

11/9/07

Laplace Transform Properties

$$\mathcal{L}\{f\} = \int_0^{\infty} f(t) e^{-st} dt = \tilde{f}(s)$$

linearity: $\mathcal{L}\{\alpha f\} = \alpha \tilde{f}(s)$

differentiation: $\mathcal{L}\{f'\} = s\tilde{f}(s) - f(0)$

$$\mathcal{L}\{f''\} = s^2 \tilde{f}(s) - sf(0) - f'(0)$$

integration: $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \tilde{f}(s)$

$$\mathcal{L}\left\{\int_0^t \int_0^{\tau} f(u) du d\tau\right\} = \frac{1}{s^2} \tilde{f}(s)$$

zero initial conditions $\Rightarrow \mathcal{L}\{f'\} = s\tilde{f}(s)$

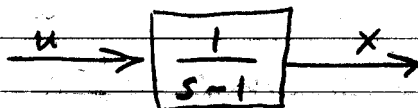
$$\Rightarrow \mathcal{L}\{f''\} = s^2 \tilde{f}(s)$$

System Example 1

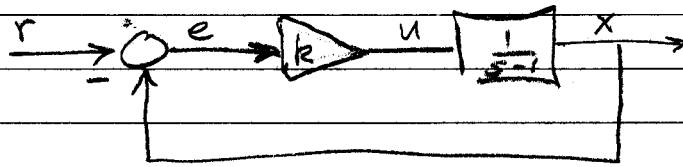
$$\dot{x} = x + u, \quad x(0) = 0$$

$$\left. \begin{array}{l} s\tilde{x} = \tilde{x} + \tilde{u} \\ (s-1)\tilde{x} = \tilde{u} \end{array} \right\} \Rightarrow \tilde{x} = \boxed{\frac{1}{s-1}} \tilde{u}$$

transfer function



close the loop: $u = k(r-x) = ke$



$$x = \frac{k}{s-1} e = \frac{k}{s-1} (r-x) = \frac{k}{s-1} r - \frac{k}{s-1} x$$

$$\Rightarrow x \left(1 + \frac{k}{s-1}\right) = \frac{k}{s-1} r$$

$$x \left(\frac{s+(k-1)}{s-1}\right) = \frac{k}{s-1} r$$

$$x = \underbrace{\frac{k}{s+(k-1)}}_{H_{xr}} r$$

closed loop stability determined by poles of H_{xr}

$$1 \text{ pole: } s = 1-k$$

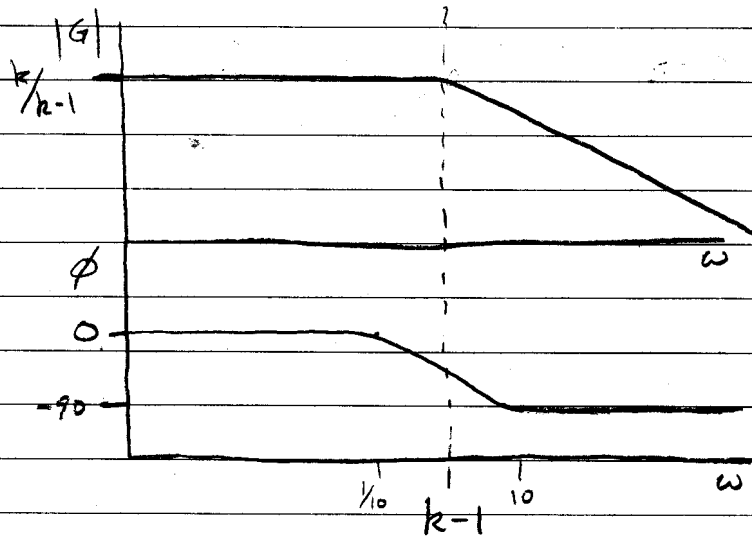
$$\text{stable} \Leftrightarrow \text{LHP} \Leftrightarrow 1-k < 0 \Leftrightarrow k > 1$$

so to stabilize $\dot{x} = x+u$ with proportional gain, need $k > 1$

closed-loop Bode plots:

