}^n}{\partial x^n}(x)) \geq 0\ \forall n\in\mathcal{N},\ x\in E\end{aligned} \tag{12}$$

Applying Lemma 1 to the last two constraints of (12), with $p_\pm(x) = \gamma \pm (\frac{\partial u^n}{\partial x^n}(x) - \frac{\partial \hat{u}^n}{\partial x^n}(x))$ yields the desired result. ∎

This SOS program can be solved efficiently using SOSTools [13].

We note that (10) has $3N$ constraints, and thus this aspect of the problem scales well with N. The issue arises in the number of optimization variables required to represent the various polynomials: a form of degree m in N variables is expressed as a sum of $\binom{N+m-1}{m}$ monomials. The number of optimization variables required grows super-exponentially in N, limiting the size of player sets that this procedure can handle in general games – however often times there will be additional structure that can be exploited (such as symmetry in the utility functions, see Section V) to alleviate this problem.

We conclude this section with an illustrative example

Example 1: We consider the two player polynomial game $\mathcal{G} = (\{1,2\},\{u^1(x^1,x^2),u^2(x^1,x^2)\},[-1,1]^2)$ with

$$\begin{aligned}u^1(x^1,x^2) &= .1x^1(x^2+\tfrac{(x^2)^2}{2}) + (x^1-.5)^2 \\ u^2(x^1,x^2) &= .1x^2(x^1+\tfrac{(x^1)^2}{2})\end{aligned} \tag{13}$$

We solved the SOS program (10) to obtain a polynomial potential game $\hat{\mathcal{G}} = (\{1,2\},\{\hat{u}^1(x^1,x^2),\hat{u}^2(x^1,x^2)\},[-1,1]^2)$ satisfying $\mathrm{MDD}(\mathcal{G},\hat{\mathcal{G}}) \leq .025$, with

$$\begin{aligned}P(x^1,x^2) &= (x^1)^2 + .1x^1x^2 - .975x^1 + .025x^2 \\ \hat{u}^1(x^1,x^2) &= (x^1)^2 + .1x^1x^2 - .975x^1 \\ \hat{u}^2(x^1,x^2) &= .1x^1x^2 + .025x^2\end{aligned} \tag{14}$$

It is easily verified that P, $\hat{u}^1$ and $\hat{u}^2$ satisfy Theorem 1. Furthermore, just to make explicit that in this case, indeed

$\mathrm{MDD}(\mathcal{G},\hat{\mathcal{G}}) = .025$, observe that

$$\frac{\partial(u^2-\hat{u}^2)}{\partial x^2}(x^1,x^2) = .1\frac{(x^1)^2}{2} - .025 \tag{15}$$

which attains its maximum value of .025 at $x^1 = \pm 1$.

C. Extensions to Finite and Mixed Polynomial Games

We begin by arguing that naive extensions of the previous framework to mixed polynomial games are intractable. In particular, since the strategy sets are not all closed intervals of the real line, Theorem 1 is no longer applicable; furthermore, the constraints imposed on P and $\{\hat{u}^n\}$ by Definition 2 are intractable, as although they are linear, they are infinite dimensional.

Motivated by this issue, we introduce the notion of a continuous relaxation of a mixed or discrete game.

Definition 5: Let $\mathcal{G} = (\mathcal{N}, \{u^n\}_{n\in\mathcal{N}}, E)$ be a mixed or discrete game. We say that $\tilde{\mathcal{G}} = (\mathcal{N}, \{u^n\}_{n\in\mathcal{N}}, [-1,1]^N)$ is the *continuous relaxation* of $\mathcal{G}$.

We can now extend the previous framework to mixed (and discrete) games.

Corollary 1: Let $\mathcal{G} = (\mathcal{N}, \{u^n\}_{n\in\mathcal{N}}, E)$ be a mixed (or discrete) polynomial game, $\tilde{\mathcal{G}}$ its continuous relaxation, and $\hat{\mathcal{G}}$ the solution to (10) satisfying $\mathrm{MDD}(\tilde{\mathcal{G}},\hat{\mathcal{G}}) \leq \gamma$. Then $\mathrm{MPD}(\mathcal{G},\hat{\mathcal{G}}) \leq 2\gamma$.

