Submitted, 2009 Conference on Decision and Control (CDC)
http://www.cds.caltech.edu/~murray/papers/tpm09-cdc.html

Compositional stability analysis based on dual decomposition

Ufuk Topcu, Andrew K. Packard, and Richard M. Murray

***Abstract*— We propose a compositional stability analysis methodology for verifying properties of systems that are interconnections of multiple subsystems. The proposed method assembles stability certificates for the interconnected system based on the certificates for the input-output properties of the subsystems. The hierarchy in the analysis is achieved by utilizing dual decomposition ideas in optimization. Decoupled subproblems establish subsystem level input-output properties whereas the "master" problem imposes and updates the conditions on the subproblems toward ensuring interconnected system level stability properties. Both global stability analysis and region-of-attraction analysis are discussed.**

I. Introduction

We propose a compositional analysis framework for verifying stability properties of systems that are formed as interconnection of multiple subsystems. The method constructs certificates of input-output properties of subsystems in isolation from other subsystems and assembles stability certificates for the interconnected system based on these subsystem certificates. The assembly of system level certificates from subsystem certificates, of course, has to account for the fact that the output of a subsystem is the input of another subsystem (i.e., for the interconnection structure). More, explicitly, consider a signal w which is the output of subsystem A and at the same time the input of subsystem B. Then, in the analysis of subsystem A the "output" signal w has to satisfy all properties that are assumed for the "input" signal w in the analysis of subsystem B. Such matching conditions at the subsystem interfaces are introduced as coupling constraints. Then, decomposition of analysis is achieved by utilizing the dual decomposition techniques where violation of these coupling constraints is allowed but iteratively reduced by adapting the subsystem level analysis questions through a subgradient type optimization scheme.

In addition to dual decomposition from optimization, the proposed scheme utilizes ideas from multiple domains. First of all the subsystem level analysis builds on the dissipation inequalities and storage functions [1] to characterize input-output properties. However, local versions of these concepts, where dissipation inequalities hold over bounded subsets of the state space and verify input-output properties for certain levels of inputs norms but not necessarily for larger norm signals, are emphasized as discussed in [2], [3]. Moreover, especially the compositional analysis framework for region-of-attraction analysis estimation (introduced in section III) is inspired by the assume-guarantee type compositional techniques that have been proposed as a partial remedy for the "state explosion" problem in software verification [4], [5]. The updates of the dual variables by the subgradient algorithm can be interpreted as automated adjustments to the assumptions in assume-guarantee schemes.

The motivation for the current work stems from the computational complexity of optimization-based analysis of nonlinear dynamical systems and in particular our earlier work on sum-of-squares (SOS) optimization [6] based quantitative local analysis of systems governed by polynomial ordinary differential equations [7], [8], [9], [10]. The growth of the problem size in SOS optimization based analysis with the state dimension and the degree of the polynomial certificates (e.g. Lyapunov or storage functions) is so fast that even the complexity of problems for systems of modest state dimension exceeds the capabilities of currently available computational resources [11]. For example, Table I shows the number of decision variables (right column in each block) and the size of the matrix (left column in each block) in the semidefinite program for checking the existence of a SOS decomposition for a degree $2d$ polynomial in n variables. Moreover, local analysis leads to bilinear, non-convex optimization problems adding to the computational complexity [9]. Therefore, the use of optimization-based analysis techniques strongly depend on the improvements in the scalability of the algorithms.

Compositional analysis and design have a long history in controls and we here give a very limited list of references mostly as it ties to the focus of the current paper. The survey [12] and the volume [13] provide an exposition to the early work. Reference [14] used the primal decomposition to decouple large-scale linear matrix inequalities that appear in the distributed analysis of systems composed of different sub-units, interconnected over an arbitrary graph and [15] employed the dual decomposition in distributed optimal control.

Next section discusses the compositional analysis for global stability and section III presents extensions for the region-of-attraction analysis. These sections are followed by a simple example and concluding remarks. We emphasize that the current paper aims at a simplified exposition of and a proof of concept for the proposed methodology and detailed analysis and demonstrations are subject to current study.

