CDS 202 Winter 2009 Solution Set 1

Problem 1 (MTA 1.2-3) Show that every separable metric space is second countable.

Solution (Contributed by Paul Skerritt [2009])

Let (X, d) be a separable metric space, and $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset. Consider the set of open discs

$$\mathcal{B} = \{D_{\frac{1}{m}}(x_n) : n, m \in \mathbb{N}\}.$$

Note that $\mathcal{B}$ is a countable union of countable sets, and so is itself countable. We claim that $\mathcal{B}$ is a basis for (X, d). Let $O \subset X$ be any open set. Then for $x \in O$, there is an $\epsilon > 0$ s.t. $D_\epsilon(x) \subset O$. Choose m s.t. $\frac{1}{m} < \frac{\epsilon}{2}$. Since $\{x_n\}$ is dense, there is some $x_n \in D_{\frac{1}{m}}(x)$. Hence $x \in D_{\frac{1}{m}}(x_n)$, and it is easy to check that $D_{\frac{1}{m}}(x_n) \subset D_\epsilon(x) \subset O$. So for every $x \in O$, there is a set $O_x \in \mathcal{B}$ s.t. $x \in O_x \subset O$. It follows that $O = \cup_{x\in O} O_x$ i.e. O can be written as a union of elements of $\mathcal{B}$. Hence $\mathcal{B}$ is a basis as claimed.

Problem 2 (MTA 1.2-8) (Solution scribe : Nawaf Bou-Rabee)

Let M denote the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Set

$$d(f, g) = \sup_{0\le x\le 1} \{|f(x) - g(x)|\}$$

(i) Show that d is a metric on M.

(ii) Show that $f_n \to f$ in $M \iff f_n$ converges uniformly to f.

(iii) By consulting theorems on uniform convergence from your advanced calculus text, show that M is a complete metric space.

Solution for (i) In order to confirm d is a metric, definiteness, symmetry, and the triangle inequality must be checked as follows.

a) $d(f, g) > 0$ unless $f(x) = g(x)$ $0 \le x \le 1$.

b) $d(f, g) = d(g, f)$.

c) Suppose $d(f, g) = |f(x_i) - g(x_i)|$. From the triangle inequality on $\mathbb{R}$,

$$|f(x_i) - g(x_i)| \le |f(x_i) - h(x_i)| + |g(x_i) - h(x_i)|$$

However,

$$|f(x_i)-g(x_i)| \le |f(x_i)-h(x_i)|+|g(x_i)-h(x_i)| \le \sup_{0\le x\le 1}\{|f(x)-h(x)|\}+\sup_{0\le x\le 1}\{|g(x)-h(x)|\}$$

Solution for (ii) Consider the sequence of functions $\{f_n(x)\}$.

f_n converges uniformly to $f \implies$ for every $\epsilon > 0$ $\exists$ a N s.t.

$$|f_n(x) - f(x)| < \epsilon$$

for $n \geq N$ and for all $x \in [0, 1]$ including the supremum of this difference. Hence, $f_n(x)$ converges in M.

Likewise, if $f_n(x)$ converges to $f(x)$ in M, then for every ϵ-ball of $f(x)$ there is an N s.t.

$$\sup_{0 \leq x \leq 1} \{|f_n(x) - f(x)|\} < \epsilon$$

for $n \geq N$. This implies that

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in [0, 1]$.

Solution for (iii) (Contributed by Paul Skerritt [2008])

Let (f_n) be a Cauchy sequence in M. Since

$$|f_n(x) - f_m(x)| \leq d(f_n, f_m), \qquad \forall x \in [0, 1]$$

it follows that $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in [0, 1]$, and so converges. Define a function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in [0, 1].$$

Now let $\epsilon > 0$. (f_n) Cauchy $\Rightarrow \exists N \in \mathbb{N}$ s.t.

$$|f_n(x) - f_m(x)| \leq d(f_n, f_m) < \frac{\epsilon}{2}, \qquad \forall n, m \geq N,\ x \in [0, 1].$$

Taking the limit as $m \to \infty$ (and using the continuity of $|\cdot|$) we get that

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon, \qquad \forall n \geq N,\ x \in [0, 1].$$

Hence f_n converges uniformly to f. Since the f_n are continuous and the convergence is uniform, a standard theorem tells us that f is continuous, i.e. $f \in M$. By part (ii), $f_n \to f$ in M, and so M is complete.

