CDS 110a Recitation 10/26/07 G. Hines

$\dot{x} = f(x)$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

consider an equilibrium point $x = x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$

change coordinates to $z = x - x^* \Rightarrow z^* = 0, \quad \dot{z} = f(z) = f(z_1, z_2) = \begin{bmatrix} f_1(z_1, z_2) \\ f_2(z_1, z_2) \end{bmatrix}$

key question: how do small disturbances propagate?
from equilibrium

use Taylor series about $z = 0$, focus on each z_i individually:

$$\cancel{\dot{z}_1} + \Delta \dot{z}_1 = \Delta \dot{z}_1 = \underbrace{f_1(0,0)}_{0} + \frac{\partial f_1}{\partial z_1} \Delta z_1 + \frac{\partial f_1}{\partial z_2} \Delta z_2 + O(z_1^2, z_2^2)$$

$\dot{z} = 0$ at eq

$$\Delta \dot{z}_1 = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \end{bmatrix} \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \end{bmatrix} + O(z_1^2, z_2^2)$$

similarly, $\Delta \dot{z}_2 = \begin{bmatrix} \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \end{bmatrix} + O(z_1^2, z_2^2)$

stack them...

$$\begin{bmatrix} \Delta \dot{z}_1 \\ \Delta \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix}}_{\text{Jacobian } (A)} \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \end{bmatrix} + \text{HOT}$$

how disturbance propagates

small disturbance about 0

$\Delta z \to z$

linearization about equilibrium given by

$$\dot{x} = A\big|_{x^*} x$$

now... given a nonlinear system

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

can define all the matrices necessary to turn this into a linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

by defining them analogously to the Jacobian

take 2D, single input

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} \qquad D = \begin{bmatrix} \frac{\partial h_1}{\partial u} \\ \frac{\partial h_2}{\partial u} \end{bmatrix}$$

for multiple controls, B & D can have more columns (~~one~~ column for each control)

behavior of the linearization *may* tell you useful info about nonlinear system:

A stable ⇒ full system is *locally* stable

A unstable ⇒ full system unstable

A has mixed stability ⇒ can't say
(e.g. saddle point)

Discrete Time Example

Predator / Prey

$$H_{k+1} = H_k + bH_k - aL_kH_k = f_1$$

$$L_{k+1} = L_k - dL_k + cL_kH_k = f_2$$

Jacobian: $\dfrac{\partial f_1}{\partial H} = 1 + b - aL_k$

$$\frac{\partial f_1}{\partial L} = -aH_k$$

$$\frac{\partial f_2}{\partial H} = cL_k$$

$$\frac{\partial f_2}{\partial L} = 1 - d + cH_k$$

so $A = \left[\begin{array}{c|c} 1+b-aL_k & -aH_k \\ \hline cL_k & 1-d+cH_k \end{array}\right]$

origin eq is trivial

nonnegative
Lynx, hare

Equilibrium Point: assumptions: $H, L > 0$

$$H_{k+1} = H_k \Rightarrow bH^* - aL^*H^* = 0$$

$$\Rightarrow L^* = \frac{b}{a}$$

$$L_{k+1} = L_k \Rightarrow H^* = \frac{d}{c}$$

so ~~the~~ eq. point, linearization is

$$A\Big|_{H^*, L^*} = \begin{bmatrix} 1 & -\frac{ad}{c} \\ \frac{bc}{a} & 1 \end{bmatrix}$$

question: is this stable?