Finally, in several places, a relationship between an algebraic condition on real variables and time domain signals is claimed, often using the same symbol for a particular real

This work is partially funded by the Boeing Corporation.

U. Topcu and R. M. Murray are with the Control and Dynamical Systems, California Institute of Technology, Pasadena, CA, 91125, USA utopcu, murray@cds.caltech.edu

A. K. Packard is with the Department of Mechanical Engineering, University of California, Berkeley, CA, 94720, USA pack@jagger.me.berkeley.edu

TABLE I

N_{SDP} (LEFT COLUMNS) AND N_{decision} (RIGHT COLUMNS) FOR DIFFERENT VALUES OF n AND $2d$.

n	2d = 4		6		8		10	
2	6	6	10	27	15	75	21	165
5	21	105	56	1134	126	6714	**252**	**2e4**
9	55	825	**220**	**1e4**	**715**	**2e5**	⋆	⋆
14	120	4200	**680**	⋆	⋆	⋆	⋆	⋆
16	**153**	**6936**	⋆	⋆	⋆	⋆	⋆	⋆

variable in the algebraic statement as well as the signal. This could be a source of confusion, so care on the reader's part is required.

II. Global stability analysis

This section presents the compositional analysis for the case where the subsystems satisfy input-output properties globally. Global input-output properties are relations between inputs and outputs of the system that hold independent of the "level" of the input. For example, for the input w and the output z of a system, the global $\mathcal{L}_2$-gain relation holds if there exists $\gamma > 0$ such that $\|z\|_2 \leq \gamma \|w\|_2$ for all values of $\|w\|_2$,[1] where $\|\cdot\|_2$ denotes $\mathcal{L}_2$ signal norm. Note that global $\mathcal{L}_2$-gain relations hold for stable linear systems whereas nonlinear systems may or may not satisfy such relations. That is for the input w and the output z of a nonlinear system, there may exist $\gamma > 0$ such that $\|z\|_2 \leq \gamma \|w\|_2$ for all w with $\|w\|_2 \leq R$ for some R but this gain relation may not hold for larger values of R.

For ease of presentation, we will focus the discussion on the interconnection of three subsystems shown in Figure 1. Extensions to more general system interconnections is straightforward and will be handled in future publications. In Figure 1, w_1, w_2, w_3, and w_4 denote the internal signals. Let $x_1 \in \mathcal{R}^{n_1}$, $x_2 \in \mathcal{R}^{n_2}$, and $x_3 \in \mathcal{R}^{n_3}$ be the states of the subsystems whose evolution is governed by the following differential equations with the output maps $h_1 : \mathcal{R}^{n_1} \to \mathcal{R}^{m_1}$, $h_2 : \mathcal{R}^{n_2} \to \mathcal{R}^{m_2+m_3}$, and $h_3 : \mathcal{R}^{n_3} \to \mathcal{R}^{m_4}$

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, w_2, w_3) \\ w_1 &= h_1(x_1) \\ \dot{x}_2 &= f_2(x_2, w_1) \\ (w_2, w_3) &= h_2(x_2) \\ \dot{x}_3 &= f_3(x_3, w_3) \\ w_4 &= h_3(x_3) \end{aligned} \tag{1}$$

with $f_1(0,0,0) = 0$, $h_1(0) = 0$, $f_2(0,0) = 0$, $h_2(0) = 0$, $f_3(0,0) = 0$, and $h_3(0) = 0$. That is $(x_1, x_2, x_3) = (0,0,0)$, $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ are equilibrium points of the interconnected system and respective subsystems.

The compositional stability analysis will build on the following Lyapunov-type result which provides a sufficient condition for the stability of the interconnected system around the equilibrium point at the origin.

[1] With the extra assumption that the system starts from rest where the output of the system vanishes.