Problem 3 (MTA 1.3-7) (Solution scribe : Nawaf Bou-Rabee)

Consider $I = [0, 2\pi)$ and $\mathbb{S}^1$ as subsets of $\mathbb{R}$ and $\mathbb{R}^2$ respectively (with the usual topology induced from $\mathbb{R}$ and $\mathbb{R}^2$, respectively). Define the map:

$$f : I \mapsto \mathbb{S}^1; x \mapsto (\cos x, \sin x).$$

Is f a homeomorphism? Why or why not?

Solution The map f is continuous but does not have a continuous inverse. Consider the image of the open set $V = [0, \pi)$ in I. $f(V)$ is the set of points on the circle with $0 \leq x < \pi$ which is not an open set in the subspace topology on $\mathbb{S}^1$, i.e., the set cannot be formed by the intersection of any open rectangle in $\mathbb{R}^2$ and $\mathbb{S}^1$.

More generally, note that the interval is not compact but, the circle is compact. Since compactness is a topological property, there cannot exist a homeomorphism between the two sets.

Even more generally, no interval in $\mathbb{R}$ is homeomorphic to the circle. Roughly, we can remove a point from the interval I so that it becomes disconnected. However, the image of this disconnected interval is connected on the circle. Since connectedness is a topological property, there cannot exist a homeomorphism between a circle and I.

Problem 4 (MTA 1.3-8) Let S be a set. Is the identity map from S with the discrete topology to S with the trivial topology continuous? A homeomorphism?

Solution If S is empty, then the identity map is vacuously continuous and homeomorphic.

If S is a one element set, then the identity map is continuous and homeomorphic since the discrete and trivial topology are the same for a one element set and the function is trivially bijective.

If S contains more than one element, then the identity map is continuous but, not homeomorphic since it does not have a continuous inverse. The identity map is one-to-one since $x \neq y \implies f(x) \neq f(y)$ and onto since $f(S) = S$. The map is continous since the pre-image of every open set in the trivial topology, namely the set itself and the empty set, is open in the discrete topology. However, the identity map does not have a continuous inverse since any subset of X that does not equal X is open in the discrete topology but, its image is not open in the trivial topology.

Problem 5 (MTA 1.4-5) Show that the mapping $f : S \to T$ is continuous iff the mapping $s \mapsto (s, f(s))$ of S to the graph $\Gamma_f = \{(s, f(s)) \mid s \in S\} \subset S \times T$ is a homeomorphism of S with Γ_f (give Γ_f the subspace topology induced from the product topology of $S \times T$).

Solution Denote the mapping $s \mapsto (s, f(s))$ by $\gamma : S \to \Gamma_f$ and let the projection of $S \times T \to T$ be denoted π_2. Note that $f = \pi_2 \circ \gamma$, so if γ is a homeomorphism, then f is continuous since π_2 is continuous and the composition of continuous maps is continuous.

For the converse, assume that f is continuous. Note that γ is a continuous bijection since each component of it is continuous. It is also an open map since if $U \subset S$ is an open subset, then $\gamma(U) = \Gamma_f \cap (U \times T)$ is open in the subspace topology of Γ_f. Thus, γ is a homeomorphism.

Alternatively, it is easily checked that the inverse of $\gamma : S \to \Gamma_f$ is $\pi_1|_{\Gamma_f} : \Gamma_f \to S$, which is continuous (since $\pi_1 : S \times T \to S$ is continuous).

Problem 6 (MTA 1.5-8 (i) and MTA 1.6-2) (Solution scribe : Nawaf Bou-Rabee)

(i) Let M_3 be the set of all 3×3 matrices with the topology obtained by regarding M_3 as $\mathbb{R}^9$. Let $\mathrm{SO}(3) = \{A \in M_3 : A^T A = I \text{ and } \det(A) = 1\}$. Show that SO(3) is compact.

(ii) Let O(3) be the set of 3×3 orthogonal matrices. Show that O(3) is not connected, and that it has two components.

