CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

# CDS 110b

John Doyle | **Problem Set #2** | Issued: 24 Jan
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In all of the following, we let

$$J_t(x_t, u_t) := x_t^\top Q x_t + u_t^\top R u_t,$$

with $Q$ and $R$ being positive semi-definite and positive definite, respectively, of appropriate dimension.

1. **Lagrange will set you free**

   1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, $F \in \mathbb{R}^{m\times n}$ with $n > m$.

      (a) Show that $\dim(\mathrm{Null}(F)) \geq n - m$.

      (b) Write the Lagrangian of

      $$\begin{array}{l} \min f(x) \\ \text{s.t. } Fx = g \end{array} \tag{1}$$

      and show that if $x^*$ is optimal, then $\nabla f(x^*) = F^\top \lambda$, for some $\lambda \in \mathbb{R}^m$.

   2) Suppose that $x$ satisfies $Fx = g$, and consider a small step in the direction $v$ to $x + hv$, where $h > 0$ is small. Show that $x + hv$ is feasible if and only if $Fv = 0$, and that

   $$f(x + hv) < f(x) \iff \nabla f(x)^\top v < f(x).$$

   *[hint: use the Taylor series expansion of f].*

   3) Conclude that if $v$ is a feasible direction, i.e., $Fv = 0$ and $x^*$ is optimal, then $\nabla f(x)^\top v = 0$.

   4) Write

   $$\begin{array}{l} \min_{x,u} \sum_{t=0}^{T-1} J_t(x_t, u_t) + x_T^\top Q x_T \\ \text{s.t. } x_{t+1} = Ax_t + Bu_t \end{array} \tag{2}$$

   as an unconstrained optimization problem in the variables $(x, u, \lambda)$.

   5) Show that the optimality conditions are given by

   $$\begin{bmatrix} x_{t+1} \\ \lambda_t \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^\top \\ Q & A^\top \end{bmatrix}}_{H} \begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix}, \; u_t = -R^{-1}B^\top \lambda_{t+1}. \tag{3}$$

   $H$ is known as the Hamiltonian of the system, and $\lambda_t$ is referred to as the co-state, or adjoint system.

   6) Show that

   $$\lambda_t = A^\top (I + P_{t+1} B R^{-1} B^\top)^{-1} P_{t+1} A x_t + Q x_t = P_t x_t$$

with $P_t = A^\top P_{t+1} A + Q - A^\top P_{t+1} B(R + B^\top P B)^{-1} B^\top P_{t+1} A$ and thus, that

$$u_t = -(R + B^\top P_{t+1} B)^{-1} B^\top P_{t+1} A x_t.$$

*[hint: Refer to the matrix fact sheet posted on the course website, courtesy of Sanjay Lall at Stanford.]*

2. **Analog to digital and back again**

The continuous time LQR problem is given by

$$\begin{aligned} &\min_{u:[0,T]\to\mathbb{R}^m} \int_{t=0}^{T} x_t^\top Q x_t + u_t^\top R u_t + x_T^\top Q_T x_T \\ &\text{s.t. } \dot{x} = Ax + Bu, x(0) = x_0. \end{aligned} \tag{4}$$

This is an infinite dimensional problem, where our optimization variable is a function. There are many ways to solve this problem, (as you should by now expect), but we will solve it via discretization.

1) Let $h > 0$ be a small step size, and $N$ a positive integer satisfying $Nh = T$. Show that, up to first order:

$$\begin{aligned} x((k+1)h) &\approx (I + hA)x(kh) + hBu(kh) \\ J &\approx \tfrac{h}{2} \textstyle\sum_{k=0}^{N-1} x(kh)^\top Q x(kh) + u(kh)^\top R u(kh) + \tfrac{1}{2} x(kh)^\top Q_T x(kh) \end{aligned} \tag{5}$$

2) Show that this leads to a discrete-time LQR problem with horizon $N$ and with parameters:

$$A_d = I + hA,\ B_d = hB,\ Q_d = hQ,\ R_d = hR \text{ and } \ Q_T.$$

3) Conclude that the optimal discretized control input is given by:

$$\begin{aligned} u(kh) &= \tilde{K}_k x(kh) \\ \tilde{K}_k &= -(R_d + B_d^\top \tilde{P}_{k+1} B_d)^{-1} B_d^\top \tilde{P}_{k+1} A_d \\ \tilde{P}_k &= Q_d + A_d^\top \tilde{P}_{k+1} A_d + A_d^\top \tilde{P}_{k+1} B_d \tilde{K}_k,\ P_N = Q_T. \end{aligned} \tag{6}$$