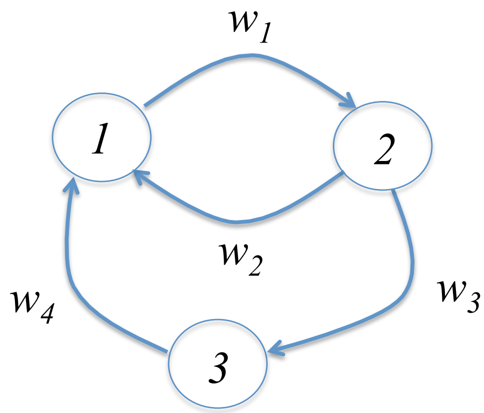

Fig. 1. Interconnection of 3 systems.

Theorem 1: If there exists a continuously differentiable positive definite function $V : \mathcal{R}^{n_1+n_2+n_3} \to \mathcal{R}$ such that $V(0) = 0$ and $\dot{V}(x_1, x_2, x_3) < 0$ for all nonzero $(x_1, x_2, x_3) \in \mathcal{R}^{n_1+n_2+n_3}$, then the system interconnection in Figure 1 is internally asymptotically stable around $(x_1, x_2, x_3) = (0,0,0)$. ◁

In the above theorem and hereafter, for a map $x \to V(x)$, $\dot{V}(x)$ denotes $\nabla V(x)\dot{x}$.

The goal is to construct a Lyapunov function (i.e., a function that satisfies the conditions in Theorem 1) to verify the internal stability of the interconnected system but through isolated analysis of the input-output properties of subsystems 1, 2, and 3. To this end, we resort to Willems' dissipation inequalities theory [1] and use the following proposition to obtain sufficient conditions for asymptotic stability.

Proposition 1: If there exist continuously differentiable positive definite functions $V_1 : \mathcal{R}^{n_1} \to \mathcal{R}$, $V_2 : \mathcal{R}^{n_2} \to \mathcal{R}$, and $V_3 : \mathcal{R}^{n_3} \to \mathcal{R}$ such that $V_1(0) = 0$, $V_2(0) = 0$, and $V_3(0) = 0$ and positive real numbers γ_{11}, γ_{12}, γ_{14}, γ_{21}, γ_{22}, γ_{23}, γ_{33}, and γ_{34} such that

$$\begin{aligned} \dot{V}_1(x_1, w_2, w_4) &< -\gamma_{11} w_1^T w_1 + \gamma_{12} w_2^T w_2 + \gamma_{14} w_4^T w_4 \\ &= -\gamma_{11} h_1(x_1)^T h_1(x_1) + \gamma_{12} w_2^T w_2 + \gamma_{14} w_4^T w_4 \\ &\quad \text{for all } \; w_2, w_4, \text{ and nonzero } x_1 \end{aligned} \tag{2a}$$

$$\begin{aligned} \dot{V}_2(x_2, w_1) &< -\gamma_{22} w_2^T w_2 - \gamma_{23} w_3^T w_3 + \gamma_{21} w_1^T w_1 \\ &= -h_2(x_2)^T \begin{bmatrix} \gamma_{22} & 0 \\ 0 & \gamma_{23} \end{bmatrix} h_2(x_2) + \gamma_{21} w_1^T w_1 \\ &\quad \text{for all } \; w_1, \text{ and nonzero } x_2 \end{aligned} \tag{2b}$$

$$\begin{aligned} \dot{V}_3(x_3, w_3) &< -\gamma_{34} w_4^T w_4 + \gamma_{33} w_3^T w_3 \\ &= -\gamma_{34} h_3(x_3)^T h_3(x_3) + \gamma_{33} w_3^T w_3 \\ &\quad \text{for all } \; w_3 \text{ and nonzero } \; x_3 \end{aligned} \tag{2c}$$

and

$$\begin{aligned} -\gamma_{11} + \gamma_{21} &\leq 0 \\ -\gamma_{22} + \gamma_{12} &\leq 0 \\ -\gamma_{23} + \gamma_{33} &\leq 0 \\ -\gamma_{34} + \gamma_{14} &\leq 0, \end{aligned} \tag{3}$$

then $(x_1, x_2, x_3) = (0,0,0)$ is an asymptotically stable equilibrium point of the interconnected system shown in Figure 1. ◁

