CDS 202 Winter 2009 Solution Set 7

Problem 1 (Nawaf Bou-Rabee)

The one-parameter subgroups generated by A and B are provided by the matrix exponential as follows,

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \qquad e^{Bt} = I + Bt = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

a basic result from linear odes.

The corresponding actions on $\mathbb{R}^2$ are the restriction of the natural action of $GL(2, \mathbb{R})$ to the subgroups generated by A and B. For $\mathbf{x} \in \mathbb{R}^2$, written component-wise as $\mathbf{x} = (x_1, x_2)$, these actions are given by:

$$\theta_A(e^{At}, \mathbf{x}) = \begin{bmatrix} \cos(t)x_1 + \sin(t)x_2 \\ -\sin(t)x_1 + \cos(t)x_2 \end{bmatrix} \qquad \theta_B(e^{Bt}, \mathbf{x}) = \begin{bmatrix} x_1 + x_2 t \\ x_2 \end{bmatrix}$$

with corresponding infinitesimal generators:

$$\psi_A(\mathbf{x}) = \left.\frac{d}{dt}\right|_{t=0} \big(\theta_A(e^{At}, \mathbf{x})\big) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \quad \psi_B(\mathbf{x}) = \left.\frac{d}{dt}\right|_{t=0} \big(\theta_B(e^{Bt}, \mathbf{x})\big) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

Problem 2 (MTA 5.3-1) (Arash Yavari 2003)

$G \times V$ is a product of manifolds and therefore is itself a manifold. It remains to show that the group structure on $G \times V$ satisfies the properties of group multiplication. Note that the group multiplication as given is smooth because the products and sums of smooth functions are smooth.

- Associativity:

$$\begin{aligned} &(g_2, v_2) \cdot (g_3, v_3) = (g_2 g_3, g_2 v_3 + v_2) \\ &(g_1, v_1) \cdot ((g_2, v_2) \cdot (g_3, v_3)) = (g_1 g_2 g_3, g_1 g_2 v_3 + g_1 v_2 + v_1) \\ &= ((g_1, v_1) \cdot ((g_2, v_2)) \cdot (g_3, v_3) \end{aligned}$$

- Identity: The identity element is $(e, 0)$ where e and 0 are identities in G and V, respectively, because,

$$\begin{aligned} (g, v) \cdot (e, 0) = (ge, g0 + v) = (g, v) \\ (e, 0) \cdot (g, v) = (eg, ev + 0) = (g, v) \end{aligned}$$

- Inverse: The (unique) inverse for (g, v) is $(g^{-1}, -g^{-1}v)$, because

$$\begin{aligned}(g, v) \cdot (g^{-1}, -g^{-1}v) &= (gg^{-1}, -gg^{-1}v + v) = (e, 0)\\ (g^{-1}, -g^{-1}v) \cdot (g, v) &= (g^{-1}g, g^{-1}v - g^{-1}v) = (e, 0)\end{aligned}$$

The Lie algebra of $G \circledS V$ is the tangent space at the identity, that is, $T_eG \times T_0V$, therefore every element in $\mathfrak{g} \circledS V$ has the form $\mu = (\xi, v)$ with $\xi \in \mathfrak{g}$ and $v \in V$. We will compute the Lie bracket in $\mathfrak{g} \circledS V$ using the equation

$$T_e\varphi^\eta \cdot \xi = [\xi, \eta]$$

from the text.

Let $\nu = (\eta, w)$ and consider the curves

$$c_1(t) = (\exp t\xi, tv) \quad \text{and} \quad c_2(t) = (\exp t\eta, tw),$$

which satisfy $c_1(0) = c_2(0) = e_{G \circledS V}$ and $c_1'(0) = \mu$, $c_2'(0) = \nu$; then

$$\begin{aligned}[\mu, \nu] &= \frac{d}{dt}\frac{d}{ds} c_1(t)c_2(s)c_1(t)^{-1}\bigg|_{s=0,t=0}\\ &= \frac{d}{dt}\frac{d}{ds}(\exp t\xi \exp s\eta \exp -t\xi,\\ &\qquad (-\exp t\xi \exp s\eta \exp -t\xi)tv + (\exp t\xi)sw + tv)|_{s=0,t=0}\\ &= \frac{d}{dt}(\mathrm{Ad}_{\exp t\xi} \cdot\eta, -(\mathrm{Ad}_{\exp t\xi} \cdot\eta)tv + \exp t\xi \cdot w)\bigg|_{t=0}\\ &= ([\xi, \eta], \xi w - \eta v)\end{aligned}$$

where

$$\xi w = \frac{d}{dt}\exp t\xi \cdot w\bigg|_{t=0}$$

for $\xi \in \mathfrak{g}$ and $w \in V$.

