CDS 202 Winter 2009 Solution Set 2

Problem 1 (MTA 2.3-1) (PS 2009)

Note: we are taking the vector space structure on $\mathbf{E} \times \mathbf{F}$ to be $\mathbf{E} \oplus \mathbf{F}$, so for clarity I'll use this notation explicitly.

Using the bilinearity of B we have that

$$B(u+e, v+f) = B(u,v) + B(u,f) + B(e,v) + B(e,f)$$

If $S : \mathbf{E} \oplus \mathbf{F} \to \mathbf{G}$ is the function defined by

$$S((e,f)) = B(u,f) + B(e,v)$$

then S is linear in $\mathbf{E} \oplus \mathbf{F}$ (i.e. $S(\alpha_1(e_1,f_1) + \alpha_2(e_2,f_2)) = \alpha_1 S((e_1,f_1)) + \alpha_2 S((e_2,f_2))$: note, it is not required to be linear in $\mathbf{E}$ and $\mathbf{F}$ separately). Also

$$\begin{aligned} ||S((e,f))||_{\mathbf{G}} &\le ||B(u,f)||_{\mathbf{G}} + ||B(e,v)||_{\mathbf{G}} \\ &\le ||B||_{L(\mathbf{E},\mathbf{F};\mathbf{G})} ||u||_{\mathbf{E}} ||f||_{\mathbf{F}} + ||B||_{L(\mathbf{E},\mathbf{F};\mathbf{G})} ||e||_{\mathbf{E}} ||v||_{\mathbf{F}} \\ &\le ||B||_{L(\mathbf{E},\mathbf{F};\mathbf{G})} \max\{||u||_{\mathbf{E}}, ||v||_{\mathbf{F}}\}(||e||_{\mathbf{E}} + ||f||_{\mathbf{F}}) \\ &= ||B||_{L(\mathbf{E},\mathbf{F};\mathbf{G})} \max\{||u||_{\mathbf{E}}, ||v||_{\mathbf{F}}\} ||(e,f)||_{\mathbf{E}\oplus\mathbf{F}} \end{aligned}$$

so S is bounded. Finally

$$\frac{||B(e,f)||_{\mathbf{G}}}{||(e,f)||_{\mathbf{E}\oplus\mathbf{F}}} \le \frac{||B||_{L(\mathbf{E},\mathbf{F};\mathbf{G})} ||e||_{\mathbf{E}} ||f||_{\mathbf{F}}}{||e||_{\mathbf{E}} + ||f||_{\mathbf{F}}} \to 0, \qquad (e,f) \to (0,0)$$

so $B(e,f)$ is $o((e,f))$. This implies that S is the derivative of B at (u,v)

$$\mathbf{D}B(u,v) \cdot (e,f) = B(u,f) + B(e,v).$$

Now

$$\begin{aligned} \mathbf{D}B(u+h, v+k) &= B(u+h, \cdot) + B(\cdot, v+k) \\ &= B(u,\cdot) + B(\cdot, v) + B(h,\cdot) + B(\cdot,k) \\ &= \mathbf{D}B(u,v) + B(h,\cdot) + B(\cdot,k). \end{aligned}$$

Let $T : \mathbf{E} \oplus \mathbf{F} \to L(\mathbf{E} \oplus \mathbf{F}, \mathbf{G})$ be defined by

$$T((h,k)) = B(h,\cdot) + B(\cdot,k)$$

T is linear in $\mathbf{E} \oplus \mathbf{F}$ and

$$\begin{aligned} ||T((h,k))||_{L(\mathbf{E}\oplus\mathbf{F},\mathbf{G})} &\le ||B(h,\cdot)||_{L(\mathbf{E}\oplus\mathbf{F},\mathbf{G})} + ||B(\cdot,k)||_{L(\mathbf{E}\oplus\mathbf{F},\mathbf{G})} \\ &\le ||B||_{L(\mathbf{E},\mathbf{F};\mathbf{G})} ||h||_{\mathbf{E}} + ||B||_{L(\mathbf{E},\mathbf{F};\mathbf{G})} ||k||_{\mathbf{F}} \\ &= ||B||_{L(\mathbf{E},\mathbf{F};\mathbf{G})} ||(h,k)||_{\mathbf{E}\oplus\mathbf{F}} \end{aligned}$$

implies that T is bounded. Also, there is no $o((h,k))$ term. So T is the derivative of $\mathbf{D}B$ at (u,v)

$$\mathbf{D}^2 B(u,v)\cdot(h,k)\cdot(e,f) = B(h,f)+B(e,k).$$

Since $\mathbf{D}^2 B$ is constant as a function of (u,v), all of its derivatives vanish, i.e.

