CDS 110a Recitation

11/9/07

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Laplace Transform Properties

$$\mathcal{L}\{f\} = \int_0^\infty f(t)\, e^{-st}\, dt = \tilde{f}(s)$$

linearity: $\mathcal{L}\{\alpha f\} = \alpha \tilde{f}(s)$

differentiation: $\mathcal{L}\{f'\} = s\tilde{f}(s) - f(0)$

$$\mathcal{L}\{f''\} = s^2 \tilde{f}(s) - s f(0) - f'(0)$$

integration: $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\tilde{f}(s)$

$$\mathcal{L}\left\{\int_0^t \int_0^\tau f(u)\, du\, d\tau\right\} = \frac{1}{s^2}\tilde{f}(s)$$

zero initial conditions $\Rightarrow \mathcal{L}\{f'\} = s\tilde{f}(s)$

$\Rightarrow \mathcal{L}\{f''\} = s^2 \tilde{f}(s)$

System Example 1

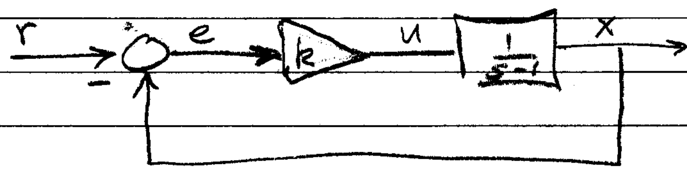
$$\dot{x} = x + u \quad , \quad x(0) = 0$$

$$\left.\begin{aligned} s\tilde{x} &= \tilde{x} + \tilde{u} \\ (s-1)\tilde{x} &= \tilde{u} \end{aligned}\right\} \Rightarrow \tilde{x} = \boxed{\frac{1}{s-1}}\, \tilde{u}$$

transfer function

$$u \longrightarrow \boxed{\frac{1}{s-1}} \xrightarrow{\;x\;}$$

close the loop: $\quad u = k(r-x) = ke$



$$x = \frac{k}{s-1} e = \frac{k}{s-1}(r-x) = \frac{k}{s-1} r - \frac{k}{s-1} x$$

$$\Rightarrow \quad x\left(1 + \frac{k}{s-1}\right) = \frac{k}{s-1} r$$

$$x\left(\frac{s+(k-1)}{\cancel{s-1}}\right) = \frac{k}{\cancel{s-1}} r$$

$$x = \underbrace{\frac{k}{s+(k-1)}}_{H_{xr}} r$$

closed loop stability determined by poles of H_{xr}

1 pole: $\quad s = 1-k$

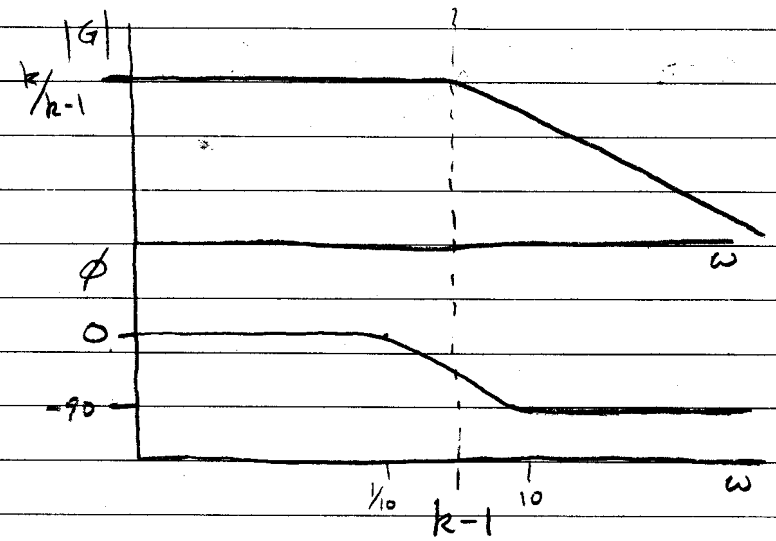
stable $\Leftrightarrow$ LHP $\Leftrightarrow$ $1-k<0 \Leftrightarrow k>1$

so to stabilize $\dot{x} = x+u$ with proportional gain, need $k>1$

closed-loop Bode plots:

