CDS 202 Winter 2009 Solution Set 4

Problem 1 (MTA 3.3-1) (PS 2009)

For conciseness I will write Γ_f for graph(f).

(i) (a) Let $\phi : U \to \mathbf{E}$ be a chart in M about m, and $\psi : V \to \mathbf{F}$ be a chart in N about $f(m)$. Define a chart $\chi : U \times V \to \mathbf{E} \times \mathbf{F}$ by

$$\chi(m, n) = (\phi(m), \psi(n) - \psi(f(m))).$$

Then it is easily seen that χ is injective, and that $\chi((U \times V) \cap \Gamma_f) = \chi(U \times V) \cap (\mathbf{E} \times \{0\})$. Also

$$(\chi \circ (\phi \times \psi)^{-1})(u, v) = (u, v - (\psi \circ f \circ \phi^{-1})(u))$$
$$((\phi \times \psi) \circ \chi^{-1})(u, v) = (u, v + (\psi \circ f \circ \phi^{-1})(u)).$$

Both of these overlap maps are C^∞ (since f is C^∞), so χ is compatible with the usual charts on $M \times N$, and verifies the submanifold property of Γ_f.

(b) The property we want to demonstrate is

$$T_{(m,f(m))}(M \times N) = T_{(m,f(m))}\Gamma_f \oplus T_{(m,f(m))}(\{m\} \times N).$$

We just need to verify that $T_{(m,f(m))}(M \times N) = T_{(m,f(m))}\Gamma_f \cap T_{(m,f(m))}(\{m\} \times N) = \{0\}$, since counting dimensions then gives the result.

So let $X_{(m,f(m))}$ be an element of this intersection. Then $X_{(m,f(m))} \in T_{(m,f(m))}\Gamma_f$ implies

$$X_{(m,f(m))} = (v_m, T_m f(v_m)) \qquad \text{for some } v_m \in T_m M \text{ (cf part (iii))}$$

while $X_{(m,f(m))} \in T_{(m,f(m))}(\{m\} \times N)$ implies

$$X_{(m,f(m))} = (0_m, w_{f(m)}) \qquad \text{for some } w_{f(m)} \in T_{f(m)}N.$$

Combining these tells us that $X_{(m,f(m))} = (0_m, 0_{f(m)})$.

(c) Take $f(x) = x^{1/3}$. Then

$$\Gamma_f = \{(x, x^{1/3}) |\ x \in \mathbb{R}\} = \{(y^3, y) |\ y \in \mathbb{R}\}.$$

So use the chart $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by

$$\zeta(x, y) = (y, x - y^3).$$

ζ is easily shown to be compatible with the usual chart $\mathrm{id}_{\mathbb{R}} \times \mathrm{id}_{\mathbb{R}}$, and $\zeta(\Gamma_f) = \mathbb{R} \times \{0\}$. So Γ_f is a submanifold of $\mathbb{R} \times \mathbb{R}$. However f is not C^∞ at $x = 0$.

(d) $\pi_M|_{\Gamma_f} : \Gamma_f \to M$ is clearly a bijection, and we can write

$$f = \pi_N \circ (\pi_M|_{\Gamma_f})^{-1}.$$

Now π_N, π_M are both C^∞, and since Γ_f is a submanifold (condition (a)), so is $\pi_M|_{\Gamma_f}$. Also $T_{(m,f(m))}(\pi_M|_{\Gamma_f})$ is injective since

$$\begin{aligned}\ker T_{(m,f(m))}(\pi_M|_{\Gamma_f}) &= T_{(m,f(m))}\Gamma_f \cap \ker T_{(m,f(m))}\pi_M \\ &= T_{(m,f(m))}\Gamma_f \cap T_{(m,f(m))}(\{m\} \times N) = \{0\}\end{aligned}$$

by condition (b). Hence $T_{(m,f(m))}(\pi_M|_{\Gamma_f})$ is bijective, since $\dim T_{(m,f(m))}\Gamma_f = \dim T_m M$. By the inverse function theorem, $(\pi_M|_{\Gamma_f})^{-1}$ is C^∞. It follows that $f = \pi_N \circ (\pi_M|_{\Gamma_f})^{-1}$ is C^∞.

(ii) f is C^∞, so (i)(a) and (i)(b) hold. Then it was proved in (i)(d) that $\pi_M|\Gamma_f : \Gamma_f \to M$ is a diffeomorphism of Γ_f and M.

Or more directly, it can be checked that the local representation of $\pi_M|_{\Gamma_f}$, wrt the charts $\phi : U \to \mathbf{E}$ and $\pi_{\mathbf{E}} \circ (\chi|_{\Gamma_f}) : \Gamma_f \to \mathbf{E}$, where ϕ and χ are as defined in (i)(a) above, is just the identity mapping.

(iii) Using the definition of vectors as equivalence classes of curves,

$$\begin{aligned}
&[(\gamma_M, \gamma_N)]_{(m,f(m))} \in T_{(m,f(m))}\Gamma_f \\
\Leftrightarrow&(\gamma_M, \gamma_N) \text{ is a curve in } \Gamma_f \text{ with } (\gamma_M(0), \gamma_N(0)) = (m, f(m)) \\
\Leftrightarrow&\gamma_N = f \circ \gamma_M,\ \gamma_M(0) = m \\
\Leftrightarrow&[(\gamma_M, \gamma_N)]_{(m,f(m))} = [(\gamma_M, f \circ \gamma_M)]_{(m,f(m))} \\
&\quad = T_{(m,m)}(\mathrm{id}_M \times f)[(\gamma_M, \gamma_M)]_{(m,m)} \\
&\quad = T_m\mathrm{id}_M \times T_m f([\gamma_M]_m, [\gamma_M]_m) \qquad \text{using the natural identification (cf Prop 3.3.13)} \\
&\quad = (T_m\mathrm{id}_M[\gamma_M]_m, T_m f[\gamma_M]_m) \\
&\quad \in \{(v_m, T_m f(v_m)) | v_m \in T_m M\}.
\end{aligned}$$

So $T_{(m,f(m))}\Gamma_f = \{(v_m, T_m f(v_m)) | v_m \in T_m M\}$.