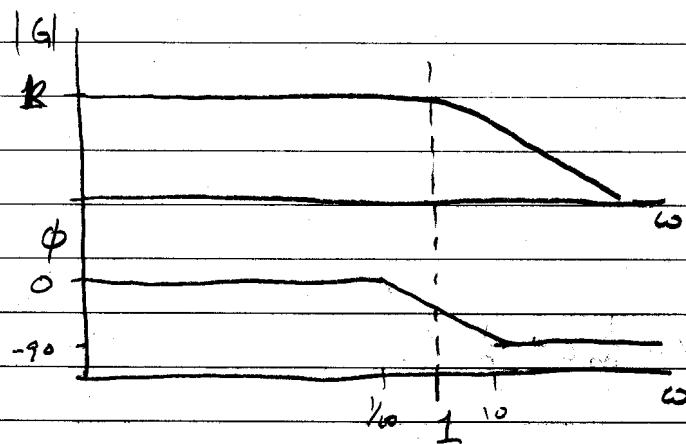
$$\text{zero-frequency output: } s=0 \Rightarrow x = \frac{k}{k-1}$$

(assume $k > 1$)

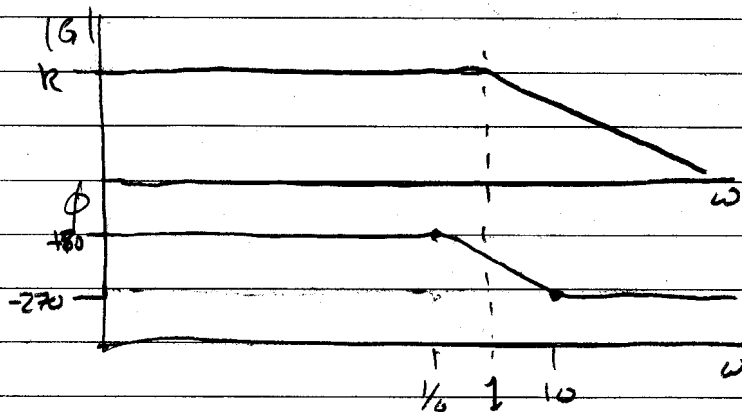


Bode Plot examples

a) $\frac{k}{s+1}$ $s=0 \Rightarrow G = k \Rightarrow \phi = 0$



b) $\frac{k}{s-1}$ $s=0 \Rightarrow |G| = -k \Rightarrow \phi = -180$



Important:
 in this case the phase starts at 180° ! If $k > 1$, this perfect phase shift causes instability.

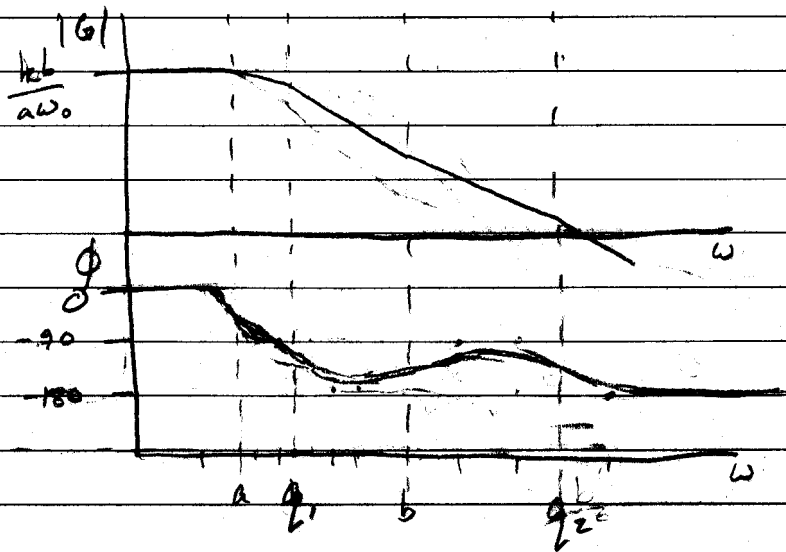
$$c) \frac{k(s+b)}{(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)}$$

$$a, b, \omega_0, \zeta > 0$$

$$s=0 \Rightarrow G = \frac{kb}{a\omega_0} \Rightarrow \phi = 0$$

$$\text{zeros: } s = -b$$

$$\text{poles: } s = -a, s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} = p_1, p_2$$



System Example 2

$$\dot{v} = -\frac{c}{m}v + \frac{b}{m}\tau + \frac{1}{m}F_{\text{hill}}$$

$$d = \frac{F_{\text{hill}}}{m}$$

$$\gamma = \frac{c}{m}$$

$$\beta = \frac{b}{m}$$

$$\dot{v} = -\gamma v + \beta\tau + d$$

Laplace

$$s\tilde{v} = -\gamma\tilde{v} + \beta\tilde{\tau} + \tilde{d}$$

$$(s+\gamma)\tilde{v} = \beta\tilde{\tau} + \tilde{d}$$

$$\tilde{v} = \underbrace{\frac{\beta}{s+\gamma}}_{H_{v\tau}} \tilde{\tau} + \underbrace{\frac{1}{s+\gamma}}_{H_{vd}} \tilde{d}$$

