CDS 140a Winter 2015 Homework 2

From MurrayWiki

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In class or to box across 107 STL

(PDF)

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. **Perko, Section 2.2, problem 5:** Let V be a normed linear space. If $T : V \to V$ satisfies
$$\|T(u) - T(v)\| \leq c\|u - v\|$$
for all $u, v \in V$ with $0 < c < 1$ then T is called a *contraction mapping*. It can be shown that contraction mappings give rise to unique solutions of the equation $T(u) = v$:

   Theorem (Contraction Mapping Principle) Let V be a complete normed linear space and $T : V \to V$ a contraction mapping. Then there exists a unique $u \in V$ such that $T(u) = v$.

   Let $f \in C^1(E)$ and $x_0 \in E$. For $I = [-a, a]$ and $u \in C(I)$, let
$$T(u)(t) = x_0 + \int_0^t f(u(s))ds.$$
   Define a closed subset V of $C(I)$ and apply the Contraction Mapping Principle to show that the integration equation (7) in Perko, Section 2.2 has a unique solution $u(t)$ for all $t \in [-a, a]$ provided the constant $a > 0$ is sufficiently small.

2. **Perko, Section 2.3, problem 1**: Use the fundamental theorem for linear systems in Chapter 1 of Perko to solve the initial value problem
$$\dot{x} = Ax, \qquad x(0) = y.$$
   Let $u(t, y)$ denote the solution and compute
$$\Phi(t) = \frac{\partial u}{\partial y}(t, y).$$
   Show that $\Phi(t)$ is the fundamental matrix solution of
$$\dot{\Phi} = A\Phi, \qquad \Phi(0) = I.$$
   - Note: this problem works through the more general result for nonlinear systems (Corallary on page 83) for the special case of a linear system.
3. **Perko, Section 2.5, problem 4**: Sketch the flow of the linear system
$$\dot{x} = Ax \quad \text{with} \quad A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

and describe $\phi_t(N_\epsilon(x_0))$ for $x_0 = (-3, 0), \epsilon = 0.2$.

4. **Perko, Section 2.5, problem 5**: Determine the flow $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ for the nonlinear system

$$\dot{x} = f(x) \quad \text{with} \quad f(x) = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}$$

and show that the set $S = \{x \in \mathbb{R}^2 | x_2 = -x_1^2/4\}$ is invariant with respect to the flow ϕ_t.

5. Choose *one* of the following systems and determine all of the equilibrium points for the system, indicating whether each is a sync, source, or saddle.

(a) Moore-Greitzer model: The Moore-Greitzer equations model rotating stall and surge in gas turbine engines are given by

$$\begin{aligned}
\frac{d\psi}{dt} &= \frac{1}{4B^2 l_c}(\phi - \Phi_T(\psi)), \\
\frac{d\phi}{dt} &= \frac{1}{l_c}\left(\Psi_c(\phi) - \psi + \frac{J}{8}\frac{\partial^2 \Psi_c}{\partial \phi^2}\right), \\
\frac{dJ}{dt} &= \frac{2}{\mu + m}\left(\frac{\partial \Psi_c}{\partial \phi} + \frac{J}{8}\frac{\partial^3 \Psi_c}{\partial \phi^3}\right) J,
\end{aligned}$$

where

$$\begin{aligned}
B &= 0.2, & \Phi_T(\psi) &= \sqrt{\psi}, \\
l_c &= 6, & \Psi_c(\phi) &= 1 + 1.5\phi - 0.5\phi^3, \\
\mu &= 1.256, & m &= 2.
\end{aligned}$$

This is a model for the dynamics of the compression system (first part of a jet engine) with ψ representing the pressure rise across the compressor, ϕ representing the mass flow through the compressor and J representing the amplitude squared of the first modal flow perturbation (corresponding to a rotating stall disturbance).

(b) Genetic toggle switch: Consider the dynamics of two transcriptional repressors connected together in a cycle. It can be shown that the normalized dynamics of the system can be written as

$$\frac{dz_1}{d\tau} = \frac{\mu}{1 + z_2^n} - z_1 - v_1, \qquad \frac{dz_2}{d\tau} = \frac{\mu}{1 + z_1^n} - z_2 - v_2.$$

where z_1 and z_2 represent scaled versions of the protein concentrations, v_1 and v_2 represent external inputs and the time scale has been changed. Let $\mu = 2.16$, $n = 2$ and $v_1 = v_2 = 0$.

(c) Congestion control: A simplified model for congestion control between N computers connected by a router is given by the differential equation

$$\frac{dx_i}{dt} = -b\frac{x_i^2}{2} + (b_{\max} - b), \qquad \frac{db}{dt} = \left(\sum_{i=1}^{N} x_i\right) - c,$$

where $x_i \in \mathbb{R}, i = 1, \ldots, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_{\max} > 0$ is the maximum buffer size and $c > 0$ is the capacity of the link connecting the router to the computers. The $\dot{x}_i$ equation represents the control law that the individual computers use to determine how fast to send data across the network and the $\dot{b}$ equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, x_1, x_2 and b) and take $b_{\max} = 1$ Mb and $c = 2$ Mb/s.

Notes:

- The problems are transcribed above in case you don't have access to Perko. However, in the case of discrepancy, you should use Perko as the definitive source of the problem statement.
