CDS 270-2: Lecture 6-1
Towards a Packet-based Control Theory

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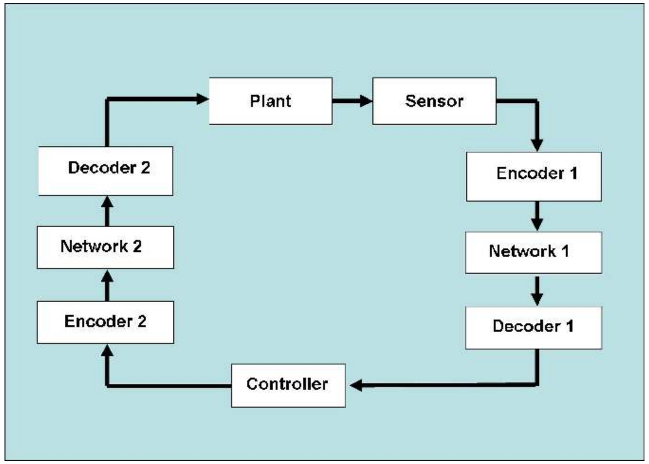
- **Goals:**
  - Describe main issues with a packet-based control system
  - Introduce common models for a packet-based control system
  - Learn basic techniques to solve the issues

- **Reading:**
  - Ling Shi, Michael Epstein and Richard Murray, "*Networked Control Systems with Norm Bounded Uncertainties: A Stability Analysis*", to appear in ACC 06
  - Ling Shi, Michael Epstein and Richard Murray, "*Towards Robust Control Over a Packet-based Network* ", to appear in MTNS 06

Issues and Models of NCS



Issues

- Packet Drops
  - network congestion
  - poor link quality (Wed's lectures)
- Finite Data Rate
  - Physical constraints
- Network Induced Delays
  - Token Ring network
  - Ethernet

Models

- Packet Drops
  - i.i.d drops
  - hidden Markov Chain
- Finite Data Rate
  - finite bits per time
  - coding & decoding
  - quantization
- Network Induced Delays
  - fixed delays
  - bounded delays
  - random delays

Some Background

- Lyapunov stability

$$|x_0| \le \delta \Rightarrow |x_k| \le \epsilon \ \forall\ k$$

- Almost sure stability:

$$\mathbf{Pr}[\lim_{k\to\infty} |x_k(x_0, \omega)| = 0] = 1$$

- For almost sure stability, $|x_k| > \epsilon$ can occur but only with small probability.

- Law of large numbers helps to rule out the "bad event"

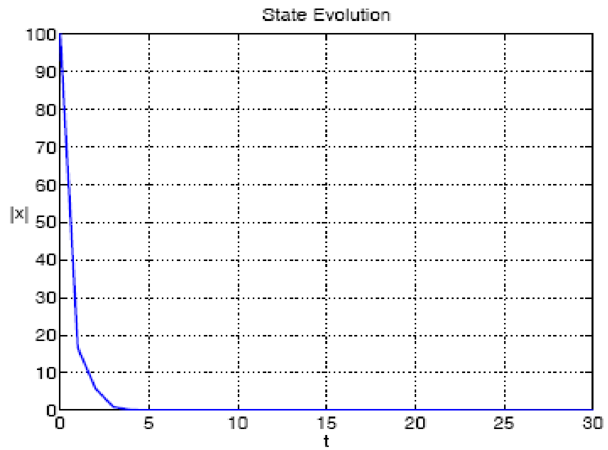


Fig. 6. System in Example 14 with $R_1 = R_2 = 5, \lambda = \gamma = 1$.