- for a discrete-time system to be stable, we need

$$x_{k+1} < x_k \Rightarrow \text{eigenvalues of } A \underline{\underline{< 1}}$$

(as opposed to continous case, where we needed $\dot{x} < 0 \Rightarrow$ eigenvalues of $A < 0$)

numerical values: $a = c = 0.014$
$b = 0.6$
$d = 0.7$

$$A\Big|_{H^*, L^*} = \begin{bmatrix} 1 & -0.7 \\ 0.6 & 1 \end{bmatrix}$$

$(1-\lambda)^2 + 0.42$

$= 1 - 2\lambda + \lambda^2 + 0.42$

$\lambda^2 - 2\lambda + 1.42 = 1 \pm \sqrt{-0.42}$

unstable

two methods: - diagonalization
- express x_0 in the basis of eigenvectors

Harder example

HW #4 (Å, M ~~Sect~~ 5:10)

Linear discrete-time system:

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

a) ~~Find~~ general form of output y_k (show it's equal to...)

to find y_k, need x_k

to find x_k, keep ~~start~~ writing ~~and the~~ until you can see a recursion in terms of IC x_0,

$$x_1 = Ax_0 + Bu_0$$
$$x_2 = Ax_1 + Bu_1 = A(Ax_0 + Bu_0) + Bu_1$$
$$x_3 = Ax_2 + Bu_2 = A(A(Ax_0 + Bu_0) + Bu_1) + Bu_2$$
$$x_4 = Ax_3 + Bu_3 = \underbrace{A(A(A(Ax_0}_{} + Bu_0) + Bu_1) + Bu_2) + Bu_3$$

pop out the IC response: $A^k x_0$

write out the control response to make it clearer

$$x_4: \quad A^3 Bu_0 + A^2 Bu_1 + ABu_2 + Bu_3$$

$k \qquad k-1$

highest power is k-1

so write $\sum_{i=0}^{k-1} A^{k-1-i} Bu_i$

final form: $x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu_i$

→

but we want output $y_k = Cx_k + Du_k$

$$y_k = C\left[A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i\right] + Du_k$$

$$= CA^k x_0 + \left[\sum_{i=0}^{k-1} CA^{k-i-1} B u_i\right] + Du_k$$

b) show asymptotic stability for eigenvalues of $A < 1$

NOTE: use unforced response WLOG (eg. shift)

<u>method 1)</u>

diagonalization

$$A = TA^d T^{-1}, \quad A^d = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$$

assumption of full basis of eigenvectors allows this diagonalization

$$T = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

then $x_{k+1} = (TA^d T^{-1})^k x_0$

$$= T(A^d)^k T^{-1} x_0$$

$$= T \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \lambda_n^k \end{bmatrix} T^{-1} x_0$$

if $\lambda_i < 0$, then $\lim_{k \to \infty} x_{k+1} = 0$ //

method 2

express initial condition as a linear combination of eigenvectors.
(can do this because the basis of eigenvectors is full $\Rightarrow$ span the state space)

$$x_0 = \sum \alpha_i v_i$$

$$\text{so} \quad x_{k+1} = A^k x_0$$

$$= A^k \sum \alpha_i v_i$$

$$= \sum \alpha_i A^k v_i$$

but recall the definition of eigenvector/eigenvalue pair:

$$A v_i = \lambda_i v_i$$

$$x_{k+1} = \sum \alpha_i \lambda_i^k v_i$$

for $\lambda_i < 0$, $\lim_{k \to \infty} x_{k+1} = 0$

c) $u_k = \sin \omega k \implies$ WLOG $u_k = e^{j\omega k}$

Linear systems preserve frequency, but may shift phase in steady state

let $x_k = N e^{j\alpha} e^{j\omega k}$
$y_k = M e^{j\theta} e^{j\omega k}$

then from the state eqn:

$$x_{k+1} = N e^{j\alpha} e^{j\omega(k+1)} = A N e^{j\alpha} e^{j\omega k} + B e^{j\omega k}$$

$$N e^{j\alpha} e^{j\omega} = A N e^{j\alpha} + B$$

$$(e^{j\omega} I - A) N e^{j\alpha} = B$$

$$N e^{j\alpha} = (e^{j\omega} I - A)^{-1} B$$

$$y_k = M e^{j\theta} \cancel{e^{j\omega k}} = C N e^{j\alpha} \cancel{e^{j\omega k}} + D \cancel{e^{j\omega k}}$$

$$= C (e^{j\omega} I - A)^{-1} B + D$$