$H_{v\tau}$

H_{vd}

$$\dot{z} = -az + Tau$$

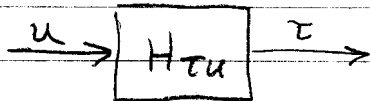
Laplace

$$s\tilde{z} = -a\tilde{z} + aT\tilde{u}$$

$$(s+a)\tilde{z} = aT\tilde{u}$$

$$\tilde{z} = \underbrace{\frac{aT}{s+a}}_{H_{Tz}u} \tilde{u}$$

block diagram?



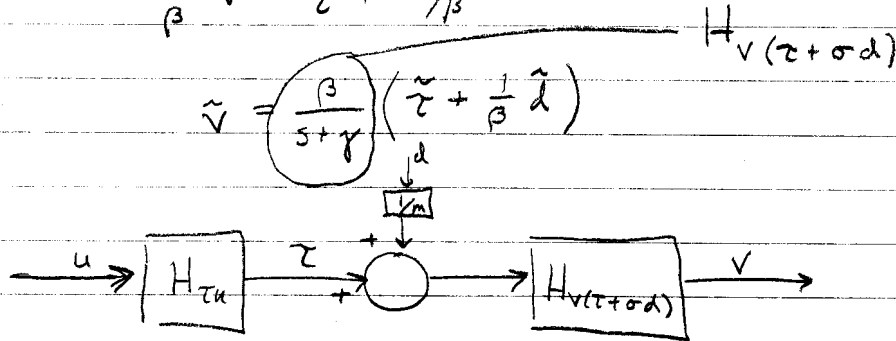
oops! how do we put in d without doing weird division?

find a new transfer function $H_{V(z+\sigma d)}$

$$(s+\gamma)\tilde{v} = \beta\tilde{z} + d$$

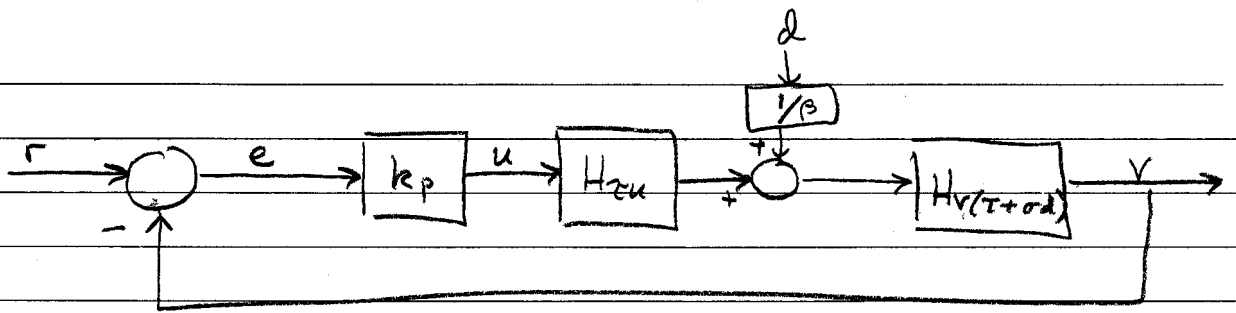
$$\frac{s+\gamma}{\beta}\tilde{v} = \tilde{z} + \frac{d}{\beta}$$

$$\tilde{v} = \left(\frac{\beta}{s+\gamma}\right) \left(\tilde{z} + \frac{1}{\beta}d\right)$$



close the loop?

$$\text{try } u = k_p e = k_p (r - v)$$



now use block diagram algebra to find all sorts of interesting TFS

Her : how the error changes wrt change in reference

to find Her, we neglect d ($d=0$)

$$e = r - y = r - \frac{\beta}{s+\gamma} r = r - \frac{\beta}{s+\gamma} \cdot \frac{aT}{s+a} u$$

$$= r - \frac{\beta}{s+\gamma} \cdot \frac{aT}{s+a} \cdot k_p e$$

$$\Rightarrow e \left(1 + \frac{\beta a T k_p}{(s+\gamma)(s+a)} \right) = r$$

$$H_{er} = \frac{(s+\gamma)(s+a)}{(s+\gamma)(s+a) + \beta a T k_p}$$

$$\left. \begin{array}{l} m=1000 \\ c=50 \\ b=25 \end{array} \right\} \Rightarrow \begin{array}{l} \gamma = 0.05 \\ \beta = 0.025 \end{array}$$

