# Lecture 4: Kalman Filtering with Intermittent Observations

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The update equation of $P_k$:

$$P_k = (1-\gamma_k)h(P_{k-1}) + \gamma_k g(P_{k-1}).$$

$\{P_k\}$ is a stochastic process depending on $\{\gamma_k\}$.

## 1 Critical Value

We want to characterize $\mathbb{E}P_k$. The main difficulty is that $g$ is not affine, hence

$$\mathbb{E}P_k \neq (1-\gamma_k)h(\mathbb{E}P_{k-1}) + \gamma_k g(\mathbb{E}P_{k-1}).$$

As a result, we want to approximate the function $g$ using linear functions:

$$0 \le g(X) \le \varphi(X, K).$$

1. lower bound for $P_k$:

   $$L_k = (1-\gamma_k)h(L_{k-1}),\ L_0 = P_0.$$

   **Theorem 1.** *$L_k \le P_k$, for all $k$.*

   *Proof.* Use the monotonicity and non-negativity of $h$ and $g$. □

2. upper bound for $P_k$:

   $$U_k = (1-\gamma_k)h(U_{k-1}) + \gamma_k\varphi(U_{k-1}, K),\ U_0 = P_0.$$

   **Theorem 2.** *$U_k \ge P_k$, for all $k$.*

   *Proof.* Use the monotonicity of $h$ and $g$ and the fact that $g(X) \le \varphi(X, K)$ for any $K$. □

Now we have

$$\mathbb{E}L_k = (1-\lambda)h(\mathbb{E}L_{k-1}),$$

and

$$\mathbb{E}U_k = (1-\lambda)h(\mathbb{E}U_{k-1}) + \lambda\varphi(\mathbb{E}U_{k-1}, K).$$

By Theorem 1 and 2, we have

$$\mathbb{E}L_k \leq \mathbb{E}P_k \leq \mathbb{E}U_k.$$

Now we consider whether $\mathbb{E}P_k$ is bounded.

- If $A$ is stable, then the following trivial estimator $\hat{x}_{k|k} = 0$ is stable. Hence $\mathbb{E}P_k$ is bounded.
- If $A$ is unstable, $(A, C)$ is observable and $(A, Q^{1/2})$ is controllable:
  - If $\lambda = 1$, we goes back to the classical case, $\mathbb{E}P_k$ is bounded.
  - If $\lambda = 0$, then the estimator does not receive any measurements, $\mathbb{E}P_k$ is unbounded.

Therefore, if $A$ is unstable, $(A, C)$ is observable and $(A, Q^{1/2})$ is controllable, then there exists a critical value $\lambda_c$, such that

- If $\lambda > \lambda_c$, then $\mathbb{E}P_k$ is bounded.
- If $\lambda < \lambda_c$, then $\mathbb{E}P_k$ is unbounded.

## 1.1 Boundedness of $\mathbb{E}L_k$

**Theorem 3.** *$\mathbb{E}L_k$ is bounded if and only if there exists an $X > 0$, such that*

$$(1-\lambda)h(X) \leq X. \tag{1}$$

*Proof.* If $\lambda = 1$, then theorem is trivial. Consider the case where $\lambda < 1$.

First suppose that (1) is true. For any $\mathbb{E}L_0$, there exists an $\alpha \geq 1$, such that $\alpha X \geq \mathbb{E}L_0$. Define $X_0 = \alpha X$ and $X_k = (1-\lambda)h(X_{k-1})$. Therefore

$$X_k \geq \mathbb{E}L_k.$$

On the other hand, by (1)

$$X_1 = \alpha(1-\lambda)AXA' + (1-\lambda)Q = \alpha h(X) - (1-\lambda)(\alpha-1)Q \leq \alpha X = X_0$$

Thus $\{X_k\}$ is decreasing(**why?**). Hence, $X_k$ is bounded, which implies that $\mathbb{E}L_k$ is bounded.

Now suppose that $\mathbb{E}L_k$ is bounded. As a result, let us choose $\mathbb{L}_0 = 0$. Hence,

$$\mathbb{L}_1 = (1-\lambda)h(\mathbb{E}L_0) \geq 0 = \mathbb{L}_0.$$

Thus $\{\mathbb{E}L_k\}$ is increasing(**why?**). On the other hand $\{\mathbb{E}L_k\}$ is bounded. Hence the following limit is well defined

$$X = \lim_{k\to\infty} \mathbb{E}L_k = \sum_{k=0}^{\infty}(1-\lambda)^{k+1}A^kQ(A^k)^T.$$

and

$$X = (1-\lambda)h(X).$$

Only need to prove that $X > 0$. If $(A, Q^{1/2})$ is controllable, then $(\sqrt{1-\lambda}A, (1-\lambda)Q)$ is also controllable. Hence, $X > 0$ is full rank. □

$$X \geq (1-\lambda)h(X) = (1-\lambda)AXA^T + (1-\lambda)Q,$$

has a positive definite solution if and only if $\sqrt{1-\lambda}A$ is stable, i.e.,

$$\sqrt{1-\lambda}\rho(A) < 1 \implies \lambda > 1 - \frac{1}{\rho(A)^2}.$$

Hence, $\lambda_c \geq 1 - \rho(A)^{-2}$.