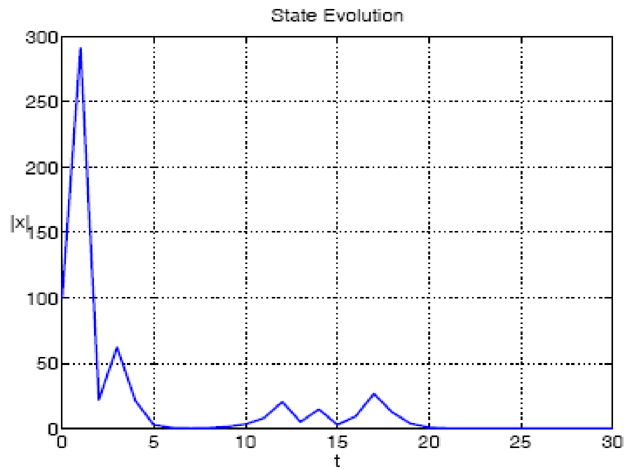
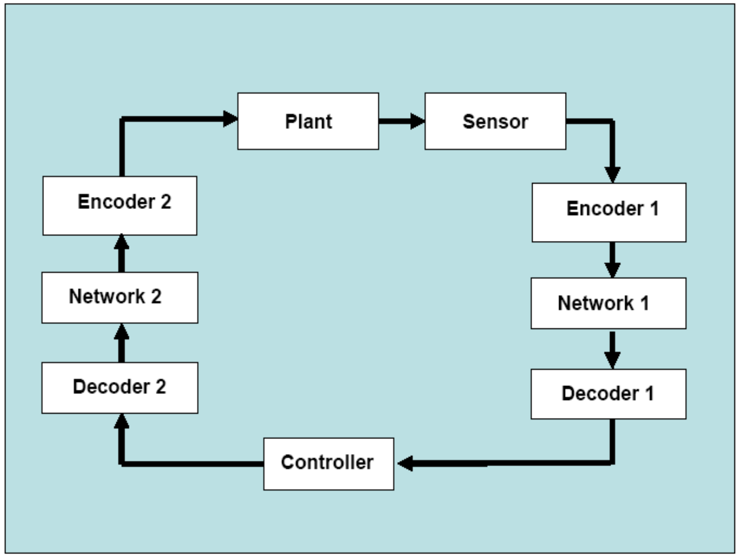


Fig. 8. System in Example 14 with $R_1 = R_2 = 5, \lambda = 0.85, \gamma = 0.8$.

When *B* and *C* are Invertible



NCS Type II

$$\begin{aligned} x_{k+1} &= (A + \Delta_k)x_k + \gamma_k B \bar{u}_k, \\ y_k &= \lambda_k C x_k, \\ u_k &= u_k(\bar{y}_k) \end{aligned}$$

Assumptions:

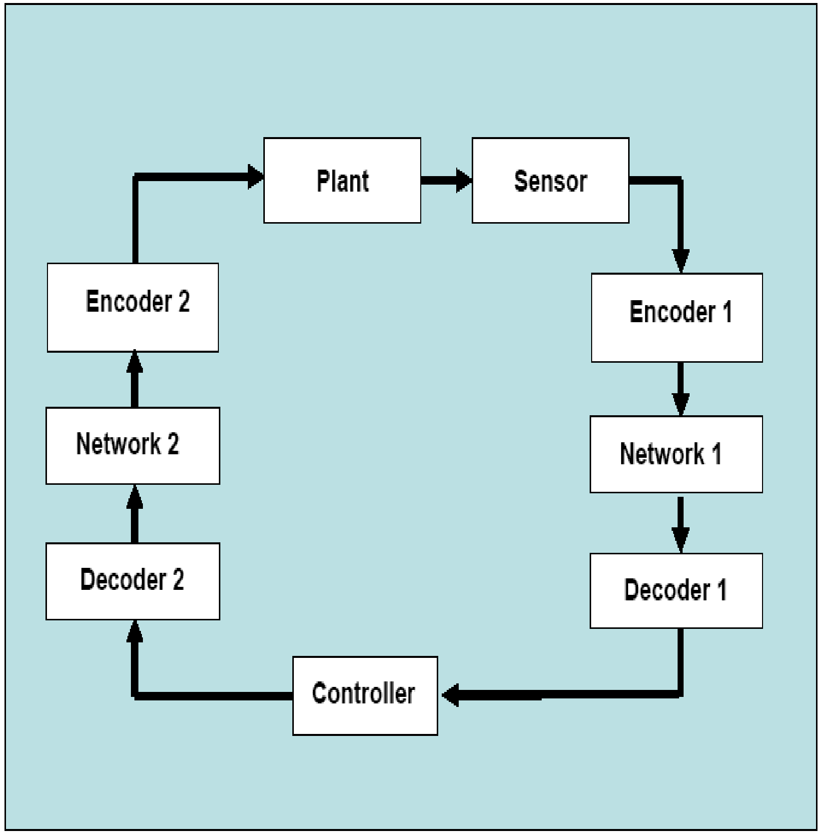
- γ_k, λ_k are i.i.d Bernoulli RVs and have mean γ and λ
- $\Delta_k^T \Delta_k \leq K^2 I$
- Network 1 has data rate R_1
- Network 2 has data rate R_2
- Everything is synchronized
- No packet reordering or delays
- *B* and *C* are invertible

Notations:

- $|A|$: Induced matrix norm
- log is of base 2
- Almost sure stability

What are the conditions on the network and system parameters such that closed loop stability is guaranteed?

