Dynamical Systems, Optimal Control, and Microsat Formation Flight

Jerrold E. Marsden

AFOSR Contractors and Grantees Meeting, August, 2000
Five Objectives of Microsat

☐ 1. Dynamics of multiple spacecraft in nearby orbits
☐ 2. Formation stabilization strategies
☐ 3. Formation reconfiguration algorithms
☐ 4. Mathematical modeling and simulation tools
☐ 5. Power requirements and limitations
Information

URL’s

- http://www.cds.caltech.edu/microsat/
- http://www.cds.caltech.edu/~marsden/
- http://www.cds.caltech.edu/~murray/
Information

■ **URL’s**

- http://www.cds.caltech.edu/microsat/
- http://www.cds.caltech.edu/~marsden/
- http://www.cds.caltech.edu/~murray/

■ **Team**

- **Faculty:** Jerry Marsden, Richard Murray, CIT, Meir Pachter, Air Force Institute of Technology
- **Postdocs:** Wang-Sang Koon, David Chichka,
- **Graduate Students:** Dong-Eui Chang, Alex Fax, Mark Milam, Bill Dunbar, plus other student help
Related Activities with JPL

- **Genesis Discovery Mission**
  - Uses 3-body problem libration point dynamics and heteroclinic connections

- **Jovian moon missions**
  - Uses transfers between different 3-body problem libration point dynamics

- **Terrestrial Planet Finder**
  - Uses coordinated control and 3-body dynamics

- **Development of LTool**
  - Software for libration point missions
Nonlinear Dynamics & Formation Flight

- Study of candidate reference orbits whose nearby orbits may support formation flight
- Development of a geometric mechanics framework for the Kepler-\(J_2\) dynamics
Posters

**Nonlinear Dynamics & Formation Flight**

- Study of candidate reference orbits whose nearby orbits may support formation flight
- Development of a geometric mechanics framework for the Kepler-$J_2$ dynamics

**Control and Optimal Control of Formation Flight**

- Optimal control for formation reconfiguration
- Cooperative control
- Lyapunov-based global orbit transfer
Three Topics

- Dynamics of satellites in Earth orbit
- Cluster reconfiguration
- Orbital transfer.
A Few Key References


Basic $J_2$ dynamics

- **Satellite in Earth orbit**
  - motion in Kepler potential plus $J_2$ perturbation
  - $J_2 = \text{bulge of the Earth}$

- **$J_2$ causes**
  - a drift in the orbit plane (geometric phase)
  - a drift in the major axis of the ellipse within the (approximate) orbital plane (direction of drift depends on the angle of inclination)
Basic $J_2$ dynamics

Satellite

Perigee

Equatorial Plane

Vernal Equinox

Ascending Node

Apogee

$\Omega$: Right Ascension of Ascending Node

$\omega$: Argument of the Perigee

$i$: True Anomaly

$i$: Orbit Inclination
Basic $J_2$ dynamics

*Example:* inclination = 28 degrees, altitude 300km
period of $\Omega$ = roughly 50 days (left)
period of $\omega$ = roughly 30 days (right).
Shows rich nonlinear dynamics

Relative to rotations around the $z$-axis

- Involves Routh reduction; one studies orbits for various energies $h$ and angular momenta $\nu$ about the $z$-axis.

- Original problem 6 dimensional: constant $h$ and $\nu$ and remove angular variable gives 3 and a Poincaré section in that is 2-dimensional.
Poincaré Sections

Poincaré maps for the $J_2$ problem (Broucke, 1992)
Relative Motions

- Poincaré sections useful for detecting bifurcations.
- What about relative motions?
  - Are there relative motions that remain tied together?
  - Not obvious because of phase drifts, etc.
- To answer this we need to have a closer look at the dynamics and the role of reduction theory.
- Indeed, there are such interesting orbits (see the posters for details)
Relative dynamics (100 days) of a satellite in a frame moving with a reference satellite
Reduction theory

- all the crucial features ($\Omega$-drift as a geometric phase etc) come out naturally
- gives a global picture of the dynamics.
**Geometric Mechanics and \( J_2 \)**

- **Reduction theory**
  - all the crucial features (\( \Omega \)-drift as a geometric phase etc) come out naturally
  - gives a global picture of the dynamics.

- **Symmetries**
  - rotational \( S^1 \) symmetry about the vertical axis.
  - A \( \mathbb{Z}_2 \) symmetry—reflection in the equatorial plane
  - Another \( \mathbb{Z}_2 \) symmetry—reflection in vertical planes
    + time reversal—an *antisymplectic* symmetry.
Moser Regularization

- Kepler Hamiltonian $H_0$ and dynamics transferred to geodesics on $S^3$.
- Symmetry group is now $SO(4)$.
- Momentum map includes the Laplace-Runge-Lenz vector (a vector pointing to the periapsis):

$$(r, \dot{r}) \mapsto (L, A) = \left( r \times \dot{r}, \dot{r} \times (r \times \dot{r}) - \mu \frac{r}{||r||} \right).$$
On an energy surface, Keplerian orbits are closed with the same period—Kepler flow gives an $S^1$ action.