$$\mathbf{D}^k B(u,v) = 0, \qquad \forall k \geq 3.$$

Problem 2 (MTA 2.3-4)

We will first show that $A\circ \mathbf{D}f(u)$ is the derivative of $(A\circ f)$ at u. By the linearity of A, we get

$$\begin{aligned}\lim_{v\to 0}\frac{\|A\circ f(u+v)-A\circ f(u)-(A\circ \mathbf{D}f(u))\cdot v\|}{\|v\|} &= \lim_{v\to 0}\frac{\|A\circ(f(u+v)-f(u)-\mathbf{D}f(u)\cdot v)\|}{\|v\|}\\ &\leq \lim_{v\to 0}\|A\|\,\frac{\|f(u+v)-f(u)-\mathbf{D}f(u)\cdot v\|}{\|v\|}\\ &= 0.\end{aligned}$$

Furthermore, $(A\circ\mathbf{D}f(u))(v)$ is linear in v, since

$$\begin{aligned}(A\circ\mathbf{D}f(u))(\alpha_1 v_1+\alpha_2 v_2) &= A(\mathbf{D}f(u)(\alpha_1 v_1+\alpha_2 v_2))\\ &= A(\alpha_1\mathbf{D}f(u)(v_1)+\alpha_2\mathbf{D}f(u)(v_2))\\ &= \alpha_1 A(\mathbf{D}f(u)(v_1))+\alpha_2 A(\mathbf{D}f(u)(v_2))\\ &= \alpha_1(A\circ\mathbf{D}f(u))(v_1)+\alpha_2(A\circ\mathbf{D}f(u))(v_2)\end{aligned}$$

and bounded since both A and $\mathbf{D}f(u)$ are. Therefore we conclude that $(A\circ f)$ is differentiable, and the derivative is $\mathbf{D}(A\circ f)(u) = A\circ\mathbf{D}f(u)$. If $\mathbf{D}f(u)$ is continuous (i.e., if f is C^1), then so is $\mathbf{D}(A\circ f)(u)$.

Now suppose that f is C^r. We prove that $(A\circ f)$ is also C^r and $\mathbf{D}^r(A\circ f)(u) = A\circ\mathbf{D}^r f(u)$ by an induction, i.e., by showing for $k=1,2,\cdots,r$, that if

$$f \text{ is } C^k \text{ and } \mathbf{D}^{k-1}(A\circ f)(u) = A\circ\mathbf{D}^{k-1}f(u)$$

then $(A\circ f)$ is also C^k and $\mathbf{D}^k(A\circ f)(u) = A\circ\mathbf{D}^k f(u)$. The case for $k=1$ has been prove above. Suppose that the premises are fulfilled, then

$$\begin{aligned}&\lim_{v\to 0}\frac{\left\|\mathbf{D}^{k-1}(A\circ f)(u+v)-\mathbf{D}^{k-1}(A\circ f)(u)-(A\circ\mathbf{D}^k f(u))\cdot v\right\|}{\|v\|}\\ &= \lim_{v\to 0}\frac{\left\|A\circ(\mathbf{D}^{k-1}f(u+v)-\mathbf{D}^{k-1}f(u)-\mathbf{D}^k f(u)\cdot v)\right\|}{\|v\|}\\ &\leq \lim_{v\to 0}\|A\|\,\frac{\left\|\mathbf{D}^{k-1}f(u+v)-\mathbf{D}^{k-1}f(u)-\mathbf{D}^k f(u)\cdot v\right\|}{\|v\|}\\ &= 0.\end{aligned}$$

The linearity and boundedness of $A \circ \mathbf{D}^k f(u)$ follows by the same argument as for $k = 1$. This shows that $(A \circ f)$ is C^k and $\mathbf{D}^k(A \circ f)(u) = A \circ \mathbf{D}^k f(u)$. Since this holds for $k = r$, we conclude that $(A \circ f)$ is also C^r.