Proof: As $E \subset [-1,1]^N$, $\mathrm{MPD}(\mathcal{G},\hat{\mathcal{G}}) \leq \mathrm{MPD}(\tilde{\mathcal{G}},\hat{\mathcal{G}})$. Given is $\mathrm{MDD}(\tilde{\mathcal{G}},\hat{\mathcal{G}}) \leq \gamma$, and so by Lemma 2, $\mathrm{MPD}(\tilde{\mathcal{G}},\hat{\mathcal{G}}) \leq 2\gamma$. ■

Remark 1: Corollary 1 is equally applicable to finite polynomial games, and is especially useful for those in which $|E^n| >> 1$. In particular, if we assume that for all n, $|E^n| = C >> 1$, then Definition 2 imposes $\mathcal{O}(C^N)$ equality constraints on any optimization problem aimed at finding a nearby potential game, a number that quickly becomes intractable for even small player sets. Corollary 1 provides an attractive alternative to this.

Example 2: We consider the same game $\mathcal{G}$ as in Example 1, except now restrict the action set of player x^2 to be $E^2 = \{-1, -.99, -.98, ..., .98, .99, 1\}$. Invoking Corollary 1, we know that the polynomial potential game $\hat{\mathcal{G}}$ obtained in Example 1 satisfies $\mathrm{MPD}(\mathcal{G},\hat{\mathcal{G}}) \leq .05$. Furthermore, we note that in this case $\mathrm{MPD}(\mathcal{G},\hat{\mathcal{G}}) = .05$, as

$$\begin{aligned}&(u^1(x^1,x^2) - u^1(z,x^2)) - (\hat{u}^1(x^1,x^2) - \hat{u}^1(z,x^2))\\ &\quad = \tfrac{(2(x^2)^2-1)(x^1-z)}{40}\end{aligned} \tag{16}$$

which achieves its maximum at the allowable strategy $(x^1,x^2,z) = (1,1,-1)$. This also demonstrates that the bound in Lemma 2 is indeed tight.

IV. Static and Dynamic Properties of Near-Potential Polynomial Games

A. Static Properties

We begin with a lemma relating the ϵ-equilibria of a game to those of its continuous relaxation.

Lemma 3: Let $\mathcal{G} = (\mathcal{N}, \{u^n\}_{n\in\mathcal{N}}, E)$ be a polynomial game, $\tilde{\mathcal{G}}$ its continuous relaxation and y be an ϵ_1-equilibrium of $\tilde{\mathcal{G}}$. Then $z(y)$ is an ϵ-equilibrium of $\mathcal{G}$ with $\epsilon \leq \epsilon_1 + 2D||z-y||_1$, where

$$\begin{aligned}D &:= \sup_{n,m\in\mathcal{N}, x\in[-1,1]^N} \frac{\partial u^n}{\partial x^m}(x)\\ z(y) &:= \arg\min_{z\in E} ||z-y||_1\end{aligned} \tag{17}$$

Proof: Let $s_1 : [0,1] \to E$ be defined by $s_1(t) = y + t(z-y)$, and similarly let $s_2(t) = \tilde{x} + t(x - \tilde{x})$. Notice then that for any $x = (x^n, z^{-n})$, $\tilde{x} = (x^n, y^{-n})$

$$\begin{aligned}&u^n(z) - u^n(x)\\ &\quad = u^n(y) - u^n(\tilde{x}) + \sum_{m\in\mathcal{N}} \int_0^1 \frac{\partial u^n}{\partial x^m}(s_1(t))(z^m - y^m)\mathrm{d}t\\ &\qquad - \sum_{m\in\mathcal{N}} \int_0^1 \frac{\partial u^n}{\partial x^m}(s_2(t))(\tilde{x}^m - x^m)\mathrm{d}t\\ &\quad \leq \epsilon_1 + 2D||z-y||_1\end{aligned} \tag{18}$$

where the last inequality follows from the definitions of D, z and $\tilde{x}$, and the fact that y is an ϵ_1-equilibrium of $\tilde{\mathcal{G}}$. As any increase due to a unilateral deviation from z is bounded above by $\epsilon_1 + 2D||z-y||_1$, the result is proved. ■

Remark 2: Although in general computing D is intractable, an upper bound can be obtained using Lemma 1 and SOS methods [12].

The $2D||z-y||_1$ term can thus be interpreted as the "quantization penalty" incurred by using the continuous relaxation of the original game in our analysis.