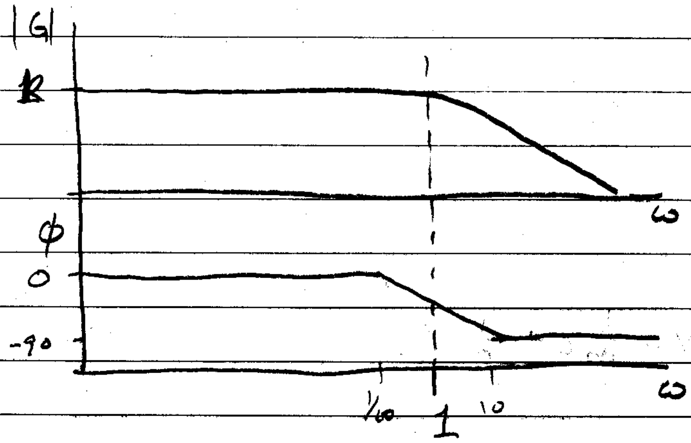
zero-frequency output: $\quad s=0 \Rightarrow x = \frac{k}{k-1}$

(assume $k>1$)

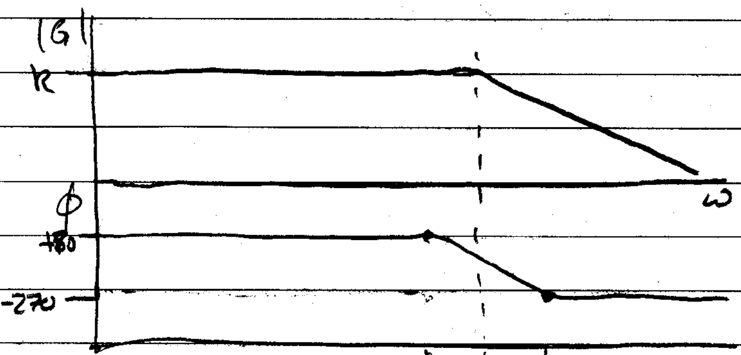


Bode Plot examples

a) $\frac{k}{s+1}$ $\quad s=0 \Rightarrow G = k \Rightarrow \phi = 0$



b) $\frac{k}{s-1}$ $\quad s=0 \Rightarrow G = -k \Rightarrow \phi = -180$



Important:
in this case the phase starts at 180°! If $k>1$, this perfect phase shift causes instability

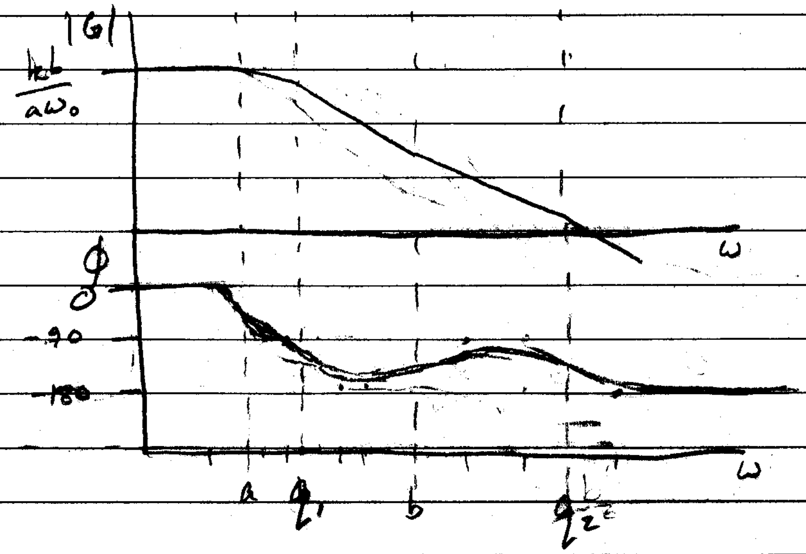
c) $\dfrac{k(s+b)}{(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)}$ $\quad a, b, \omega_0, \zeta > 0$

$$s=0 \Rightarrow \quad G = \frac{kb}{a\omega_0} \quad \Rightarrow \quad \phi = 0$$

zeros: $s = -b$

poles: $s = -a, \; s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} = q_1, q_2$