Problem 3 (MTA 3.1-4 (i))

(i) First consider the Mobius band $\mathbb{M}$ which is obtained from $[0, 2\pi[\times\mathbb{R}$ by identifying $(0, x)$ with $(2\pi, -x)$. Use two charts:

$$\begin{aligned}(U_1, \varphi_1) &= (]0, 2\pi[\times\mathbb{R},\ \text{identity})\\ (U_2, \varphi_2) &= (([0, \pi[\cup]\pi, 2\pi[) \times \mathbb{R},\ \varphi_2 \text{ is as defined above}).\end{aligned}$$

Each φ_i is a bijection from U_i to an open subset of $\mathbb{R}^2$. We see that $U_1 \cup U_2 = [0, 2\pi[\times\mathbb{R} = \mathbb{M}$. Furthermore,

$$\begin{aligned}\varphi_1(U_1 \cap U_2) &= (]0, \pi[\cup]\pi, 2\pi[) \times \mathbb{R}\\ \varphi_2(U_1 \cap U_2) &= (] - \pi, 0[\cup]0, \pi[) \times \mathbb{R}\end{aligned}$$

are open in $\mathbb{R}^2$, and the overlap map

$$(\varphi_2 \circ \varphi_1^{-1} \big| \varphi_1(U_1 \cap U_2))(\theta, x) = \begin{cases} (\theta, x), & \text{if } 0 < \theta < \pi \\ (\theta - 2\pi, -x), & \text{if } \pi < \theta < 2\pi \end{cases}$$

is a diffeomorphism, where the inverse is given by

$$(\varphi_1 \circ \varphi_2^{-1} \big| \varphi_2(U_1 \cap U_2))(\theta, x) = \begin{cases} (\theta, x), & \text{if } 0 < \theta < \pi \\ (\theta + 2\pi, -x), & \text{if -}\pi < \theta < 0 \end{cases}$$

All of these imply that (U_1, φ_1) and (U_2, φ_2) define an atlas on $\mathbb{M}$. Therefore $\mathbb{M}$ is a manifold.

Problem 4 (MTA 3.1-5) (PS 2009)

(i) Clearly $\mathbb{R}^n_c = U \cup U_\infty$, and ϕ and ϕ_∞ are easily seen to be bijective. Also

$$U \cap U_\infty = \mathbb{R}\backslash\{0\} \Rightarrow \phi(U \cap U_\infty) = \phi_\infty(U \cap U_\infty) = \mathbb{R}\backslash\{0\}.$$

On $\mathbb{R}\backslash\{0\}$

$$(\phi_\infty \circ \phi^{-1})(x) = \frac{x}{||x||^2}, \quad (\phi \circ \phi_\infty^{-1})(y) = \frac{y}{||y||^2}$$

which are C^∞ since the denominator is never zero. So $\mathcal{A}_c$ defines a smooth manifold structure.

(ii) Let $\{O_\alpha\}_{\alpha\in\mathcal{I}}$ be an open cover of $\mathbb{R}^n_c$. At least one O_α contains the point ∞. Every open subset of $\mathbb{R}^n_c$ is the union of inverses of open sets in $\mathbb{R}^n$ via ϕ, ϕ_∞. Since $\infty \in O_\alpha$, there exists an $\epsilon > 0$ such that

$$\begin{aligned} &\phi_\infty^{-1}(D_\epsilon(0)) \subset O_\alpha \\ \Rightarrow &\mathbb{R}^n_c\backslash O_\alpha \subset \mathbb{R}^n_c\backslash\phi_\infty^{-1}(D_\epsilon(0)) = \phi_\infty^{-1}(\mathbb{R}^n\backslash D_\epsilon(0)) \cup \{0\}. \end{aligned}$$

It's straightforward to check that $\phi_\infty^{-1}(\mathbb{R}^n\backslash D_\epsilon(0)) = D_{\frac{1}{\epsilon}}(0)\backslash\{0\}$, so

$$\mathbb{R}^n_c\backslash O_\alpha \subset D_{\frac{1}{\epsilon}}(0).$$

So $\mathbb{R}^n_c\backslash O_\alpha$ is bounded. It is also closed in $\mathbb{R}^n_c$ and doesn't contain ∞, so it is closed in $\mathbb{R}^n$. By the Heine-Borel theorem, it is compact. $\{O_\beta\}_{\beta\in\mathcal{I}\backslash\{\alpha\}}$, and hence $\{O_\beta \cap \mathbb{R}^n\}_{\beta\in\mathcal{I}\backslash\{\alpha\}}$, is an open cover of $\mathbb{R}^n_c\backslash O_\alpha$, and so by compactness must contain a finite subcover $\{O_{\beta_1}\cap\mathbb{R}^n, O_{\beta_2}\cap\mathbb{R}^n, \ldots, O_{\beta_s}\cap\mathbb{R}^n\}$. Then $\{O_\alpha, O_{\beta_1}, O_{\beta_2}, \ldots, O_{\beta_s}\}$ is a finite subcover of $\mathbb{R}^n_c$. Hence $\mathbb{R}^n_c$ is compact.