Total Hamiltonian is a sum:

$$ H = H_0 + \epsilon H_1; \quad \epsilon = J_2 \text{ size} $$

Average with respect to the Keplerian $S^1$ action.

Averaged Hamiltonian has symmetry

$$ S^1 \times S^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2. $$

*Double* reduction—a Keplerian $S^1$ symmetry and an axial $S^1$ symmetry

Discrete symmetries pass to the reduced space.
Symmetry & Reduction

Reduction map
(gets rid of the angular variable)

$S^3$ \rightarrow \Omega

$S^2$
flow parametrized by (Keplerian) energy $h$ and $\nu$

bifurcations as $h$ and $\nu$ are varied; bifurcation occurs at the critical inclination: $h^2 - 5\nu^2 = 0$

fixed points on $S^2 = \text{periodic orbits on } S^3$

singularity if angular momentum $\nu = 0$; singular points are orbits that head directly into poles

can use energy-momentum methods for stability and bifurcation

figures show a series of reduced phase portraits for fixed $h$ as $\nu$ decreases to zero.
Final Reduced Space $S^2$

1. $h/\sqrt{5} < \nu < h$
2. $0 < \nu < h/\sqrt{5}$, (bifurcation at critical inclination)
3. $\nu$ near zero, 4. $\nu = 0$ (singular case).
Variational integrators

- Variational (symplectic) algorithms are remarkably good in many problems, especially for long term, sensitive integrations.
- Need to use symmetry properly.
- We have developed a reduction theory for discrete mechanics and have applied the associated variational integrators to the $J_2$ problem.
Spacecraft Clusters

Loose control

For example, the simultaneous in situ measurement of the magneto-sphere may require a loose constellation scattered all over the magneto-sphere.
Spacecraft Clusters

- **Loose control**
  - For example, the simultaneous in situ measurement of the magneto-sphere may require a loose constellation scattered all over the magneto-sphere.

- **Precision control**
  - For interferometry, the shape and orientation of the formation must be maintained to some degree.
  - Either
    - one has to maintain it very precisely or
    - one has to *know* the relative positions precisely so that this information can be managed in software.
Spacecraft Clusters

Performance Metrics

1. **Coverage analysis**—estimate of the amount of data collected. Based on this metric, one could then make quantitative tradeoff studies and compare different designs.
2. *Propulsion requirements*—the use of natural dynamics greatly affects this.

3. *Power requirements*—e.g., battery size contributes significantly to the mass of the spacecraft.

4. *Spacecraft mass*—depends not only on the trajectory design, control algorithm, propulsion and power subsystems, but also on the launch vehicle capability and launch deployment strategy.
Natural dynamics plays a critical role. Since only small changes are presumably needed, standard linear control techniques should be adequate for maintenance.

One still needs to deal with cooperative techniques as well as possible combinatorial explosions. Graph theoretic methods probably useful.

Utilize the fact that this is a problem with two greatly differing scales: the relative distances between the satellites (the shape dynamics) and the absolute position of the cluster as a whole. Geometric mechanics can help in the separation of these effects.
Formation Reconfiguration

More challenging

- Larger dynamic motions are involved
- Example: reorient the whole formation
- Fuel usage is potentially critical

Optimal control

- Requires a good first guess
- Works best when used with the natural dynamics

Two examples

- Trajectory correction maneuvers (TCM) for halo orbit insertion
- Earthbound satellite reconfiguration
TCM for Halo Orbit Insertion

- Optimization software used is COOPT: (COntrol–OPTimization)
- Optimizes: cost function $= \Delta V$ subject to the constraint of the equations of motion.
- We vary the number of impulses and also consider the effect of delaying the first impulse and the launch uncertainty.
- Makes use of the dynamical systems structure of the three body problem (especially the invariant manifolds of halo orbits).
Movie-Optimal Insertion
Sensitivity Analysis

Parametric Study of the Optimal Solution

Number of maneuvers:
- Unperturbed injection velocity: 1
- Perturbed injection velocity: 2

Influence of:
- Delay in TCM1
- Perturbation in launching velocity

Optimal solutions found for all cases

Varying launch velocity and first manoeuvre delay
Alternative software for the reconfiguration problem

- NTG (nonlinear trajectory generation)
- Also uses direct (brute force) optimization
- Key difference with COOPT: substitutes the forces using the equations of motion, so one gets a higher order cost function
- Avoids treating the equations of motion as constraints (horrors!)
Sample problem

Consider the toy problem of the equations linearized about a circular orbit without any $J_2$ effect. This is not realistic, but it is a demonstration that the software is effective.