$a < \omega_0 < b$



System Example 2

$$\dot{v} = -\frac{c}{m}v + \cancel{\frac{b}{m}\tau + \frac{1}{m}F_{hill}}, \qquad d = F_{hill}/m$$

$$\gamma = \frac{c}{m}$$

$$\beta = b/m$$

$$\dot{v} = -\gamma v + \beta\tau + d$$

Laplace

$$s\tilde{v} = -\gamma\tilde{v} + \beta\tilde{\tau} + \tilde{d}$$

$$(s+\gamma)\tilde{v} = \beta\tilde{\tau} + \tilde{d}$$

$$\tilde{v} = \underbrace{\frac{\beta}{s+\gamma}}_{H_{v\tau}}\tilde{\tau} + \underbrace{\frac{1}{s+\gamma}}_{H_{vd}}\tilde{d}$$

$$\dot{\tau} = -a\tau + Tau$$

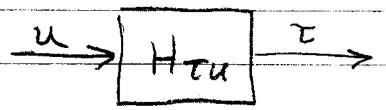
Laplace ↓

$$s\tilde{\tau} = -a\tilde{\tau} + aT\tilde{u}$$

$$(s+a)\tilde{\tau} = aT\tilde{u}$$

$$\tilde{\tau} = \underbrace{\frac{aT}{s+a}}_{H_{\tau u}}\tilde{u}$$

block diagram?



oops! how do we put in d without doing weird division?

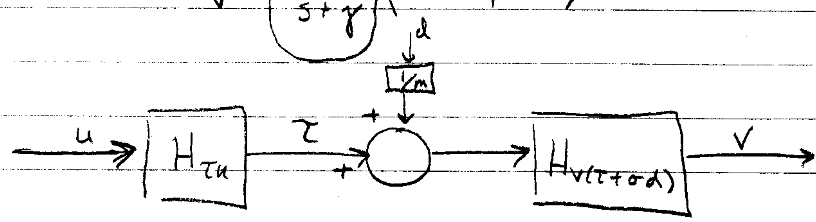
find a new transfer function $H_{v(\tau+\sigma d)}$

$$(s+\gamma)\tilde{v} = \beta\tilde{\tau} + \tilde{d}$$

$$\frac{s+\gamma}{\beta}\tilde{v} = \tilde{\tau} + \tilde{d}/\beta$$

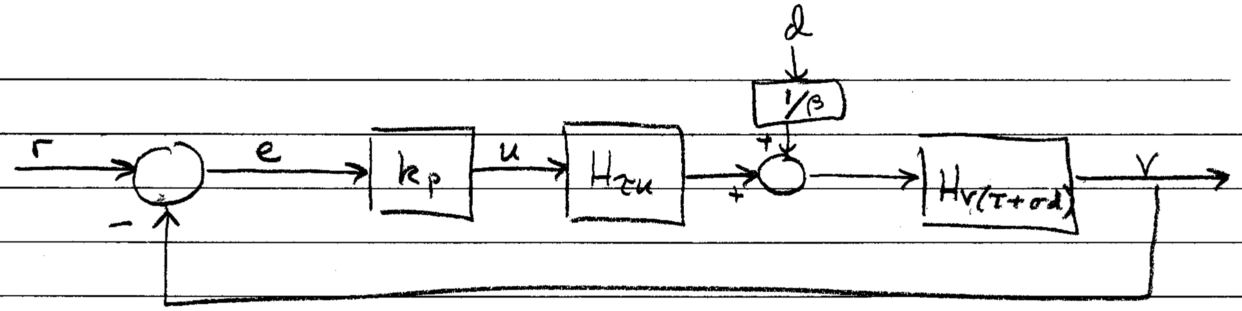
$H_{v(\tau+\sigma d)}$

$$\tilde{v} = \frac{\beta}{s+\gamma}\left(\tilde{\tau} + \frac{1}{\beta}\tilde{d}\right)$$



close the loop?

try $u = k_p e = k_p(r-v)$



now use block diagram algebra to find all sorts of interesting TFs

H_{er} : how the error changes wrt change in reference

to find H_{er}, we neglect d ($d = 0$)

$$e = r - v = r - \frac{\beta}{s+\gamma}\tau = r - \frac{\beta}{s+\gamma}\cdot\frac{aT}{s+a}u$$

$$= r - \frac{\beta}{s+\gamma}\cdot\frac{aT}{s+a}\cdot k_p e$$

$$\Rightarrow e\left(1 + \frac{\beta a T k_p}{(s+\gamma)(s+a)}\right) = r$$

$$H_{er} = \frac{(s+\gamma)(s+a)}{(s+\gamma)(s+a) + \beta a T k_p}$$

$$\left.\begin{array}{l} m = 1000 \\ c = 50 \\ b = 25 \end{array}\right\} \Rightarrow \begin{array}{l} \gamma = 0.05 \\ \beta = 0.025 \end{array}$$