4) Using the expressions you found in part 2), and taking $h \to 0$, (and consequently, $N \to \infty$), show that

$$\begin{aligned} &u(t) = -R^{-1} B^\top P_t x_t \\ &\text{where } -\dot{P} = Q + A^\top P + PA - PBR^{-1}B^\top P,\ P(T) = Q_T \end{aligned} \tag{7}$$

*[hint: use the fact that as $h \to 0$ and $N \to \infty$, for any $t \in [0, T]$, there exist $k$ such that $\tilde{P}_k = P(kh) \to P(t)$ ].*

5) Derive the optimal infinite horizon continuous time LQR control law.

3. **Flight of the Martingale**

A stochastic process $\{X_t\}$ is called a Martingale if it satisfies $\mathbb{E}[|X_t|] < \infty$, and $\mathbb{E}[X_{t+1} \mid X_t, \ldots, X_0] = X_t$ for any time $t \geq 0$.

Consider the stochastic infinite horizon discrete LQR problem

$$\begin{array}{l} \min_u \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[J_t(x_t, u_t) + x_T^\top Q x_T] \\ \text{s.t. } x_{t+1} = Ax_t + Bu_t + w_t \end{array} \tag{8}$$

where we now assume $w_t$ to be a martingale. In particular, we assume that the noise process $\{w_t\}$ satisfies

- $\mathbb{E}[w_{t+1}] = w_t$, and
- $\mathbb{E}[w_t w_t^\top] = W$
- $\mathbb{E}w_0 = 0$.

Use dynamic programming to solve optimization (8).

4. **One step behind (or separated, but not alone)**

In this problem we will solve the stochastic discrete time LQR problem assuming a one step actuation delay.

1) Given the stochastic discrete time system

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{9}$$

where now $w_t$ satisfies our usual assumptions (i.i.d., zero mean with covariance $W$), compute $\mathbb{E}[x_t \mid x_{t-1}, u_{t-1}]$.

2) Consider now the uncontrolled system

$$\hat{x}_{t+1} = A\hat{x}_t + w_t. \tag{10}$$

Show that if both systems (9) and (10) are driven by the same noise process, then

$$x_t - \mathbb{E}[x_t \mid x_{t-1}, u_{t-1}] = \hat{x}_t - \mathbb{E}[\hat{x}_t \mid \hat{x}_{t-1}]. \tag{11}$$

This shows that your control actions do not affect the quality of your state estimates: this is referred to as the control input having *no dual effect*, or the system enjoying a *separation principle.*

3) Use the fact that your control action does not affect the quality of your state estimate, and hence can be designed independently of the estimator that you computed in part 1), to solve the following stochastic discrete time LQR:

$$\begin{array}{ll} \min_u & \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[J_t(x_t, u_t) + x_T^\top Q x_T] \\ & \text{s.t. } x_{t+1} = Ax_t + Bu_t + w_t \\ & u_t = g(x_{t-1}, t), \end{array} \tag{12}$$

where the constraint on $u_t$ means that it may only depend on $x_{t-1}$.

*[note: we will prove a more general separation principle in class]*

5. **Tracking error**

1) Modify the continuous time LQR problem so that instead of regulating to the origin, it tracks a reference signal $x_r(t)$ with dynamics

$$\dot{x}_r = A_r x_r + r, \; x_r(0) = r_0 \tag{13}$$

*[hint: the name of the problem]*

2) Apply this algorithm to either the linearized inverted pendulum model given in the class notes, or the model that you are planning on using for your project, and make your system perform two tracking tasks involving sinuisodal reference signals. For example, for the inverted pendulum, you could make it:

(a) act as much like a metronome as possible, i.e. stay at $x = 0$ but make the rod oscillate between $\theta = -\frac{\pi}{16}$ and $\theta = \frac{\pi}{16}$ at a regular frequency, or

(b) sway back and forth, i.e. have $x$ track a sinusoidal trajectory while maintaining the rod as close to $\theta = 0$ as possible.

For this problem, you may assume full state feedback, an are free to choose the system's regulated output.

Submit your Matlab code, and simulations showing state trajectories and control effort.

3) Let $A_r = 0$ and $x_r(0) = 0$, so that we are back to stabilizing at the origin. Assume no process noise, but a small random initial condition.

Implement your controller and model in simulink, and insert a delay in the control channel. How large can you make the delay and still be able to stabilize the linear system? The non-linear system?

*[hint: A controller with small bandwith is more robust to delays, so choose your R matrix accordingly.]*