Remarks 1: Note that the conditions in (2) and (3) are homogenous in the decision variables in V_i's and γ_{ij}'s (i.e., one can choose γ_{ij}'s arbitrarily by properly scaling V_i's). This can be avoided by setting some of the γ_{ij}'s (for example γ_{11}, γ_{21}, and γ_{34}) to 1. By such normalization, the number constraints in (3) can be reduced. However, we don't employ this normalization in order to keep the conditions in (2) and (3) notationally symmetric. Instead, we avoid this homogeneity by properly normalizing one of the decision variables in each V_i. $\triangleleft$

Proof: (of Proposition 1) Let $V : \mathcal{R}^{n_1+n_2+n_3} \to \mathcal{R}$ be defined through $V(x_1, x_2, x_3) = V_1(x_1) + V_2(x_2) + V_3(x_3)$ and note that $V(x_1, x_2, x_3) > 0$ for all nonzero (x_1, x_2, x_3) and $V(0, 0, 0) = 0$. Then, $\dot{V}(x_1, x_2, x_3) < (-\gamma_{11} + \gamma_{21})w_1^T w_1 + (-\gamma_{22} + \gamma_{12})w_2^T w_2 + (-\gamma_{23} + \gamma_{33})w_3^T w_3 + (-\gamma_{34} + \gamma_{14})w_4^T w_4 \leq 0$ where the first inequality follows from (2) and the second from (3). Consequently, V satisfies the conditions in Theorem 1 and the closed-loop system is asymptotically stable around the origin. ■

The inequalities in (2) are dissipation inequalities with the quadratic supply rates $-\gamma_{11}w_1^T w_1 + \gamma_{12}w_2^T w_2 + \gamma_{14}w_4^T w_4$, $-\gamma_{22}w_2^T w_2 - \gamma_{23}w_3^T w_3 + \gamma_{21}w_1^T w_1$, and $-\gamma_{34}w_4^T w_4 + \gamma_{33}w_3^T w_3$, respectively [1]. In fact, conditions in (2) are slight generalizations of the dissipation inequalities corresponding to $\mathcal{L}_2$-gain inequalities. V_1, V_2, and V_3 are called the storage functions associated with the respective subsystems and the supply rates. Roughly speaking conditions in (2) constrain the (weighted) norms of the outputs of the subsystems in terms of the (weighted) norms of the inputs and conditions in (3) ensure the internal stability of the interconnection of these subsystems. Proposition 1 can be considered as a generalization of the small-gain theorem [16], [17] for the interconnection of multiple subsystems (with specified governing equations in (1) and "gain" relations in (2)).

One would typically use a specific finite (linear) parameterization for V_i's (e.g. quadratic or polynomial functions) and search for V_i's and γ_{ij}'s satisfying the conditions in (2) and (3) through numerical optimization (or feasibility search). For example, for linear dynamics and output maps and quadratic parameterizations for the storage functions conditions in (2) and (3) lead to standard linear matrix inequality (LMI) constraints [18] and for polynomial dynamics and storage functions this search can be performed through sum-of-squares programming [6]. Note that conditions in (2) are only coupled through γ_{ij}'s. Typically, the number of decision variables in V_i's[2] dominates the total number of decision variables in the corresponding numerical optimization problem for the search of V_i's and γ_{ij}'s that satisfy the conditions in Proposition 1.

Toward decoupling the conditions in (2), consider the following related optimization problem:

[2]One of the main difficulties of sum-of-squares programming is that the number of the extra decision variables that are introduced to parameterize the Gram matrices corresponding to the polynomial inequalities rapidly grows with the state dimension.