Problem 2 (Nawaf Bou-Rabee)

By the *submersion theorem*, $T_m F^{-1}(0) = \ker(T_m F)$. Therefore, X is a tangent vector to M provided $\mathbf{d}F \cdot X_{F(x^1,x^2,x^3)} = 0$, i.e.,

$$X[F]|_{F(x^1,x^2,x^3)=0} = \left(v_1 \frac{\partial F}{\partial x^1} + v_2 \frac{\partial F}{\partial x^2} + v_3 \frac{\partial F}{\partial x^3} \right)\Bigg|_{F(x^1,x^2,x^3)=0} = 0.$$

Problem 3 (PS 2008)

Let $h : \mathbb{R} \to \mathbb{R}$ be a C^∞ function which is 0 at $x = 1$ and vanishes outside $]1/2, 3/2[$ (such a function exists by Lemma 4.2.13). Define a vector field X on $\mathbb{R}^n$ by

$$X(x) = \begin{cases} h(\|x\|)Y(\frac{x}{\|x\|}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The function $\frac{\cdot}{\|\cdot\|} : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is C^∞ and $\|\cdot\|$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$, so X is clearly C^∞ on $\mathbb{R}^n \setminus \{0\}$. Also X is 0 in a neighborhood of $x = 0$, and so is C^∞ there also.

Problem 4 (MTA 4.1-5) (PS 2009)

Let $x(t)$ be the solution of $\dot{x} = Xx + f(x)$ with initial condition $x(0) = x_0$. Define the function

$$y(t) = e^{tX}x_0 + \int_0^t e^{(t-s)X} f(x(s))ds.$$

Then

$$\begin{aligned} \dot{y}(t) &= Xe^{tX}x_0 + f(x(t)) + \int_0^t Xe^{(t-s)X} f(x(s))ds \\ &= Xy(t) + f(x(t)). \end{aligned}$$

So

$$\dot{x}(t) - \dot{y}(t) = Xx(t) - Xy(t) = X(x(t) - y(t)).$$

Since $x(0) - y(0) = x_0 - x_0 = 0$, uniqueness of solutions of the equation $\dot{z} = Xz$ implies that $x(t) = y(t)$, i.e.,

$$x(t) = e^{tX}x_0 + \int_0^t e^{(t-s)X} f(x(s))ds.$$

Problem 5 (Nawaf Bou-Rabee)

Clearly, ϕ_t is smooth as a map from $\mathbb{R}^3 \to \mathbb{R}^2$ because each of its coordinate functions are infinitely differentiable. Moreover,

$$\begin{aligned} \phi_0(x, y) &= (x, y) = id_{\mathbb{R}^2} \\ (\phi_t \circ \phi_s)(x, y) &= (xe^{2s}e^{2t}, ye^{-3s}e^{-3t}) \\ &= (xe^{2(t+s)}, ye^{-3(t+s)}) \\ &= \phi_{t+s}(x, y) \end{aligned}$$

Thus, ϕ_t defines a smooth flow on M.

The infinitesimal generator of this flow is by definition:

$$\begin{aligned}\psi(x,y) &= \left(\frac{d}{dt}\phi_t(x,y)\right)_{t=0}\\ &= (2x,-3y)\end{aligned}$$

Let $q=(x,y)$. Then in directional derivative form:

$$X(q) = 2x(\partial/\partial x) - 3y(\partial/\partial y).$$

X is ϕ-invariant because:

$$\begin{aligned}(\phi_{t*}X)(q) &= T\phi_t \circ X \circ \phi_t^{-1}(q)\\ &= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t}\end{bmatrix} \circ X \circ (xe^{-2t}, ye^{3t})\\ &= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t}\end{bmatrix} \circ (2xe^{-2t}\partial/\partial x - 3ye^{3t}\partial/\partial y)\\ &= 2x\partial/\partial x - 3y\partial/\partial y\\ &= X(q).\end{aligned}$$

Problem 6 (PS 2009)

(a)

$$\hat{\omega}v = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0\end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3\end{bmatrix} = \begin{bmatrix}\omega_2 v_3 - \omega_3 v_2 \\ \omega_3 v_1 - \omega_1 v_3 \\ \omega_1 v_2 - \omega_2 v_1\end{bmatrix} = \omega \times v$$

(b) As shown in Example 3.5.5 C, O(3) is a 3-dimensional submanifold of $L(\mathbb{R}^3,\mathbb{R}^3)$, and it was demonstrated in HW1 that SO(3) is the component of O(3) containing to the identity. The tangent space to SO(3) at R is given by $\ker T_R f = \{H \in L(\mathbb{R}^3,\mathbb{R}^3) | RH^T + HR^T = 0\} = \{H \in L(\mathbb{R}^3,\mathbb{R}^3) | (HR^T)^T = -HR^T\}$. Any skew skew-symmetric matrix HR^T can be written as $\hat{\omega}$ for some $\omega \in \mathbb{R}^3$, and so we see that this set is equal to $\{\hat{\omega}R : \omega \in \mathbb{R}^3\}$.

(c)

$$\frac{d}{dt}\phi_t(R) = \hat{\omega}\exp\hat{\omega}tR = \hat{\omega}\phi_t(R) = g(\phi_t(R))$$

implies that ϕ_t is the flow of the vector field $g(R)$.