## 1.2 Boundedness of $\mathbb{E}U_k$

**Theorem 4.** *$\mathbb{E}U_k$ is bounded if and only if there exists an $X > 0$, such that*

$$(1-\lambda)h(X) + \lambda\varphi(X, K) \leq X. \tag{2}$$

Define

$$\overline{\lambda} \triangleq \inf\{\lambda \in [0,1] : \text{there exists } K, X > 0 \text{ such that Eq (2) holds}\}.$$

Then $\lambda_c \leq \overline{\lambda}$.

### 1.2.1 Special Case: $C$ invertible

If $C$ is invertible (or in general if $C$ is of rank $n$), then we can choose $K = C^{-1}$. As a result,

$$\varphi(X, C^{-1}) = (I - C^{-1}C)h(X)(I - C^{-1}C)^T + C^{-1}RC^{-T} = C^{-1}RC^{-T}.$$

Therefore, (2) becomes

$$(1-\lambda)h(X) + \lambda C^{-1}RC^{-T} \leq X.$$

Therefore, if $\lambda > 1 - \rho(A)^{-2}$, then the above equation has a positive definite solution. Hence, $\overline{\lambda} \leq 1 - \rho(A)^{-2}$, which implies that $\lambda_c = 1 - \rho(A)^{-2}$.

## 2 Observability in NCS

If $C$ is invertible, then the critical value $\gamma_c = 1 - \rho(A)^{-2}$.

However, for general systems, this may not be true. For the following system, one can prove that critical value is $1 - \rho^{-4}$.

$$A = \begin{bmatrix} \rho & 0 \\ 0 & -\rho \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{3}$$

The reason being that although $(A, C)$ is observable, $(A^2, C)$ (or in general $(A^{2k}, C)$) is not observable.

- $C$ invertible implies that we can reconstruct the state $x_k$ using only 1 measurements: $\hat{x}_{k|k} = C^{-1} y_k$.
- $(A, C)$ observable implies that we can reconstruct the state $x_k$ using at most $n$ sequential measurements $y_k, y_{k-1}, \ldots, y_{k-n+1}$.

**Theorem 5.** *If the linear system satisfies:*

1. *$A$ is diagonalizable;*
2. *$(A^r, C)$ is observable for any $r \in \mathbb{R}^+$,*

*then the critical value is given by*

$$\lambda_c = 1 - \frac{1}{\rho(A)^2}.$$

Condition 1 and 2 are called non-degeneracy condition and essentially they implies that we can reconstruct the state $x_k$ using any $n$ measurements.

## 3 References on Kalman Filter with Intermittent Observations

- The original paper on critical value: [6]
- The concept of non-degeneracy and its relationship with critical value: [5]
- A counter example where the critical value of an observable but degenerate system is not the lower bound: [4]
- A special case where one can drive the pdf of $P_k$: [1]
- Contraction properties of Riccati and Lyapunove equation: [2]
- A survey paper on networked control problem: [3]

# References


[1] Andrea Censi. On the performance of Kalman filtering with intermittent observations: a geometric approach with fractals. In *Proceedings of the American Control Conference (ACC)*, St. Louis, Missouri, June 2009.

[2] Andrea Censi. Kalman filtering with intermittent observations: convergence for semi-markov chains and an intrinsic performance measure. *IEEE Transactions on Automatic Control*, February 2011.

[3] Joao P Hespanha, Payam Naghshtabrizi, and Yonggang Xu. A survey of recent results in networked control systems. *PROCEEDINGS-IEEE*, 95(1):138, 2007.

[4] Yilin Mo and B. Sinopoli. A characterization of the critical value for kalman filtering with intermittent observations. In *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pages 2692–2697, Dec 2008.

[5] Yilin Mo and Bruno Sinopoli. Kalman filtering with intermittent observations: tail distribution and critical value. *Automatic Control, IEEE Transactions on*, 57(3):677–689, 2012.

[6] Bruno Sinopoli, Luca Schenato, Massimo Franceschetti, Kameshwar Poolla, Michael I Jordan, and Shankar S Sastry. Kalman filtering with intermittent observations. *Automatic Control, IEEE Transactions on*, 49(9):1453–1464, 2004.