We now relate the ϵ-equilibria of two nearby games.

Proposition 1: Let $\mathcal{G} = (\mathcal{N}, \{u^n\}_{n\in\mathcal{N}}, E)$ be a polynomial game, $\tilde{\mathcal{G}}$ its continuous relaxation and $\hat{\mathcal{G}} = (\mathcal{N}, \{\hat{u}^n\}_{n\in\mathcal{N}}, [-1,1]^N)$ a continuous potential polynomial game such that $\mathrm{MDD}(\tilde{\mathcal{G}},\hat{\mathcal{G}}) \leq \gamma$. Then for every ϵ_1-equilibrium y of $\hat{\mathcal{G}}$, $z(y)$ as defined in (17) is an ϵ-equilibrium of $\mathcal{G}$ with $\epsilon \leq 2\gamma + \epsilon_1 + 2D||z-y||_1$.

Proof: Let y be an ϵ_1-equilibrium of $\hat{\mathcal{G}}$, and let $x = (x^n, z^{-n})$ for some player $n \in \mathcal{N}$ and some unilateral deviation $x^n \in E^n$. Then by Lemma 3

$$(\hat{u}^n(z) - \hat{u}^n(x)) \leq \epsilon_1 + 2D||z-y||_1 \tag{19}$$

We now write

$$\begin{aligned}u^n(z) - u^n(x) &= (u^n(z) - \hat{u}^n(z)) - (u^n(x) - \hat{u}^n(x))\\ &\quad + (\hat{u}^n(z) - \hat{u}^n(x))\\ &\leq \mathrm{MPD}(\mathcal{G},\hat{\mathcal{G}}) + \epsilon_1 + 2D||z-y||_1\\ &\leq 2\mathrm{MDD}(\tilde{\mathcal{G}},\hat{\mathcal{G}}) + \epsilon_1 + 2D||z-y||_1\\ &\leq 2\gamma + \epsilon_1 + 2D||z-y||_1\end{aligned} \tag{20}$$

As the improvement due to a unilateral deviation from z by an arbitrary player is bounded above by $2\gamma+\epsilon_1+2D||z-y||_1$, z is indeed an ϵ-equilibrium of $\mathcal{G}$ with $\epsilon \leq 2\gamma+\epsilon_1+2D||z-y||_1$. ■

In the case where the original game $\mathcal{G}$ is continuous, one can always pick $z = y$, and Proposition 1 reduces to

Corollary 2: Let $\mathcal{G}$ be a continuous game, $\hat{\mathcal{G}}$ be as in Proposition 1, and y be an ϵ_1-equilibrium of $\hat{\mathcal{G}}$. Then y is an ϵ-equilibrium of $\mathcal{G}$ with $\epsilon \leq 2\gamma + \epsilon_1$.

Using Theorem 3, given an arbitrary polynomial game $\mathcal{G}$, we can generate a nearby potential game $\hat{\mathcal{G}}$ satisfying $\mathrm{MDD}(\mathcal{G}, \hat{\mathcal{G}}) \leq \gamma$ for some finite $\gamma \geq 0$. For continuous games, the local maxima of P correspond to NE of $\hat{\mathcal{G}}$, which by Corollary 2 are then 2γ-equilibria of $\mathcal{G}$.

B. Dynamic Properties

We next show that dynamics in an arbitrary polynomial game can be characterized using a nearby polynomial potential game. We first define ϵ-better response dynamics [1]:

Definition 6: In *ϵ-better response dynamics*, updates take place in a round robin manner, and at any update only a single user can modify its strategy. If there exist strategies which improve the updating player's utility by at least $\epsilon > 0$, then said player updates to one such strategy – otherwise it does not modify its strategy.

In the following proposition (analogous to Prop. 7 in [1]), we show that for an arbitrary polynomial game, ϵ-better response dynamics converge to an ϵ-equilibrium, where ϵ is determined by the game's MDD to a polynomial potential game.

Proposition 2: Let $\mathcal{G}$ be an arbitrary polynomial game, $\tilde{\mathcal{G}}$ its continuous relaxation, and let $\hat{\mathcal{G}}$ be a nearby polynomial potential game satisfying $\mathrm{MDD}(\tilde{\mathcal{G}}, \hat{\mathcal{G}}) \leq \gamma$. Then after a finite number of round-robin iterations, the ϵ-better response dynamics will be confined to the ϵ-equilibrium set of $\mathcal{G}$, with $\epsilon = 2\gamma + \delta$, for arbitrary $\delta > 0$.

