Reading: Abraham, Marsden, and Ratiu (MTA), sections 3.3, 4.1, 4.2

Problems:

1. MTA 3.3-1: tangent spaces/maps for graphs

2. If $F(x_1, x_2, x_3) = 0$ is a submersion defining a 2-dimensional manifold in $\mathbb{R}^3$, under what conditions is $X = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3}$ (evaluated at a point where $F(x_1, x_2, x_3) = 0$) a tangent vector to $M$?

3. [Boothby, page 119, #12]
   Show that any smooth vector field $Y$ on $S^{n-1} \subset \mathbb{R}^n$ is the restriction of a smooth vector field $X$ on $\mathbb{R}^n$.

4. MTA 4.1-5: convolution equation

5. [Boothby, page 126, #6]
   Show that $\phi_t(x, y)$ defined by
   $$\phi_t(x, y) = (xe^{2t}, ye^{-3t})$$
   defines a $C^\infty$ flow on $M = \mathbb{R}^2$. Determine the vector field that generates this flow (called the infinitesimal generator of the flow) and show that it is $\phi$ invariant.

6. Let $SO(3)$ be the set of $3 \times 3$ orthogonal matrices with determinant $+1$. The tangent space of $SO(3)$ at the identity is given by the set of skew-symmetric matrices of the form
   $$\hat{\omega} = \omega^\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$
   (we’ll show this later in the course).
   
   (a) Show that if $v \in \mathbb{R}^3$, $\hat{\omega}v = \omega \times v$, where $\times$ is the cross product in $\mathbb{R}^3$.
   
   (b) Show that the tangent space $T_RSO(3)$ consists of matrices of the form $\hat{\omega}R$ where $\hat{\omega}$ is skew-symmetric.
   
   (c) Show that the flow of a vector field $g(R) = \hat{\omega}R$ is given by $\phi_t(R) = \exp(\hat{\omega}t)R$ where exp is the matrix exponential.