Stability Analysis

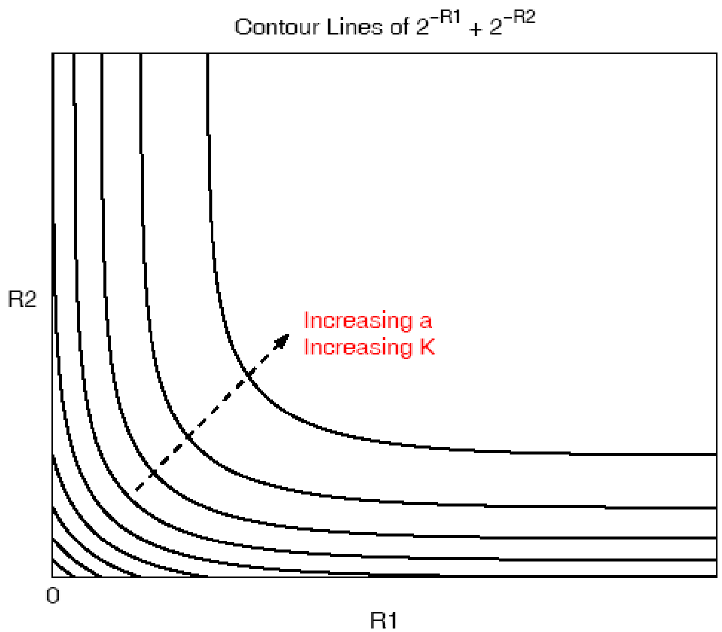


NCS Type II

- Lemma 1: λ = 1, γ = 1 and K = 0 imply closed loop stability if

$$a(2^{-R_1} + 2^{-R_2}) < 1$$

Proof:



Contour plot of $2^{-R1}+2^{-R2}$. The lines depict fixed values of a and K. Regions above the lines are stable for these fixed values.

Stability Analysis

- Lemma 2: $\lambda = 1$, $\gamma = 1$ and $K < 1$ imply closed loop stability if

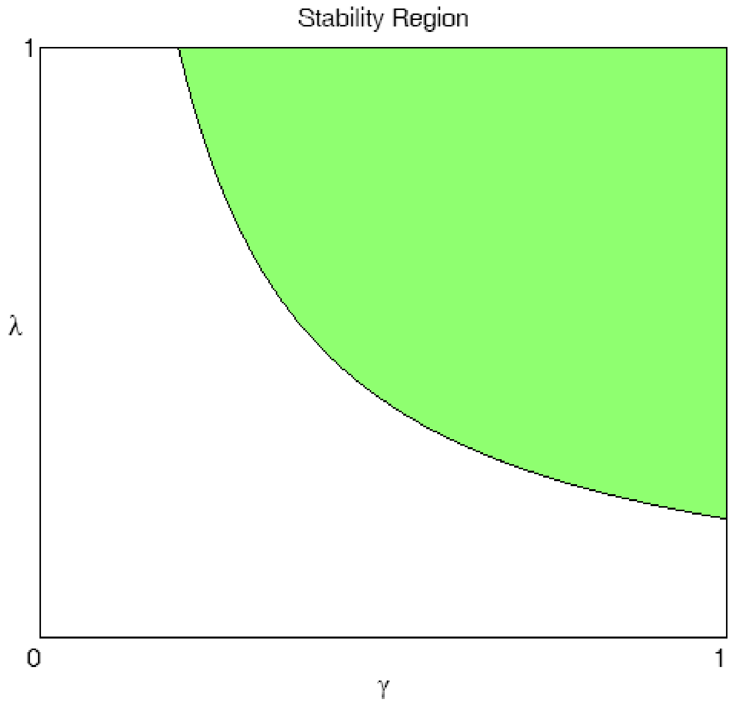
$$a(2^{-R_1} + 2^{-R_2}) < 1 - K$$

- Lemma 3: $R_1 = R_2 = \infty$, $K < 1$ imply closed loop stability if

$$\lambda\gamma > \frac{\log(a+K)}{\log(a+K) - \log K}$$

- Lemma 4: A sufficient condition for closed loop stability is that

$$(a+K)^{1-\lambda\gamma}(a2^{-R_1} + a2^{-R_2} + K)^{\lambda\gamma} < 1$$



Stability plot for NCS II with packet arrival rates γ and λ

Stability Analysis

• Theorem 2: Assume *B*, *C* are invertible and the system dimension is *n*. Then a sufficient condition for closed loop stability is that

$$(|A|+K)^{1-\lambda\gamma}(|A|2^{-\frac{R_1}{n}}+|B||B^{-1}A|2^{-\frac{R_2}{n}}+K)^{\lambda\gamma} < 1$$

• Remark: All previous lemmas and theorems can be derived from this master inequality including NCS I.