$$\begin{array}{ll} \max\limits_{\lambda_1,\ldots,\lambda_4\geq 0} \min\limits_{\gamma_{ij}\text{'s},\ V_i\text{'s}} & \lambda_1(\gamma_{21}-\gamma_{11}) + \lambda_2(\gamma_{12}-\gamma_{22}) \\ & \quad + \lambda_3(\gamma_{33}-\gamma_{23}) + \lambda_4(\gamma_{14}-\gamma_{34}) \\ \text{subject to} & \text{constraints in (2)} \\ & V_1,\ V_2,\ V_3 \text{ positive definite and} \\ & V_1(0)=0,\ V_2(0)=0,\ V_3(0)=0. \end{array} \tag{4}$$

and re-write the problem in (4) as

$$\max_{\lambda_1,\ldots,\lambda_4\geq 0} \ \varphi_1(\lambda) + \varphi_2(\lambda) + \varphi_3(\lambda), \tag{5}$$

where

$$\begin{array}{lll} \varphi_1(\lambda) := & \min\limits_{\gamma_{11},\gamma_{12},\gamma_{14},\ V_1} & -\lambda_1\gamma_{11} + \lambda_2\gamma_{12} + \lambda_4\gamma_{14} \\ & \text{subject to} & \text{constraints in } (2a) \\ & & V_1 \text{ positive definite,} \\ & & V_1(0)=0,\ \gamma_{11},\gamma_{12},\gamma_{14} > 0 \end{array} \tag{6a}$$

$$\begin{array}{lll} \varphi_2(\lambda) := & \min\limits_{\gamma_{22},\gamma_{23},\gamma_{21},\ V_2} & \lambda_1\gamma_{21} - \lambda_2\gamma_{22} - \lambda_3\gamma_{23} \\ & \text{subject to} & \text{constraints in } (2b) \\ & & V_2 \text{ positive definite} \\ & & V_2(0)=0,\ \gamma_{22},\gamma_{23},\gamma_{21} > 0 \end{array} \tag{6b}$$

$$\begin{array}{lll} \varphi_3(\lambda) := & \min\limits_{\gamma_{33},\gamma_{34},\ V_3} & \lambda_3\gamma_{33} - \lambda_4\gamma_{34} \\ & \text{subject to} & \text{constraints in } (2c) \\ & & V_3 \text{ positive definite} \\ & & V_3(0)=0,\ \gamma_{33},\gamma_{34} > 0. \end{array} \tag{6c}$$

For given λ, one can compute $\varphi_1(\lambda)$, $\varphi_2(\lambda)$, and $\varphi_3(\lambda)$ by solving (6a)-(6c) independently and if the optimizing value of γ_{ij}'s satisfy the inequality constraints in (3), then $V := V_1 + V_2 + V_3$ satisfies the conditions in Proposition 1 and the internal stability of the interconnected system is certified. If the optimizing values of γ_{ij}'s do not satisfy the inequality constraints in (3), then λ needs to be updated. Following [19], we attempt to solve (5) using a subgradient algorithm. Define $\lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and, for given λ, let $\gamma_{ij}^*(\lambda)$ denote the optimal value of γ_{ij} in (6a)-(6c). Let $g_1(\lambda) = (-\gamma_{11}^*(\lambda), \gamma_{12}^*(\lambda), 0, \gamma_{14}^*(\lambda))^T$, $g_2(\lambda) = (\gamma_{21}^*(\lambda), -\gamma_{22}^*(\lambda), -\gamma_{23}^*(\lambda), 0)^T$, and $g_3(\lambda) = (0, 0, \gamma_{33}^*(\lambda), -\gamma_{34}^*(\lambda))^T$. Then, by the following inequalities that hold for all $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$

$$\begin{aligned} \varphi_1(\mu) \geq \varphi_1(\lambda) &- \gamma_{11}^*(\lambda)(\mu_1 - \lambda_1) \\ &+ \gamma_{12}^*(\lambda)(\mu_2 - \lambda_2) + \gamma_{14}^*(\lambda)(\mu_4 - \lambda_4) \end{aligned}$$

$$\begin{aligned} \varphi_2(\mu) \geq \varphi_2(\lambda) &+ \gamma_{21}^*(\lambda)(\mu_1 - \lambda_1) \\ &- \gamma_{22}^*(\lambda)(\mu_2 - \lambda_2) - \gamma_{23}^*(\lambda)(\mu_3 - \lambda_3) \end{aligned}$$

$$\varphi_3(\mu) \geq \varphi_3(\lambda) + \gamma_{33}^*(\lambda)(\mu_3 - \lambda_3) - \gamma_{34}^*(\lambda)(\mu_4 - \lambda_4),$$

