Modeling of the Caltech Ducted Fan

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1 Introduction

The Caltech Ducted Fan is a small flight control experiment whose dynamics are representative of either a Harrier in hover mode or a thrust vectored aircraft (such as the F18-HARV or X-31) in forward flight. A diagram of the experimental systems is shown in Figure 1. It consists of a ducted fan engine with a high-efficiency electric motor and 6-inch diameter blade, capable of generating up to 9 Newtons of thrust. Flaps on the fan allow the thrust to be vectored from side to side and even reversed. The engine is mounted on a three degree of freedom stand which allows horizontal and vertical translation as well as unrestricted pitch angle.

This system has been used for a number of studies and papers. A description of the overall design and control considerations is given in [2]. A comparison of several different linear and nonlinear controllers was performed by Kantner et al.[3] and a more focused comparison on LPV controllers is given in [1]. The application of differential flatness based controllers is reported in [4].

In this note we describe different models for the ducted fan engine and give parameter values corresponding to the current experimental setup at Caltech.

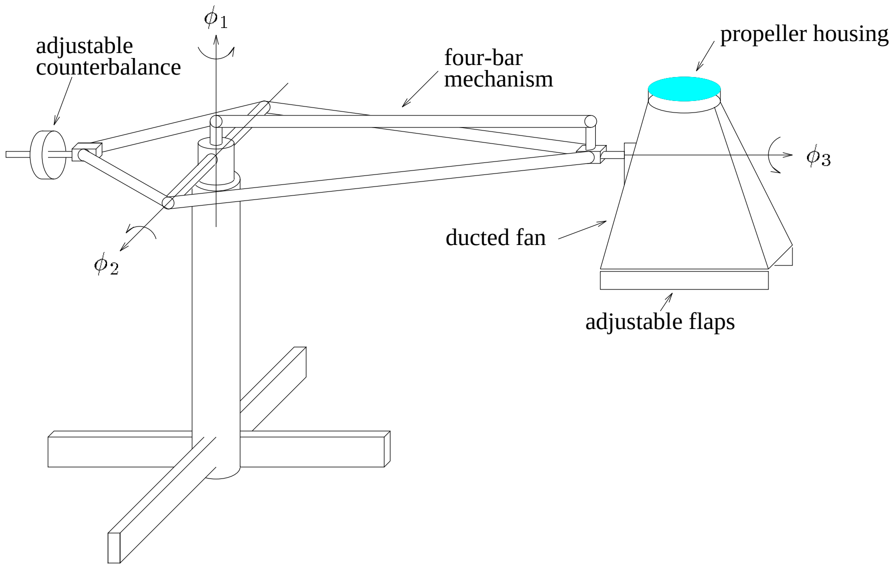

Figure 1: Caltech ducted fan with support stand.

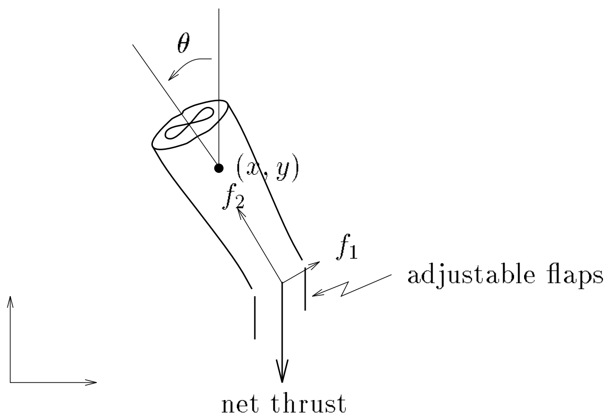

Figure 2: Planar ducted fan model.

2 Planar model with gravity and drag (`planar`)

We begin with a simple planar model, shown in Figure 2, that ignores the stand dynamics. This model is useful for determining basic characteristics of the model and testing initial control designs.

Let (x, y, θ) denote the position and orientation of a point on the main axis of the fan that is a distance l from the center of mass. (This point is where the stand would normally attach and hence it is useful to write the equations of motion in terms of the coordinates of this point rather than the center of mass.) We assume that the forces acting on the fan consist of a force f_1 perpendicular to the axis of the fan acting at a distance r and a force f_2 parallel to the axis of the fan. Let m be the mass of the fan, J the moment of inertia, g the gravitational constant, and $c_{d,x}(\theta, \dot{x})$ a function which models the drag coefficient as a function of orientation. Then the equations of motion for the fan are given by:

$$\begin{aligned} m\ddot{x} &= f_1 \cos\theta - f_2 \sin\theta - c_{d,x}(\theta, \dot{x}) \\ m\ddot{y} &= f_1 \sin\theta + f_2 \cos\theta - mg - c_{d,y}(\theta, \dot{y}) \\ J\ddot{\theta} &= r f_1 - mgl \sin\theta - c_{d,\theta}(\theta, \dot{\theta}) \end{aligned} \tag{1}$$