Simulation Examples

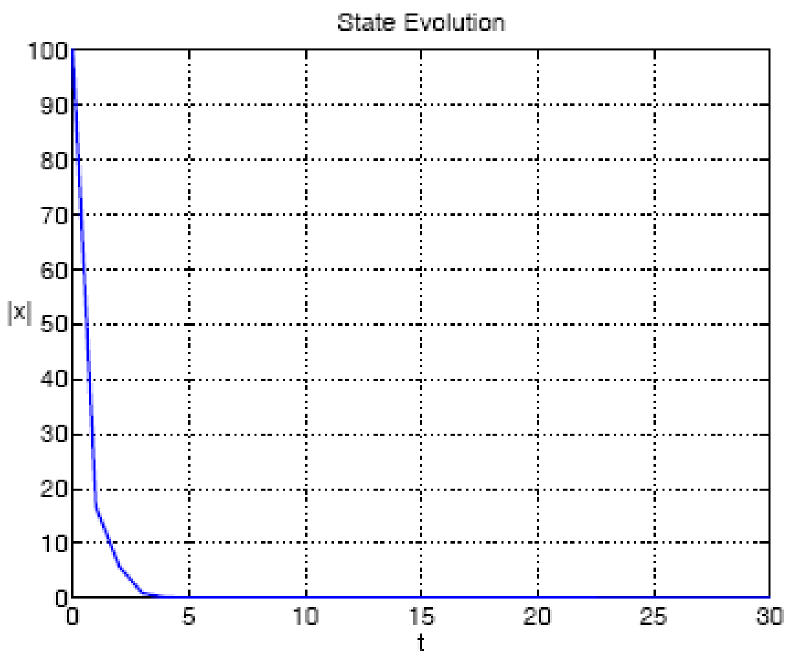


Fig. 6. System in Example 14 with $R_1 = R_2 = 5, \lambda = \gamma = 1$.

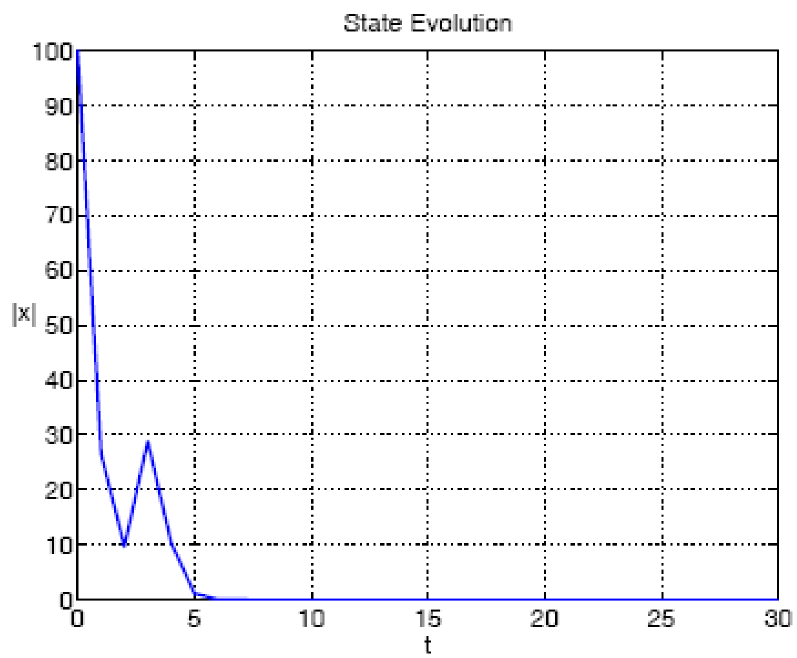


Fig. 7. System in Example 14 with $R_1 = R_2 = \infty, \lambda = 0.85, \gamma = 0.8$.