Solution for (i) Define

$$F : \mathbb{R}^9 \to \mathbb{R}^{10}; A \to (A^T A, \det(A))$$

SO(3) is closed since the function F is continuous, and hence, the pre-image of the closed set (or point) $\{\mathbf{Id}, 1\}$ is closed in $\mathbb{R}^9$. It is bounded since each matrix entry must be in between $[-1, 1]$, otherwise the rows would have norm greater than one. By the Heine-Borel theorem, SO(3) is compact.

Solution for (ii) (Contributed by Paul Skerritt [2009])

We can write O(3) as the disjoint union of SO(3) and $P{\cdot}\mathrm{SO}(3)$, where

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is reflection in the y-z plane. Note that for $\det : \mathrm{O}(3) \to \{\pm 1\}$, $\mathrm{SO}(3) = \det^{-1}(\{1\})$ while $P{\cdot}\mathrm{SO}(3) = \det^{-1}(\{-1\})$. So SO(3) and $P{\cdot}\mathrm{SO}(3)$ are both open, and hence both closed in O(3). This proves that O(3) is not connected. We will show that SO(3) and $P{\cdot}\mathrm{SO}(3)$ are both arcwise connected, hence connected, and so are the components of O(3).

It's a well known result that an orthogonal matrix is completely diagonalizable. Considering $A \in \mathrm{O}(3)$ as a linear operator on the inner product space $\mathbb{C}^3$ with inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\dagger \mathbf{y}$, it is straightforward to prove from the orthogonality condition that its eigenvalues must all lie on the complex unit circle. Also, since the entries of A are real, $e^{i\theta}$ is an eigenvalue $\Leftrightarrow$ $e^{-i\theta}$ is an eigenvalue. The possible sets of eigenvalues are $\{1, e^{i\theta}, e^{-i\theta}\}$ and $\{-1, e^{i\theta}, e^{-i\theta}\}$. The first corresponds to elements of SO(3), the second to elements of P·SO(3).

Consider $A \in \mathrm{SO}(3)$, and let $\mathbf{v}$ be a (real) normalized eigenvector with eigenvalue 1, and $\mathbf{x}, \bar{\mathbf{x}}$ be (complex) normalized eigenvectors with eigenvalues $e^{i\theta}, e^{-i\theta}$ which are orthonormal to $\mathbf{v}$ and each other (for generic θ this follows from the orthogonality of A, and for the degenerate cases $\theta = 0, \pi$ it can be arranged so). So we have

$$\begin{aligned} A\mathbf{v} &= \mathbf{v} \\ A\mathbf{x} &= e^{i\theta}\mathbf{x} \\ A\bar{\mathbf{x}} &= e^{-i\theta}\bar{\mathbf{x}}. \end{aligned}$$

Let $\mathbf{s} = \frac{1}{\sqrt{2}}(\mathbf{x} + \bar{\mathbf{x}}), \mathbf{t} = \frac{1}{\sqrt{2}i}(\mathbf{x} - \bar{\mathbf{x}})$. Then $\mathbf{v}, \mathbf{s}, \mathbf{t}$ form an real, orthonormal triple of vectors (which by multiplying $\mathbf{v}$ by -1 if necessary we can take to be right-handed) and

$$\begin{aligned} A\mathbf{v} &= \mathbf{v} \\ A\mathbf{s} &= \cos\theta\ \mathbf{s} - \sin\theta\ \mathbf{t} \\ At &= \sin\theta\ \mathbf{s} + \cos\theta\ \mathbf{t}, \end{aligned}$$

i.e. A corresponds to (right-handed) rotation about $\mathbf{v}$ by angle θ,

$$A = S \begin{pmatrix} 1 & 0 & 0 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix} S^T = R_{\mathbf{v}}(\theta)$$

where $S = (\mathbf{v}, \mathbf{s}, \mathbf{t})$. The curve $\gamma : [0, 1] \to \mathrm{SO}(3)$ defined by $\gamma(t) = R_{\mathbf{v}}(t\theta)$ links the identity I to A in SO(3). It follows that any two points in SO(3) can be linked by a curve going through the identity, and so SO(3) is arcwise connected.

The case for $A \in P$·SO(3) is similar (connecting curves will pass through P instead of I).