It is convenient to redefine the inputs so that the origin is an equilibrium point of the system with zero input. If we let $u_1 = f_1$ and $u_2 = f_2 - mg$ and model the drag terms as viscous friction, then the equations become

$$\begin{aligned} m\ddot{x} &= -mg \sin\theta - d\dot{x} + u_1 \cos\theta - u_2 \sin\theta \\ m\ddot{y} &= mg(\cos\theta - 1) - d\dot{y} + u_1 \sin\theta + u_2 \cos\theta \\ J\ddot{\theta} &= r u_1 - mgl \sin\theta. \end{aligned} \tag{2}$$

These equations are referred to as the *planar ducted fan equations.*

For the Caltech ducted fan, representative values of the parameters are listed in Table 1. The maximum force that can be generated by the fan is approximately 10 Newtons in the axial direction and 2 Newtons in the sideways direction. A quick calculation shows that the fan cannot support its own weight with these limits on the maximum thrust, and hence a counterweight is used to offset part of the weight of the fan. This can be roughly incorporated into the dynamics given above by limiting the axial thrust to 5 Newtons instead of 10.

Symbol	Description	Value	
m	inertial mass of fan, x axis	4.0	kg
J	fan moment of inertia, ϕ_3 axis	0.0475	kg m^2
r	nominal distance of flaps from fan pivot	26.0	cm
l	center of mass offset for fan	2.13	cm
g	gravitational constant	9.8	m/sec^2

Table 1: Parameter values for the planar ducted fan model which approximate the dynamics of the Caltech ducted fan.

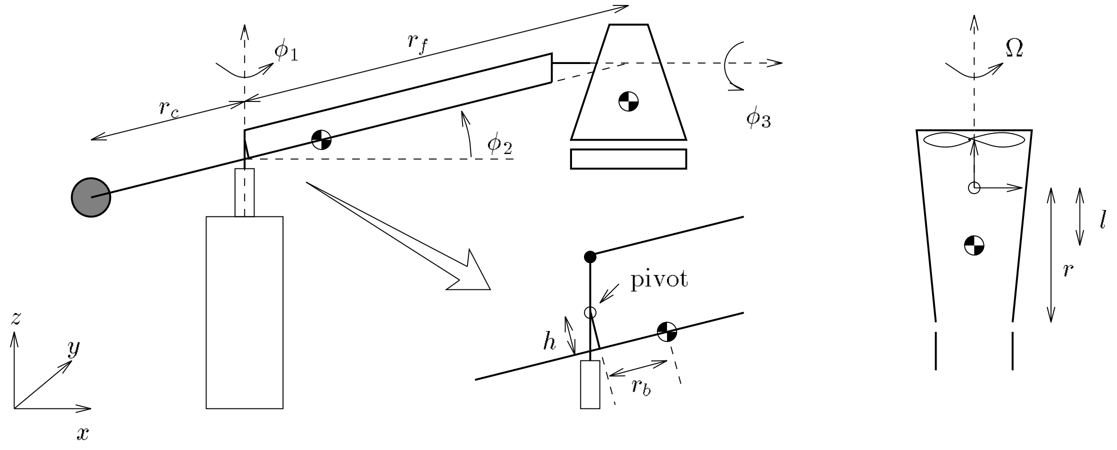

Figure 3: Ducted fan with simplified model of stand.

3 Ducted fan with simplified stand dynamics (`snlfan`)

A much more accurate description of the dynamics is available by including some of the essential stand dynamics. We consider the simplified version of the stand shown in Figure 3. This model ignores some of the details of the stand geometry but captures most of the important effects. The main difference between this model and the previous model is the effect of the fan counterweight, which changes the effective mass and weight of the fan. In addition, due to the kinematics of the stand, the inertial of the fan about the vertical axis varies as a function of the altitude of the fan.

Let (r_f, m_f) be the distance from the stand's vertical axis to the fan and the mass of the fan, respectively, and let (r_c, m_c) be the distance and mass of the counterweight. We also incorporate the mass of the supporting strut, which we take to have mass m_b and center of mass r_b from the vertical axis. We let J_1 represent the inertia of the fan about the vertical axis (which determines the position, x) and J_2 represent the inertia of the fan about the horizontal axis (which determines the altitude, y). We assume that the boom (connecting the fan to the counterweight) has a center of along a line between the fan and the counterweight and that this line is a distance h below the pivot point for the altitude axis.