Simulation Examples

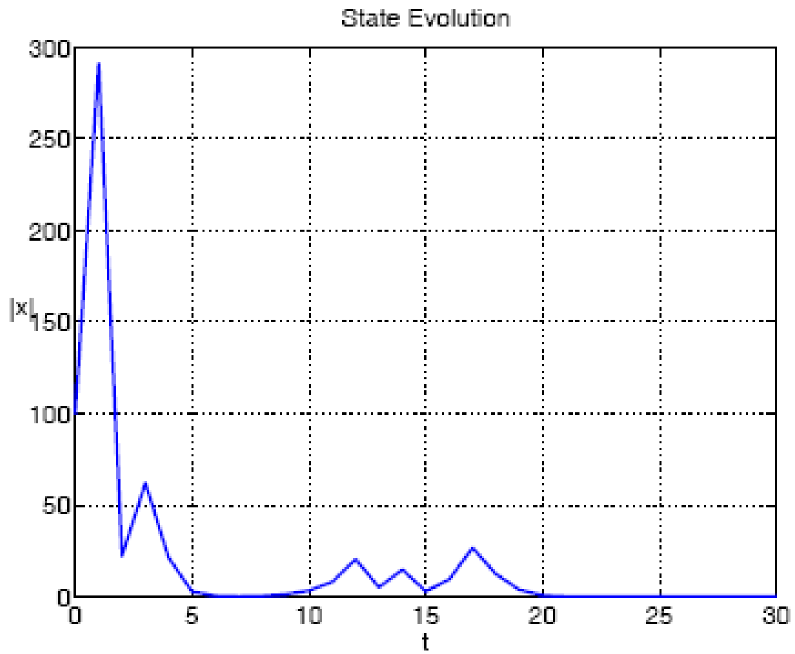


Fig. 8. System in Example 14 with $R_1 = R_2 = 5, \lambda = 0.85, \gamma = 0.8$

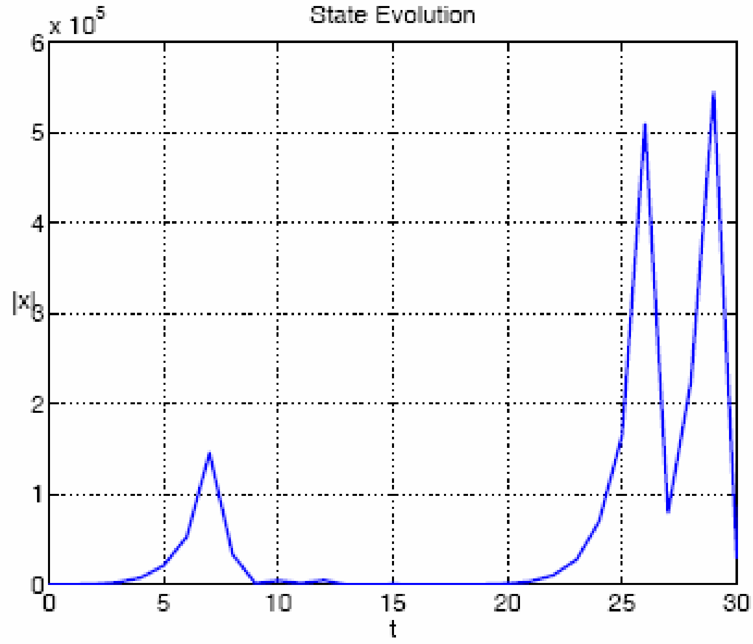
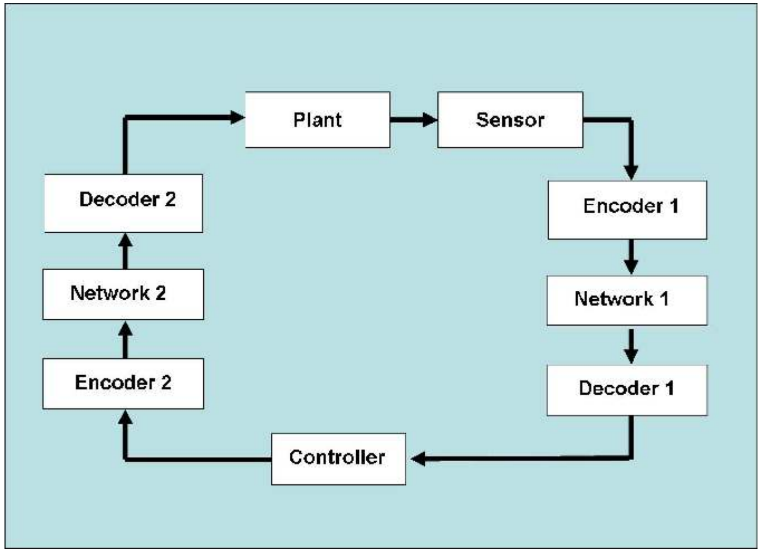


Fig. 9. System in Example 14 with $R_1 = 5, R_2 = 3, \lambda = 0.5, \gamma = 0.6$.