Proof: If the ϵ-equilibrium set is reached, by the definition of the dynamics, no player modifies its strategy – thus it is sufficient to show that the dynamics reach this set starting from an arbitrary strategy profile.

Let $\{u^n\}$ and $\{\hat{u}^n\}$ be the utility functions of $\mathcal{G}$ and $\hat{\mathcal{G}}$ respectively. Let player n be the updating player, which only modifies its strategy profile from x^n to y^n if

$$u^n(y^n, x^{-n}) - u^n(x^n, x^{-n}) \geq \epsilon > 2\gamma \tag{21}$$

Then using similar reasoning as in the proof of Proposition 1, it follows that

$$\begin{aligned}&\hat{u}^n(y^n, x^{-n}) - \hat{u}^n(x^n, x^{-n}) \\ &\quad \geq u^n(y^n, x^{-k}) - u^n(x^n, x^{-n}) - 2\mathrm{MDD}(\tilde{\mathcal{G}}, \hat{\mathcal{G}}) \\ &\quad \geq \epsilon - 2\gamma > 0\end{aligned} \tag{22}$$

Every time that an update occurs, the potential function P of $\hat{\mathcal{G}}$ increases – thus, no strategy profile can be visited twice by this update process. If the 2γ-equilibrium set has not been reached, then at each round robin a player that can improve its payoff is found. Applying Theorem 2 with f defined as the strategy updates of players under 2γ-better response dynamics, $X = E$, Y^* the 2γ-equilibrium set (which is closed and bounded and hence compact), and $\alpha = P$, we see that the dynamics will either converge to the 2γ-equilibrium set, or every limit point of these dynamics is in the 2γ-equilibrium set. In the latter case, due to the smoothness of polynomials, after finitely many iterations, the dynamics will be confined to the $(2\gamma+\delta)$-equilibrium set, for arbitrary $\delta > 0$, proving the result. ∎

V. Example: Distributed Power Minimization

Consider the N player mixed polynomial game $\mathcal{G}_N$ defined by

- $\mathcal{N} = \{1, 2, ..., N\}$
- $E^n \subset [-1, 1]$
- $u^n(x) = -\frac{1}{10}(x^n - x_0^n)^2 - \frac{1}{20}(x^n - x^{n+1})^2$

where the index set is understood to be cyclical (i.e. $x^{N+1} = x^1$; $x^{-1} = x^N$), and $x_0^n \in E^n$ are randomly assigned nominal positions. This game can be interpreted as agents or sensors attempting to minimize the power needed to communicate (via a weighted directed cyclical graph, see Figure 1) with (i) each other, and (ii) with their respective "base stations" located at x_0^n, by optimizing their positions within $E^n \subset [-1, 1]$.

Noting that there is a high degree of symmetry in the utility functions of $\mathcal{G}_N$, we solve (10) with $\tilde{\mathcal{G}}_N$ for $N = 3$, and once again exploit the symmetry present in the approximating potential game $\hat{G}_3$, to obtain a polynomial game approximation $\hat{\mathcal{G}}_N$, satisfying $\mathrm{MDD}(\tilde{\mathcal{G}}_N, \hat{\mathcal{G}}_N) \leq .1$, and by Lemma 2, $\mathrm{MPD}(\mathcal{G}_N, \hat{\mathcal{G}}_N) \leq .2$, with potential function P_N and utility functions $\hat{u}^n$ given by

- $P_N(x) = \sum_{n=1}^{N} -\frac{1}{20}\left(3(x^n)^2 - x^n x^{n+1} - 4x^n x_0^n\right)$
- $\hat{u}^n(x) = -\frac{1}{20}\left(3(x^n)^2 - x^n x^{n+1} - x^n x^{n-1} - 4x^n x_0^n\right)$

Given this polynomial potential game, we can solve for the unique maximizer x_* of $P_N(x)$ ($\nabla^2 P_N < 0$) to identify a .2-equilibria of $\mathcal{G}_N$. Alternatively, we can invoke Proposition 2, and run .2-better response dynamics – shown in Figure 1 are the final positions x_{br} computed by these dynamics for N=100.