It will be convenient to write the motion of the fan not in terms of the joint variables ϕ_i, but rather in terms of the position of the fan. Hence we write

$$x = r_f \phi_1 \qquad y = r_f \phi_2 \qquad \theta = \phi_3$$

Define

$$\alpha = \frac{J_1 + (m_f r_f^2 + m_c r_c^2 + m_b r_b^2)}{r_f^2}$$

$$\delta = \frac{J_1 - J_3 + m_f(r_f^2 - h^2) + m_b(r_b^2 - h^2) + m_c(r_c^2 - h^2)}{r_f^2}$$

$$\delta' = \frac{2h(m_f r_f - m_c r_c + m_b r_b)}{r_f^2}$$

$$\beta = \frac{J_2 + m_f(r_f^2 + h^2) + m_b(r_b^2 + h^2) + m_c(r_c^2 + h^2)}{r_f^2}$$

$$\gamma = 9.8 \frac{r_f(m_f r_f + m_b r_b - m_c r_c)}{J_2 + m_f(r_f^2 + h^2) + m_b(r_b^2 + h^2) + m_c(r_c^2 + h^2)}$$

$$\kappa = 9.8 \frac{h r_f(m_f + m_b + r_c)}{J_2 + m_f(r_f^2 + h^2) + m_b(r_b^2 + h^2) + m_c(r_c^2 + h^2)}$$

$$\sigma = m_f l^2 \qquad \lambda = m_f l / r_f \qquad \lambda' = mflh/rf^2$$
$$\eta = m_f l \qquad \eta' = m_f l h / r_f \qquad \eta'' = J_3 / r_f.$$

Using these definitions, the inertia tensor for the system is given by

$$M(q) = \begin{bmatrix} \alpha - \delta s_2^2 + \delta' c_2 s_2 + \sigma s_3^2 & \lambda s_2 s_3 - \lambda' c_2 s_3 & \eta c_2 c_3 + \eta' s_2 c_3 + \eta'' s_2 \\ \lambda s_2 s_3 - \lambda' c_2 s_3 & \beta & \eta c_2 s_3 + \eta' s_2 s_3 \\ \eta c_2 c_3 + \eta' s_2 c_3 + \eta'' s_2 & \eta c_2 s_3 + \eta' s_2 s_3 & J_3 + \sigma \end{bmatrix}$$

Assuming that δ' and η, η', η'' and λ, λ' are negligible and $\sigma \ll \alpha$, and taking into account the inertia J_m of the motor and its rotational velocity Ω, Lagrange's equations yield the following equations of motion for the system:

$$\begin{bmatrix} (\alpha - \delta s_y^2)\ddot{x} \\ \beta \ddot{y} \\ J\ddot{\theta} \end{bmatrix} = \begin{bmatrix} 2(\delta/r_f) s_y c_y \, \dot{x}\dot{y} - (J_m \Omega / r_f) \sin\theta \, \dot{x} \\ -(\delta/r_f) s_y c_y \, \dot{x}^2 - \beta\gamma c_y - \beta\kappa s_y \\ (J_m \Omega / r_f) \sin\theta \, \dot{x} - m_f g l \sin\theta \end{bmatrix} + \begin{bmatrix} c_2 \cos\theta & -c_2 \sin\theta \\ c_2 \sin\theta & c_2 \cos\theta \\ r & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{3}$$

where

$$c_y = \cos(y/rf) \qquad s_y = \sin(y/r_f)$$

The parameter values for the Caltech ducted fan are given in Table 2.

It is useful to analyze the open loop dynamics of the fan to help explain its controllability properties. In the x and y equations, the main effect of the stand is make the inertia of the system in the x direction depend on the altitude of the fan (due to the fact that the counterweight and the fan move closer to the vertical axis when the supporting strut is not horizontal). This introduces a nonconstant inertia matrix, which in turn introduces Coriolis and centrifugal forces in the x and y equations. A further complication is due to the inertia of the spinning blade on the fan, which generates Coriolis forces when the fan is moving about the ϕ_1 axis and is not pointed vertically ($\sin\theta \neq 0$). These forces can be quite significant since the fan blade spins at approximately 1000 rad/sec (10,000 rpm).