Problem 3 (MTA 5.3-2)

Solution for (a) Elements of $E(3)$ are of the form (A, v), where $A \in O(3)$, $v \in \mathbb{R}^3$, and the group multiplication in $E(3)$ is

$$(A_1, v_1) \cdot (A_2, v_2) = (A_1A_2, A_1v_2 + v_1).$$

Thus, by taking A_1v_2 as the action of A_1 on v_2, and identifying $E(3)$ with $O(3)\times\mathbb{R}^3$, (A_1, v_1) with (g_1, v_1), and (A_2, v_2) with (g_2, v_2), we see that the group multiplication in $E(3)$ is nothing but $(g_1, v_1) \cdot (g_2, v_2)$ as defined in Problem 5.3-1. This shows that $E(3)$ can be written as $O(3)\circledS\mathbb{R}^3$.

Solution for (b) Just define the mapping f as follows:

$$f : (A, v) \mapsto \begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix},$$

where $A \in O(3)$, $v \in \mathbb{R}^3$. It is clear that f is bijective. Now use the usual matrix multiplication as the group multiplication on the group of those 4×4 matrices. We get

$$\begin{aligned} f((A_1, v_1) \cdot (A_2, v_2)) &= f(A_1A_2, A_1v_2 + v_1) \\ &= \begin{bmatrix} A_1A_2 & A_1v_2 + v_1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} A_1 & v_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 & v_2 \\ 0 & 1 \end{bmatrix} \\ &= f(A_1, v_1) \cdot f(A_2, v_2), \end{aligned}$$

also

$$\begin{aligned} f^{-1}\left(\begin{bmatrix} A_1 & v_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A_2 & v_2 \\ 0 & 1 \end{bmatrix}\right) &= f^{-1}\left(\begin{bmatrix} A_1A_2 & A_1v_2 + v_1 \\ 0 & 1 \end{bmatrix}\right) \\ &= (A_1A_2, A_1v_2 + v_1) \\ &= (A_1, v_1) \cdot (A_2, v_2) \\ &= f^{-1}\left(\begin{bmatrix} A_1 & v_1 \\ 0 & 1 \end{bmatrix}\right) \cdot f^{-1}\left(\begin{bmatrix} A_2 & v_2 \\ 0 & 1 \end{bmatrix}\right) \end{aligned}$$

Thus, f is a group isomorphism, and $E(3)$ is isomorphic to the group of those 4×4 matrices.

Problem 4 (MTA 5.3-4)

We will identify TG with $G \times \mathfrak{g}$ via right translations. This puts a group structure in $G \times \mathfrak{g}$ which we will show coincides with $G Ⓢ \mathfrak{g}$ when G acts on $\mathfrak{g}$ via the adjoint map.

Let $v_g, v_h \in TG$ be vectors at g and h respectively. By definition,

$$v_g \cdot v_h = T_{(g,h)}\mu(v_g, v_h).$$

It is clear that $(v_g \cdot v_h)$ is a vector at gh, which we will call v_{gh}.

Let

$$\xi = T_gR_{g^{-1}}v_g, \quad \eta = T_hR_{h^{-1}}v_h \quad \text{and} \quad \nu = T_{gh}R_{(gh)^{-1}}v_{gh}.$$

Then in terms of $G\times\mathfrak{g}$, the relation $v_g \cdot v_h = v_{gh}$ is written $(g, \xi)(h, \eta) = (gh, \nu)$. We calculate ν:

$$\begin{aligned} v_{gh} &= T_{(g,h)}\mu(v_g, v_h) = T_{(g,h)}\mu(T_eR_g\xi, T_eR_h\eta) \\ &= T_eR_{gh} \cdot \xi + T_e(R_h \circ L_g) \cdot \eta. \end{aligned}$$

Then $\nu = T_{gh}R_{(gh)^{-1}}v_{gh} = \xi + \operatorname{Ad}_g \eta$. Therefore we obtain

$$(g, \xi) \cdot (h, \eta) = (gh, \operatorname{Ad}_g \eta + \xi)$$

as predicted.

Problem 5 (PS 2009)

(a) Taking the zero of potential energy at the center of the disc, the Lagrangian for the disc is

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_\theta\dot{\theta}^2 + \frac{1}{2}I_\phi\dot{\phi}^2$$

where M is the mass of the disc, and I_θ, I_ϕ are the moments of inertia corresponding to rotation in θ, ϕ directions (if the disc is taken as infinitesimally thin, then $I_\theta = \frac{1}{4}M\rho^2$ and $I_\phi = \frac{1}{2}M\rho^2$, where ρ is the radius of the disc).

