CDS 202 Winter 2009 Solution Set 9

Problem 1 (Nawaf Bou-Rabee)

Solution for (i),(ii) The derivative of a two-form ω in $\mathbb{R}^3$ is,

$$d\omega = (\text{div}\vec{F})dx \wedge dy \wedge dz$$

where $\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy$ and $\vec{F} = (f_1, f_2, f_3)$. Therefore, for $\omega = z^2 dx \wedge dy - (z^2 + 2y)dz \wedge dx$,

$$d\omega = (2z - 2)dx \wedge dy \wedge dz$$

since $\text{div}(\vec{F} = (0, -z^2 - 2y, z^2)) = 2z - 2$.

Solution for (iii) The derivative of the one-form $\omega = f dg$ is,

$$\begin{aligned} d(f dg) &= d\left(f(g_x dx + g_y dy + g_z dz)\right) \\ &= d(f g_x) \wedge dx + d(f g_y) \wedge dy + d(f g_z) \wedge dz \\ &= g_x df \wedge dx + g_y df \wedge dy + g_z df \wedge dz \qquad \text{since } d(dg) = 0 \\ &= df \wedge g_x dx + df \wedge g_y dy + df \wedge g_z dz \\ &= df \wedge (g_x dx + g_y dy + g_z dz) \\ &= df \wedge dg \end{aligned}$$

Solution for (iv) The derivative of a two-form $\omega = (x + 2y^3)(dz \wedge dx - \frac{1}{2} dx \wedge dy)$ is,

$$d\omega = (6y^2)dx \wedge dy \wedge dz$$

since $\text{div}(\vec{F} = (0, x + 2y^3, -1/2(x + 2y^3))) = 6y^2$.

Problem 2 (MTA 7.4-3)

For the given φ,

$$D\varphi = \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix}.$$

Consequently,

$$\begin{aligned} \varphi^* dx &= -r\sin\theta d\theta + \cos\theta dr, \\ \varphi^* dy &= r\cos\theta d\theta + \sin\theta dr, \\ \varphi^*(dx \wedge dy) &= \varphi^* dx \wedge \varphi^* dy \\ &= (-r\sin\theta d\theta + \cos\theta dr) \wedge (r\cos\theta d\theta + \sin\theta dr) \\ &= -r\sin^2\theta d\theta \wedge dr - r\cos^2\theta d\theta \wedge dr \\ &= r dr \wedge d\theta. \end{aligned}$$

On the other side,

$$\begin{aligned}\mathbf{d}(\varphi^* x) \wedge \mathbf{d}(\varphi^* y) &= \mathbf{d}(r\cos\theta) \wedge \mathbf{d}(r\sin\theta) \\ &= (-r\sin\theta d\theta + \cos\theta dr) \wedge (r\cos\theta d\theta + \sin\theta dr) \\ &= r dr \wedge d\theta \\ &= \varphi^*(dx \wedge dy)\end{aligned}$$

Problem 3 (MTA 7.5-6)

We will use the following key identities:

$$\pounds_{[X,Y]} = [\pounds_X, \pounds_Y]$$

As described in Corollary 7.4.12 this identity follows from Cartan's formula and the fact that

$$\mathbf{i}_{[X,Y]} = [\pounds_X, \mathbf{i}_Y]$$

Now we compute using Cartan's formula

$$\begin{aligned}\mathrm{div}_\mu[X,Y]\mu &= \pounds_{[X,Y]}\mu \\ &= \pounds_X \pounds_Y \mu - \pounds_Y \pounds_X \mu \\ &= \pounds_X \mathrm{div}_\mu Y \mu - \pounds_Y \mathrm{div}_\mu X \mu \\ &= (X[\mathrm{div}_\mu Y] - Y[\mathrm{div}_\mu X])\mu,\end{aligned}$$

and so $\mathrm{div}_\mu[X,Y] = X[\mathrm{div}_\mu Y] - Y[\mathrm{div}_\mu X]$.

Problem 4 (MTA 8.2-1)

Suppose that M and N are oriented n-manifolds with boundary and $f : M \to N$ is an orientation-preserving diffeomorphism. Let ω be any compactly supported $(n-1)$-form on N. Note that ∂f, the restriction of f to ∂M, is a diffeomorphism onto ∂N by Proposition 8.2.6. Then we have

$$\begin{aligned}\int_M \mathbf{d} f^*\omega &= \int_{\partial M} f^*\omega \quad \text{by Stokes' theorem} \\ &= \int_{f(\partial M)} \omega \quad \text{by change of variables} \\ &= \int_{\partial N} \omega \quad \text{since } f(\partial M) = \partial N \text{ and the two} \\ &\qquad\qquad \text{orientations on } \partial N \text{ agree (see below)} \\ &= \int_N \mathbf{d}\omega \quad \text{by Stokes' theorem} \\ &= \int_M f^*\mathbf{d}\omega \quad \text{by change of variables}\end{aligned}$$

Note that the orientation on ∂N induced by the diffeomorphism f is consistent with the Stoke's orientation: if the orientation on M is defined by the volume form μ_M, then the orientation on ∂M is defined by $\mathbf{i}_X \mu_M$, where X is any outward pointing vector field on ∂M. Then the orientation on ∂N induced by f is $f_*(\mathbf{i}_X \mu_M) = \mathbf{i}_{f_* X} f_* \mu_M$. f is orientation-preserving, and so $f_* \mu_M$ and μ_N define the same orientation on N. Also, since X is outward pointing, $X_x = [c]_x$ where c is a curve in M which ends at $x \in \partial M$. Then $T_x f(X_x) = [f \circ c]_{f(x)}$, and $f \circ c$ is a curve in N which ends at $f(x) \in \partial N$ (by lemma 8.2.4). So $f_* X$ is outward pointing also. Together, these two facts imply that the orientations on ∂N defined by $f_*(\mathbf{i}_X \mu_M)$ and $\mathbf{i}_{f_* X} \mu_N$ agree. The latter is the Stoke's orientation.

Since ω was arbitrary, the above equality implies that

$$\mathbf{d} \circ f^* = f^* \circ \mathbf{d}$$

as required.

Problem 5 8.2-2 (Steve Waydo 2003)

We can write $\mathbf{d}\alpha = \beta\mu$, where $\beta : M \to \mathbb{R}$ is a coefficient depending on location on M and μ is a choice of volume element on M. By Stokes' theorem, $\int_M \mathbf{d}\alpha = 0$ because M is boundaryless. This leaves two possibilities: either $\mathbf{d}\alpha = 0$ everywhere, in which case the result is trivially true, or β must be positive in some regions and negative in others. By the continuity of $\mathbf{d}\alpha$, this implies that $\beta = 0$ at some point and hence $\mathbf{d}\alpha$ must vanish at some point.

Problem 6

See solution set 7, problem 5 (c)