We can also compare the performance of (i) .2-better response dynamics, a completely decentralized update rule requiring no a priori knowledge of x_0, with (ii) a centralized optimization with respect to a nominal vector x_0. Specifically, we consider the cost function

$$J(x) = -\sum_{n \in \mathcal{N}} u^n(x) \tag{23}$$

and compute x_* as the solution to the convex program

$$\mathrm{minimize}_{x \in [-1,1]^N} J(x) \tag{24}$$

For this particular choice of x_0, we have $J(x_{br}) = 5.8884$ and $J(x_*) = 4.8701$ – although the centralized optimization performs approximately 20% better than the better response dynamics, we note that this method requires exact knowledge of x_0 a priori, and is limited in the number of agents it can handle. The better response dynamics, on the other hand, require no a priori or centralized knowledge of x_0, and are completely distributed, making them scalable to arbitrarily large player sets.

VI. Conclusions and Future Work

We introduced a framework for the analysis of arbitrary polynomial games in terms of "nearby" continuous potential polynomial games. We defined the Maximum Differential Difference of two continuous game and showed that the

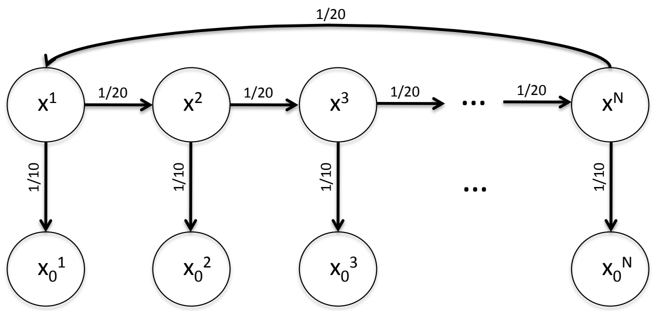

Fig. 1. Communication graph for distributed power minimization example

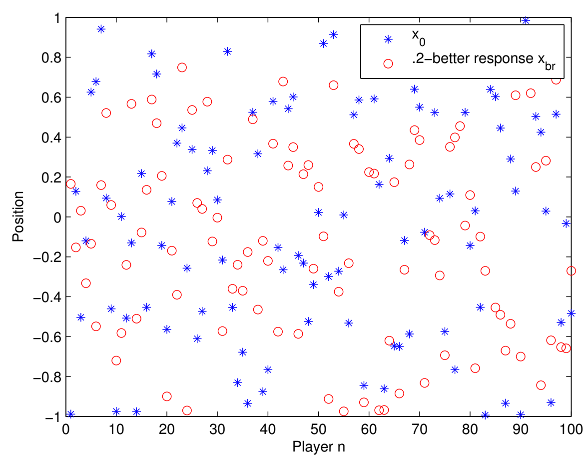

Fig. 2. Plot of base station positions x_0, and positions x_{br} computed using .2-better response dynamics.

problem of finding a nearby continuous potential game with respect to this pseudo-metric can be formulated as an SOS program, which can be solved in polynomial time. We then introduced the concept of a continuous relaxation of a mixed (or discrete) game, allowing us to extend this projection framework to mixed (and discrete) games, and used the MDD to bound the Maximum Pairwise Difference of two arbitrary games.

We then showed that the static and dynamic properties of an *arbitrary polynomial game* can be analyzed in terms of a nearby continuous polynomial potential game – not surprisingly, the closer the original game is to its continuous potential approximation, the tighter the characterization.

There are many interesting avenues for future work. It was shown in [1] that projecting onto the set of weighted potential games results in a tighter characterization of the original game's static and dynamic properties – extending our projection framework to weighted potential polynomial games would be a logical next step. Since the space of weighted potential games is inherently non-convex (product terms between the weights and the partial derivatives of the potential function appear in the constraints), the challenge will be in finding a good convex relaxation. There are also additional static properties, such as mixed-equilibria, efficiency notions (price of anarchy, price of stability, etc.), and other update rules, that we can attempt to analyze under this framework.

Finally, we would like to point out that although we have restricted our attention to games where continuous strategy sets are closed intervals, and cost/potential functions are polynomials, several fairly straightforward extensions based on standard SOS/algebraic results are possible. For example, we can directly extend the results presented to continuous strategy sets that are finite unions of intervals.

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