Let $g(\alpha, c, d) \in \mathrm{SE}(2)$ denote a counterclockwise rotation by angle α followed by a translation by (c, d). The action of SE(2) on Q is given by

$$\Phi_{g(\alpha,c,d)}(x, y, \theta, \phi) = (x\cos\alpha - y\sin\alpha + c, x\sin\alpha + y\cos\alpha + d, \theta + \alpha, \phi).$$

The tangent of this map is given by

$$T_{(x,y,\theta,\phi)}\Phi_{g(\alpha,c,d)} \cdot (v_x, v_y, v_\theta, v_\phi) = (v_x\cos\alpha - v_y\sin\alpha, v_x\sin\alpha + v_y\cos\alpha, v_\theta, v_\phi).$$

L is easily seen to be invariant under this mapping.

(b) Here we take the base to be $S^1 \times S^1$ (corresponding to angles θ and ϕ), and the fiber to be $\mathbb{R}^2$ (corresponding to position on the plane). The noslip conditions can be written either as

$$\begin{aligned} v_x\cos\theta + v_y\sin\theta - \rho v_\phi &= 0 \\ -v_x\sin\theta + v_y\cos\theta &= 0 \end{aligned}$$

(corresponding to decomposition along and perpendicular to the plane of the disc) or as

$$\begin{aligned} v_x - v_\phi\rho\cos\theta &= 0 \\ v_y - v_\phi\rho\sin\theta &= 0 \end{aligned}$$

(corresponding to decomposition along the x and y axes). The latter is in the form required for a principal connection; in the notation of the paper

$$\begin{aligned}\Gamma(q)\begin{bmatrix} v_x \\ v_y \\ v_\theta \\ v_\phi \end{bmatrix} &= \begin{bmatrix} v_x - v_\phi \rho\cos\theta \\ v_y - v_\phi \rho \sin\theta \end{bmatrix} \\ &= \begin{bmatrix} v_x \\ v_y \end{bmatrix} - \begin{bmatrix} 0 & \rho\cos\theta \\ 0 & \rho\sin\theta \end{bmatrix}\begin{bmatrix} v_\theta \\ v_\phi \end{bmatrix} \in T_{(0,0)}\mathbb{R}^2 \simeq \mathbb{R}^2.\end{aligned}$$

Note that

$$\Gamma(q)\begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

and

$$\Gamma(T_q\Phi_{(c,d)} \cdot X_q) = \Gamma(X_q) = \mathrm{Ad}_{(c,d)}\Gamma(X_q)$$

(since $\mathbb{R}^2$ is abelian), so Γ obeys the required properties to be a principal connection.

From the above we read off the local version of the connection

$$A(\theta,\phi)\begin{bmatrix} v_\theta \\ v_\phi \end{bmatrix} = -\begin{bmatrix} 0 & \rho\cos\theta \\ 0 & \rho\sin\theta \end{bmatrix}\begin{bmatrix} v_\theta \\ v_\phi \end{bmatrix}.$$

(c) There are two ways to show this:

(i) The distribution defined by the connection Γ (i.e. such that $\Gamma(q) \cdot X = 0$) is spanned by

$$\left\{ \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ \rho\cos\theta\frac{\partial}{\partial x} + \rho\sin\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta} \right\} = \{X_1, X_2\}$$

Then

$$\begin{aligned}[X_1, X_2] &= \rho\sin\theta\frac{\partial}{\partial x} - \rho\cos\theta\frac{\partial}{\partial y} \\ [[X_1, X_2], X_2] &= -\rho\cos\theta\frac{\partial}{\partial x} - \rho\sin\theta\frac{\partial}{\partial y}\end{aligned}$$

Clearly $\{X_1, X_2, [X_1, X_2], [[X_1, X_2], X_2]\}$ spans all of TQ, and so the system is totally controllable, and hence fiber controllable.

(ii) Writing A explicitly as a differential form

$$\begin{aligned} A(\theta,\phi) &= -\begin{bmatrix}1\\0\end{bmatrix}\rho\cos\theta d\phi - \begin{bmatrix}0\\1\end{bmatrix}\rho\sin\theta d\phi \\ \Rightarrow dA(\theta,\phi) &= \begin{bmatrix}1\\0\end{bmatrix}\rho\sin\theta d\theta\wedge d\phi - \begin{bmatrix}0\\1\end{bmatrix}\rho\cos\theta d\theta\wedge d\phi \end{aligned}$$

Then clearly we have that

$$\operatorname{span}\{(dA)_p(X_p,Y_p) \mid X_p,Y_p \in T_p(S^1\times S^1), p\in S^1\times S^1\} = \mathbb{R}^2$$

(e.g. take $p = (0,0),(\pi/2,0)$ and $X_p = \frac{\partial}{\partial\theta}, Y_p = \frac{\partial}{\partial\phi}$). So by the abelian version of the Ambrose-Singer theorem, the system is totally controllable, and hence fiber controllable.

(d) One possible trajectory is this: set $\theta = \arctan(y/x)$, and move the disc till it hits the origin. Then set θ to 0. Finally, ϕ can be set to 0 by executing a circle of radius $\rho(1-\frac{\phi}{2\pi})$.