(iii) We just need to check that ψ_∞ is compatible with the existing charts ϕ, ϕ_∞. This follows because on $\mathbb{R}^n\backslash\{0\}$

$$\begin{aligned} (\phi \circ \psi_\infty^{-1})(z) &= \phi(z^{-1}) = z^{-1} = \frac{\bar{z}}{||z||^2} \\ (\psi_\infty \circ \phi^{-1})(z) &= \psi_\infty(z) = z^{-1} = \frac{\bar{z}}{||z||^2} \end{aligned}$$

while on $\mathbb{R}^n$

$$\begin{aligned} (\phi_\infty \circ \psi_\infty^{-1})(z) &= \begin{cases} \phi_\infty(z^{-1}) = \frac{z^{-1}}{||z^{-1}||^2} = \bar{z} & z \neq 0 \\ \phi_\infty(\infty) = 0 & z = 0 \end{cases} \\ (\psi_\infty \circ \phi_\infty^{-1})(z) &= \begin{cases} \psi_\infty\left(\frac{z}{||z||^2}\right) = \left(\frac{z}{||z||^2}\right)^{-1} = \bar{z} & z \neq 0 \\ \psi_\infty(\infty) = 0 & z = 0 \end{cases} \end{aligned}$$

which are all smooth functions.

(iv) Using the stereographic charts ϕ_1, ϕ_2 defined on P.129, we construct two maps

$$\begin{aligned} \phi^{-1} \circ \phi_1 &: S^n\backslash\{N\} \to \mathbb{R}^n \\ \phi_\infty^{-1} \circ \phi_2 &: S^n\backslash\{S\} \to \mathbb{R}^n_c\backslash\{\infty\}. \end{aligned}$$

They agree on their common domain of definition $S^n\backslash\{N,S\}$ since

$$\begin{aligned}
(\phi^{-1}\circ\phi_1)(x^1,\ldots,x^{n+1}) &= \phi^{-1}\left(\frac{x^2}{1-x^1},\ldots,\frac{x^{n+1}}{1-x^1}\right)\\
&= \left(\frac{x^2}{1-x^1},\ldots,\frac{x^{n+1}}{1-x^1}\right)\\
(\phi_\infty^{-1}\circ\phi_2)(x^1,\ldots,x^{n+1}) &= \phi_\infty^{-1}\left(\frac{x^2}{1+x^1},\ldots,\frac{x^{n+1}}{1+x^1}\right)\\
&= \frac{(1+x^1)^2}{1-(x^1)^2}\left(\frac{x^2}{1+x^1},\ldots,\frac{x^{n+1}}{1+x^1}\right)\\
&= \left(\frac{x^2}{1-x^1},\ldots,\frac{x^{n+1}}{1-x^1}\right)
\end{aligned}$$

so together the define a bijection $F : S^n \to \mathbb{R}^n_c$. Since the open sets on S^n and $\mathbb{R}^n_c$ are derived from their respective charts, it is clear that F is a homeomorphism.

Problem 5 (PS 2009)

(i) To show that $g\circ f : M \to P$ is smooth at x, let $\phi : U \to \mathbf{E}$ be a chart on M containing x, $\psi : V \to \mathbf{F}$ a chart on N containing $f(x)$, $\chi : W \to \mathbf{G}$ a chart on P containing $g(f(x))$. Then

$$\chi\circ(g\circ f)\circ\phi^{-1} = (\chi\circ g\circ\psi^{-1})\circ(\psi\circ f\circ\phi^{-1})$$

is smooth when defined on the appropriate domain (namely $\phi((g\circ f)^{-1}(W)\cap g^{-1}(V)\cap U)\subset\mathbf{E}$), since $\chi\circ g\circ\psi^{-1}$ and $\psi\circ f\circ\phi^{-1}$ are smooth on their respective domains, and composition of smooth functions between open subsets of Euclidean spaces are smooth. By definition, this implies that $g\circ f$ is smooth.

(ii) f and g are bijections, therefore so is $g\circ f$. f and g are both smooth, so by part (i) $g\circ f$ is smooth. f^{-1} and g^{-1} are both smooth, so $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$ is smooth, again by part (i). Hence $g\circ f$ is a diffeomorphism.

Problem 6 (MTA 3.3-1)

See HW 4