$a = .2$

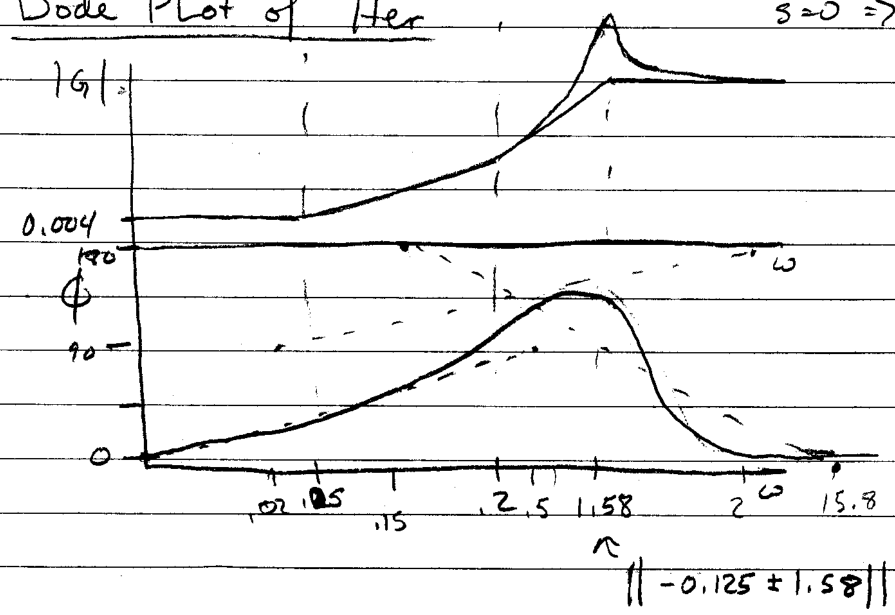
$T = 200$

$k_p = 0.5$

poles : -0.125 ± 1.58

zeros : -0.2 , -0.05

Bode PLot of Her $s=0 \Rightarrow G = .004$



Hyd? this tells how disturbances propagate through the system

here, we let $r = 0$ (big idea: we only consider inputs were interested in...all others 0)

$$\hat{v} = \frac{\beta}{s+\gamma}\left(\tau + \frac{1}{\beta} d\right)$$

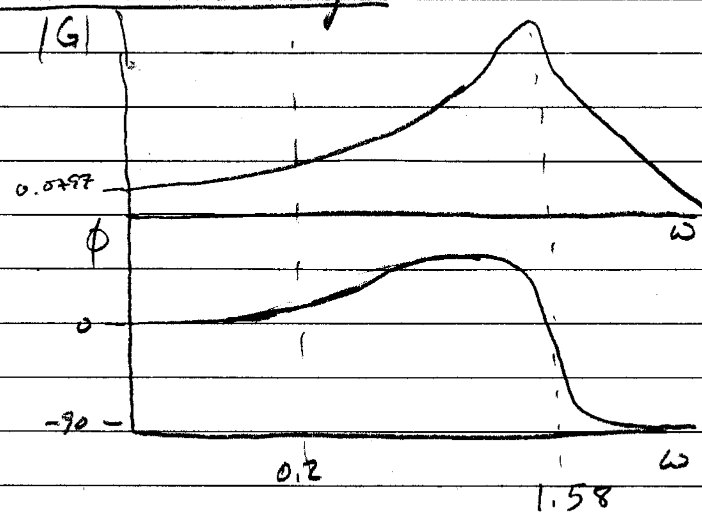
$$= \frac{\beta}{s+\gamma}\left(\frac{aT}{s+a} u + \frac{1}{\beta} d\right)$$

$$= \frac{\beta}{s+\gamma}\left(-\frac{aTk_p}{s+a} v + \frac{1}{\beta} d\right)$$

$$v\left(1 + \frac{\beta a T k_p}{(s+\gamma)(s+a)}\right) = \frac{1}{s+\gamma} d$$

$$H_{yd} = \frac{s+a}{(s+\gamma)(s+a) + \beta a T k_p}$$

Bode Plot of Hyd $\quad s=0 \Rightarrow G=0.0797 \quad \phi=0$



poles: $-0.125 \pm 1.57j$
$\|\cdot\| = 1.58$

zeros: -0.2

remember: if you have a pair of complex conjugate poles, they both act at

$$\omega = \|s\|$$