When B & C Are Not Invertible



$$x_{k+1} = (A + \Delta_k)x_k + \gamma_k B u_k,$$
$$y_k = \lambda_k C x_k$$

- (*A*,*B*) Controllable, (*C*,*A*) Observable
  - Output feedback
  - Design observer
- Assume infinite bandwidth & no delays
- $||\Delta_k|| \leq \beta$
- γ_k, λ_k are i.i.d Bernoulli RVs

Four options for this NCS

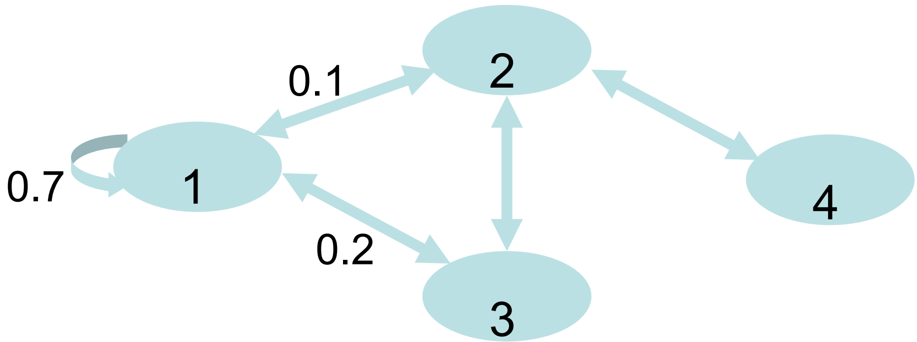
- Zero Control, Observer at the Sensor
- Zero Control, Observer at the Controller
- Predictive Control, Observer at the Sensor
- Predictive Control, Observer at the Controller

Some Background

- Consider the Jump Linear System (JLS)

$$x_{k+1} = A(\sigma_k)x_k \ , \ \sigma_k \in \{1, 2, \cdots, N\}$$

with steady state probability distribution $\mathbf{Pr}[\sigma_k = j] = \pi_j$



A Markov Chain with 4 States

$$P = [P_{ij}] = \begin{bmatrix} 0.7 & 0.1 & 0.2 & 0 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.3 & 0 \\ 0 & 0.1 & 0 & 0.9 \end{bmatrix}$$

$$\pi_j = \pi_j P$$

Some Background

- Lemma 1: If *A* is stable, then there exists a matrix norm $||\cdot||_H$ such that $||A||_H < 1$ where

$$||A||_H^2 = \sup_{x \neq 0} \frac{||Ax||_H^2}{||x||_H^2} = \sup_{x \neq 0} \frac{x^T A^T H A x}{x^T H x}$$

  Proof:

- Corollary 2: For any *A*, the following identity is true

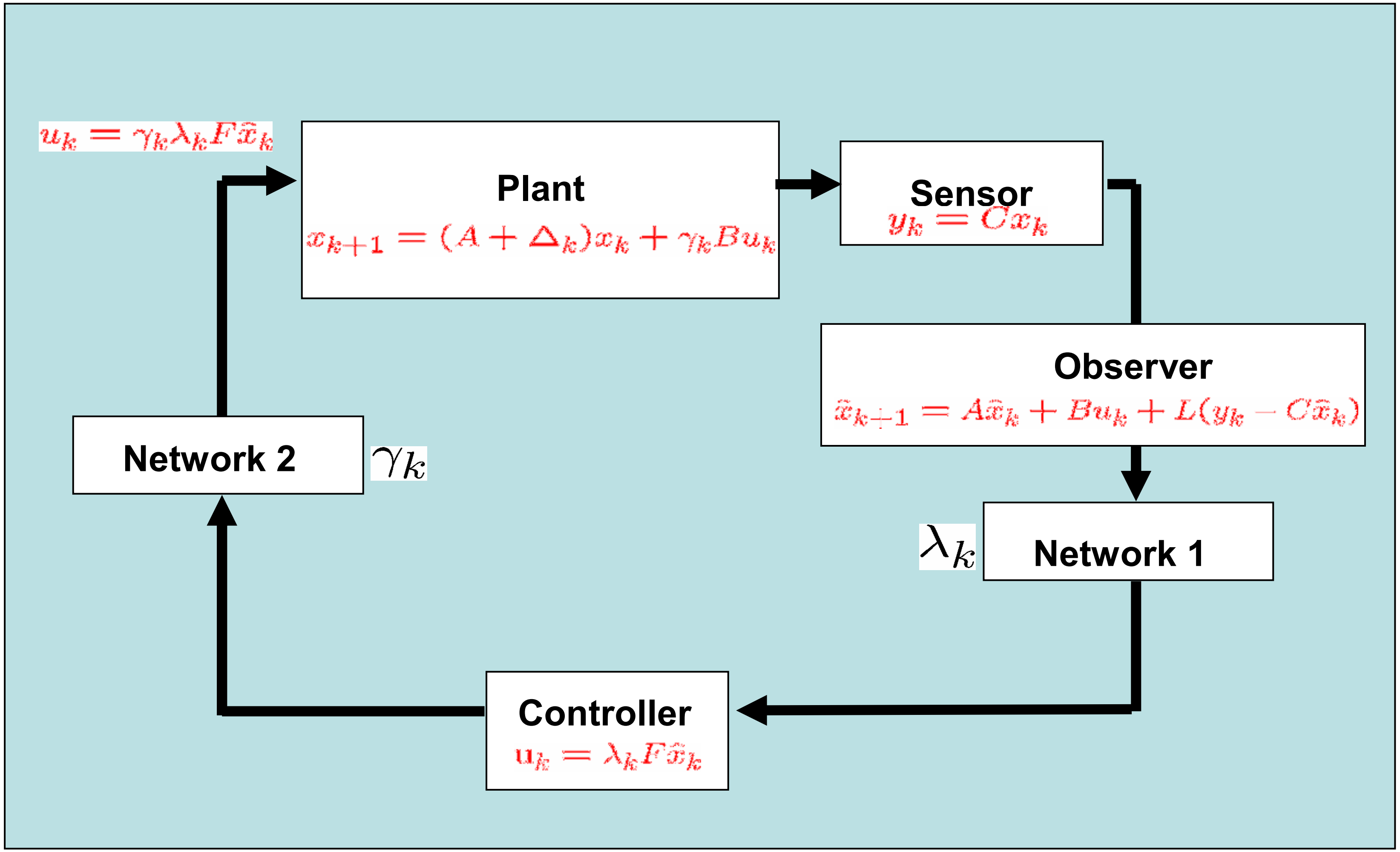
$$||A||_H = ||PAP^{-1}||, \; P = H^{1/2}.$$

- Lemma 2: The JLS above is almost sure stable if there exists a matrix norm such that

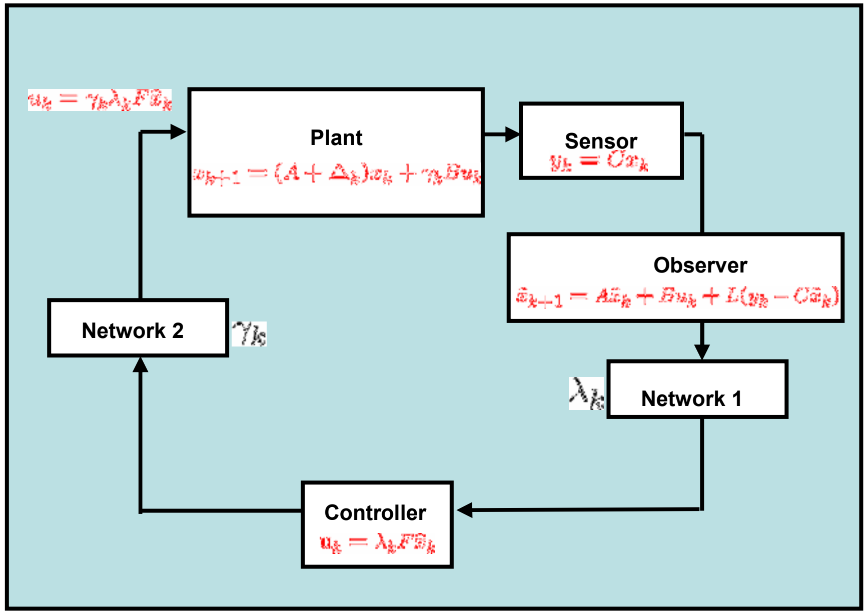
$$\prod_{i=1}^{N} ||A(i)||^{\pi_i} < 1$$

  Proof: (intuitive ideas)