Symbol	Description	Value	
r_f	distance of fan center of mass from ϕ_2 axis	143.5	cm
r_c	distance of counterweight from ϕ_2 axis	56.8	cm
r_b	offset of boom center of mass	30	cm
r	nominal distance of flaps from fan pivot	26.0	cm
h	distance of boom axis from pivot point	4.29	cm
m_c	counterweight mass	6.427	kg
m_b	mass of boom	2.872	kg
m_f	mass of fan	2.215	kg
l	center of mass offset for fan	2.13	cm
J_1	stand moment of inertia, ϕ_1 axis	1.731	kg m^2
J_2	stand moment of inertia, ϕ_2 axis	0.7537	kg m^2
$J = J_3$	fan moment of inertia, ϕ_3 axis	0.0475	kg m^2
b_1	viscous friction coefficient, ϕ_1 axis	0.595	N m sec
b_2	viscous friction coefficient, ϕ_2 axis	0.980	N m sec
b_3	viscous friction coefficient, ϕ_3 axis	0.0291	N m sec
J_m	inertia of motor + blade	0.166×10^{-3}	kg m^2
Ω	angular speed of motor	1047	rad/sec
g	gravitational constant	9.8	m/sec^2
α	inertial mass of fan, x axis	4.19	kg
β	inertial mass of fan, y axis	3.72	kg
δ	varying mass of fan, x axis	4.16	kg
δ'	varying mass of fan, x axis	0.0162	kg
γ	effective gravitational constant	0.711	m/sec^2
κ	effective gravitational constant	0.906	m/sec^2
η	cross inertia due to fan offset	0.0472	kg
η'	cross inertia due to fan offset	0.00141	kg
η''	cross inertia due to fan offset	0.0331	kg
λ	cross inertia due to fan offset	0.0329	kg
λ'	cross inertia due to fan offset	0.983×10^{-3}	kg
σ	pitch inertia due to fan offset	0.488×10^{-3}	kg
b_x	viscous friction coefficient, x axis	0.289	N sec / m
b_y	viscous friction coefficient, y axis	0.476	N sec / m
b_θ	viscous friction coefficient, θ axis	0.0291	N m sec

Table 2: Parameter values for Caltech ducted fan.

4 Simulink models of the ducted fan

Computer models of the dynamics described above are available in the form of Simulink S-functions. These functions can be used within Simulink and Matlab, or in a custom simulation program. The following models are available:

Name	Source	Section	Comments
`planar`	`planar.c`	2	Planar ducted fan model (nonlinear)
`snlfan`	`snlfan.c`	3	Ducted fan model with simplified nonlinear dynamics

Both of these models have f_1, f_2 as inputs and the full state as outputs. The `snlfan` source code contains optional code to include aerodynamic forces but this is turned off by default since it is pretty heuristic.

Appendix: Estimation of blade/motor inertia

The motor is a Cobalt 40 motor, which weighs a total of 236 grams and has an outer diameter of 3.68 cm and length of 5.46 cm. Assuming that the rotor has a mass of 236 grams uniformly distributed along a 3 cm radius, the moment of inertia of the motor is

$$J_{\text{motor}} = 2/3\ (0.236\ \text{kg})\ (0.03\ \text{m})^2 = 0.1416 \times 10^{-3}\ \text{kg m}^2.$$

Each of the two blades in the propeller has a diameter of 15.4 cm, width of approximately 1.7 cm, and mass of 6 grams. Assuming a uniform distribution along a rectangle of these dimensions, the moment of inertia of a single fan blade is

$$J_{\text{blade}} = 1/12(0.006\ \text{kg})\left((0.154\ \text{m})^2 + (0.017\ \text{m})^2\right) = 0.012 \times 10^{-3}\ \text{kg m}^2.$$

The total inertia of the blade/motor system is thus given by

$$J_m = J_{\text{motor}} + 2J_{\text{blade}} = 0.166 \times 10^{-3}\ \text{kg m}^2.$$

References

[1] B. Bodenheimer, P. Bendotti, and M. Kantner. Linear parameter-varying control of a ducted fan engine. *International Journal of Robust and Nonlinear Control*, 1996. (to appear).

[2] H. Choi, P. Sturdza, and R. M. Murray. Design and construction of a small ducted fan engine for nonlinear control experiments. In *Proc. American Control Conference*, pages 2618–2622, 1994.

[3] M. Kantner, B. Bodenheimer, P. Bendotti, and R. M. Murray. An experimental comparison of controllers for a vectored thrust, ducted fan engine. In *Proc. American Control Conference*, 1995.

[4] M. van Nieuwstadt and R. M. Murray. Real time trajectory generation for differentially flat systems. In *Proc. IFAC World Congress*, 1996.