More generally, one would have to use the full non-linear equations about a nominal trajectory family. Linearization methods here may not be adequate, but the sofware treats the nonlinear problem with no difficulty.
Specific Objective: Find a trajectory that minimizes the control energy, over a fixed time, for three microsats such that at the final configuration the microsats will remain on a circle (projection onto the $yz$ plane) of 100 m radius, 120 degrees apart with no control force, indefinitely.

Use a rotating frame with coordinate axes so that the $x$-axis is along the line of sight from the earth, while the $y$-axis is along the orbit and $z$ is perpendicular to these two. The $yz$ plane is what you see looking up from the Earth.
Earthbound Satellites

**Cost Function**: minimize

\[ J = \int_0^T \sum_{i=1}^3 (|a_{xi}|^2 + |a_{yi}|^2 + |a_{zi}|^2) \, dt \]

**Equations of Motion** in normalized coordinates:

\[
\begin{align*}
\ddot{x}_i &= 2\dot{y}_i + 3x_i + a_{xi} \\
\ddot{y}_i &= -2\dot{x}_i + a_{yi} \\
\ddot{z}_i &= z + a_{zi},
\end{align*}
\]

where \((x_i, y_i, z_i)\) is the position of the \(i\)th satellite, \(i = 1, \ldots, 3\) relative to the reference circular orbit.
Earthbound Satellites

- Note that if one substitutes from the equations of motion, then one gets a cost function that depends on second order derivatives.

- **Final Time Constraints**: Satellites should move in the known circular solutions of the equations that lie in the plane $2x + z = 0$ with line of sight radius chosen to be $R = 100\text{m}$.
Optimal Reconfiguration

3D movie insert
Optimal Reconfiguration

Earth movie insert
The optimal reconfigurations above assume, implicitly, a centralized controller with full state information.

Can formations (nearly) recover optimal trajectories through minimum transfer of information between satellites (or vehicles) when only certain avenues of communication and sensing are available?

The communication topology and information flow design are important questions.

Example: six cars asked to acquire points on a regular hexagon (relative to one another) where each car can only sense its position relative to another car and can only communicate with one other car.
Information Exchange

- Movie 1 shows the solution for the hexagon problem when the only sensing information available to each car is the relative position of the car behind.
Information Exchange

- Movie 2 shows the solution when, in addition, some message passing from cars to cars ahead of them is available. (Reasonably close to the optimal solution).
**Lyapunov Orbit Transfer**

**Key Features**

- Method of orbit transfer for the Kepler problem with thrust control, based on Lyapunov stability theory.
- Makes use of the conserved quantities of the Kepler motion; the angular momentum \( \mathbf{L} \) and the Laplace-Runge-Lenz vector \( \mathbf{A} \).
- Traditionally, orbit transfer is designed using conic-section geometry and orbital elements. No guarantee of convergence.
- The controller we design makes it possible to use continuous thrust so that we can do orbit transfer without necessarily using a large impulsive thrust.
Lyapunov Orbit Transfer

- Lyapunov-based methods are efficient, guarantees convergence and reduces to many well-known cases such as transfer between circular orbits and a one-pulse inclination change.
- Basic: $\mathbf{L}$ and $\mathbf{A}$ uniquely specify a Keplerian orbit.

**Some key points**

- Equation of the motion with control

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{||\mathbf{r}||^3} + \mathbf{F}$$

where $\mathbf{F}$ is the control force.

- Fix a Kepler orbit with a given target value $(\mathbf{L}_T, \mathbf{A}_T)$. 
Define $V : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by

$$V(L, A) = \frac{1}{2} a \| \Delta L \|^2 + \frac{1}{2} \| \Delta A \|^2,$$

where $\Delta L = L - L_T$, $\Delta A = A - A_T$ and $a > 0$.

Use the equations of motion and

$$\dot{L} = r \times F, \quad \dot{A} = F \times L + \dot{r} \times (r \times F).$$

to compute the time derivative of $V$.

Design the controller accordingly:

$$F(r, \dot{r}, t; L_T, A_T) =$$

$$- f(r, \dot{r}, t) \left( a \Delta L \times r + L \times \Delta A + (\Delta A \times \dot{r}) \times r \right)$$

with $f(r, \dot{r}, t) > 0$ suitably chosen.
Planar transfer between two coplanar circular orbits of radii 1 and 2.
Lyapunov Orbit Transfer

Orbit Transfer Movie
The End
The End

**Typesetting Software:** TeX, Textures, LaTeX, hyperref, texpower, Adobe Acrobat 4.05

**Illustrations & Movies:** Adobe Illustrator 8.0, Mathematica, MATLAB, QuickTime, and other tools

**LaTeX Slide Macro Packages:** Wendy McKay, Ross Moore