Zero Control, Observer at Sensor



Zero Control, Observer at Sensor



$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = A(\sigma_k) \begin{bmatrix} x_k \\ e_k \end{bmatrix}$$

$$A(\sigma_k) = \begin{cases} M_1 + T_k, & \text{if } \lambda_k \gamma_k = 1 \\ M_2 + T_k, & \text{if } \lambda_k \gamma_k = 0 \end{cases}$$

with

$$M_1 = \begin{bmatrix} A + BF & -BF \\ 0 & A - LC \end{bmatrix}$$

$$M_2 = \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix}$$

$$T_k = \begin{bmatrix} \Delta_k & 0 \\ \Delta_k & 0 \end{bmatrix}.$$

Let $||\Delta_k|| \leq \beta < \beta_{max} = \frac{1 - ||M_1||_H}{\sqrt{2}||P|| \cdot ||P^{-1}||}$

then the NCS is almost sure stable if

$$N_1^{\lambda\gamma} N_2^{1-\lambda\gamma} < 1$$

where

$$N_i = ||M_i||_H + \sqrt{2}\beta ||P|| \cdot ||P^{-1}||.$$

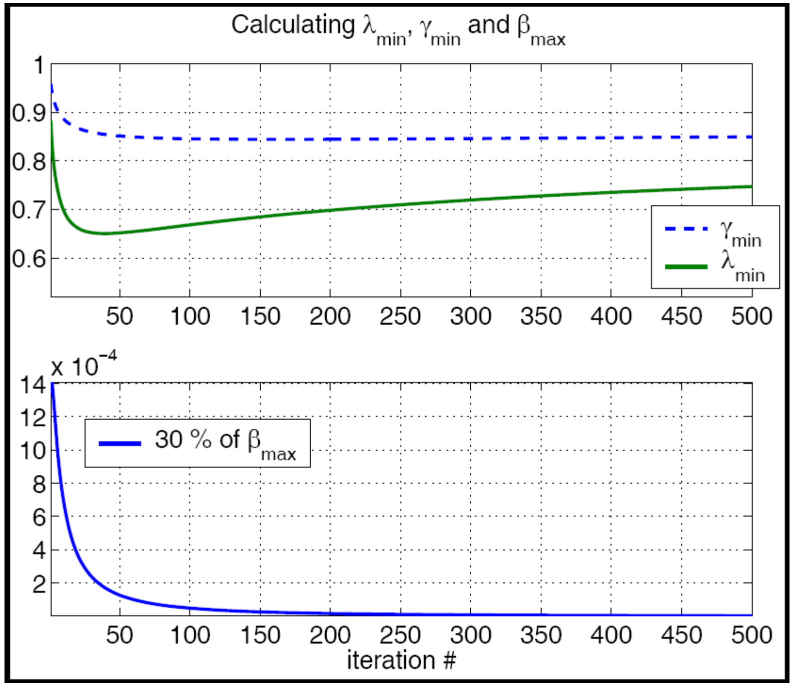
Proof:

Design Tradeoffs

- Objective: maximize β, minimize λ and γ according to

$$\beta_{max} = \frac{1-||M_1||_H}{\sqrt{2}||P||\cdot||P^{-1}||}$$

$$N_1^{\lambda\gamma} N_2^{1-\lambda\gamma} < 1$$



Algorithm II

Given (*A, B, C, F, L*)

1. **Form the matrices M_j as in Section III, where $j \in \{1,2\}$ or $\{1,2,3,4\}$ depending on different control schemes.**
2. **Set i = 1 and $Q_i = I$**
3. **Solve $A^T H_i A - H_i = - Q_i$ via standard lyapunov equation solvers to get H_i. Set $Q_i = H_i$.**
4. **Decompose H_i into $H_i = P_i P_i$ via standard algorithms.**
5. **Find maximum sufficient uncertainty according to (12) or (36) depending on different control schemes.**
6. **Find minimum sufficient packet arrival rates λ and γ for some portion of the uncertainty found in step 5 according to Theorems 5,7,8, or 9 respectively.**
7. ***i = i+1.***
8. **Repeat steps 3 to 7 until the incremental increase or decrease of these values are within a certain threshold.**

Project Ideas

1. Find necessary conditions for closed loop stability and compare the difference with the sufficient conditions obtained in the previous theorems.

2. Consider the case where actuating and sensing uncertainties are also present, ie, there are some uncertainties in *B* and *C.*

3. We have mainly considered the multiplicative uncertainty model here. Extend the ideas to other uncertainly models such as additive uncertainty model and so on.

4. Apply the theoretical results to Roboflag or MVWT testbed for design guidelines.