$$a = 0.2$$

$$T = 200$$

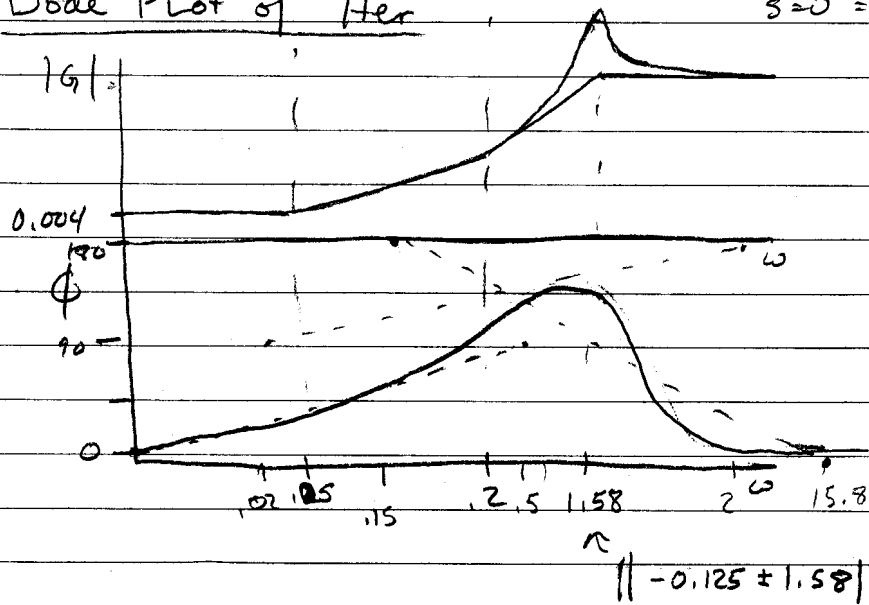
$$k_p = 0.5$$

$$\text{poles : } -0.125 \pm 1.58$$

$$\text{zeros : } -0.2, -0.05$$

Bode Plot of Her

$$s=0 \Rightarrow G=1,004$$



Hyd? this tells how disturbances propagate through the system

here, we let $r=0$ (big idea: we only consider inputs we're interested in... all others 0)

$$\tilde{v} = \frac{\beta}{s+\gamma} \left(\tau + \frac{1}{\beta} d \right)$$

$$= \frac{\beta}{s+\gamma} \left(\frac{aT}{s+a} u + \frac{1}{\beta} d \right)$$

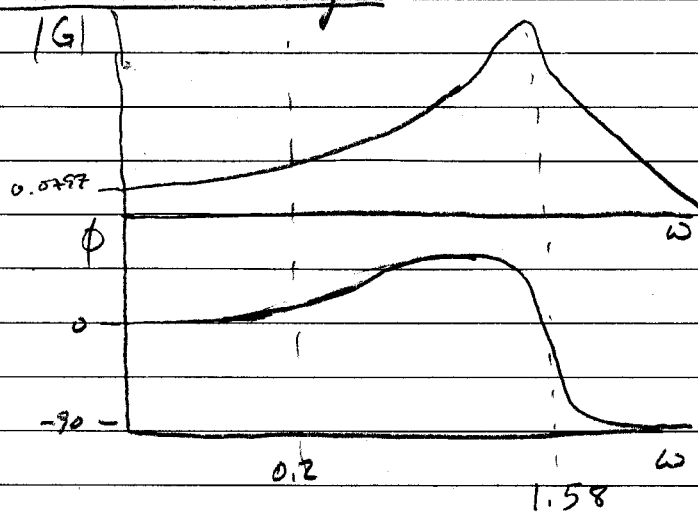
$$= \frac{\beta}{s+\gamma} \left(-\frac{aT k_p}{s+a} v + \frac{1}{\beta} d \right)$$

$$v \left(1 + \frac{\beta a T k_p}{(s+\gamma)(s+a)} \right) = \frac{1}{s+\gamma} d$$

$$Hyd = \frac{s+a}{(s+\gamma)(s+a) + \beta a T k_p}$$

Bode Plot of Hyd

$$s = 0 \Rightarrow G = 0.0797 \quad \phi = 0$$



$$\text{poles: } -0.125 \pm 1.579j$$

$$\|s\| = 1.58$$

$$\text{zeros: } -0.2$$

remember: if you have a pair of complex conjugate poles, they both act at

$$\omega = \|s\|$$