$g_1(\lambda)$, $g_2(\lambda)$, and $g_3(\lambda)$ are subgradients of φ_1, φ_2, and φ_3 at λ, respectively. Consequently, $g_1(\lambda) + g_2(\lambda) + g_3(\lambda)$ is a subgradient of φ at λ. Then, the subgradient method solves the problem in (4) iteratively by updating λ at iteration k for $\ell = 1, \ldots, 4$ using

$$\lambda_\ell^{k+1} = \lambda_\ell^k - \alpha^k \delta_\ell(\lambda^k)_\ell^k$$

where

$$\delta_1(\lambda^k) = \gamma_{11}^*(\lambda^k) - \gamma_{21}^*(\lambda^k) \quad (7a)$$
$$\delta_2(\lambda^k) = \gamma_{22}^*(\lambda^k) - \gamma_{12}^*(\lambda^k) \quad (7b)$$
$$\delta_3(\lambda^k) = \gamma_{23}^*(\lambda^k) - \gamma_{33}^*(\lambda^k) \quad (7c)$$
$$\delta_4(\lambda^k) = \gamma_{34}^*(\lambda^k) - \gamma_{14}^*(\lambda^k) \quad (7d)$$
$$(7e)$$

and $\alpha^k > 0$ denotes the step size at iteration k.

Consider now that at iteration k one of the constraints in (3), say $-\gamma_{11}^*(\lambda^k) + \gamma_{21}^*(\lambda^k) \leq 0$, is violated. Then,

$$\lambda_1^{k+1} = \lambda_1^k - \alpha^k \delta_1(\lambda^k) = \lambda_1^k - \alpha^k \left(\gamma_{11}^*(\lambda^k) - \gamma_{21}^*(\lambda^k)\right) > \lambda_1^k.$$

Consequently, if $-\gamma_{11}^*(\lambda^k) + \gamma_{21}^*(\lambda^k) > 0$, then the update rule (7) increases λ_1 (the dual variable corresponding to the constraint $-\gamma_{11} + \gamma_{21} \leq 0$) and this puts a larger penalty on the violation of the constraint $-\gamma_{11} + \gamma_{21} \leq 0$. If more than one of the constraints in (3) are violated, then the increase in the dual variables corresponding to these violated constraints is proportional to the amount of violation, i.e., the weight of the penalty on the largest violation increases at most from iteration k to iteration $k+1$. More generally, the problem in (4) penalizes the violations in the constraints in (3) and then the subgradient algorithms adjusts the level of penalty with an adjustment proportional to the relative size of violation of the "dualized" constraints, i.e., constraints in (3).

Note that $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ and therefore (for a given λ) the objective of the optimization problem in (6a) is to minimize a linear combination of γ_{12} and γ_{14} with respect to γ_{11}. Referring back to the condition in (2a), the subproblem in (6a) tries to minimize the gain from a weighted $\mathcal{L}_2$-norm of the input signals (w_2 and w_4) of subsystem 1 to the $\mathcal{L}_2$-norm of the output w_1. Similar relations can be established between the problems in (6b) and (6c) and the conditions (2b) and (2c), respectively. Hence, one reaches at an interpretation of the dual variables parallel to the "price" interpretation, for example, in network analysis [20]: the master problem sets the "prices" for the subsystems for deviating from the system level analysis objectives and given this "price" subproblems try to compute an optimal estimate for the subsystem level input-output properties.

Remarks 2: The procedure that led to the structure with the "master" problem in (5) and the decoupled subproblems in (6a)-(6c) is called the *dual decomposition* [19], [21] and has been used in many engineering applications including communication networks [20] and more recently in distributed optimal control [15]. ◁

Remarks 3: The dual decomposition based procedure employed in this section is not limited to three subsystem configuration in Figure 1 and straightforwardly generalizes for more than three subsystems. In this case, the conditions in (2) are extended to include a dissipation inequality for each subsystem and the conditions in (3) are extended to include all conditions of the form $-\gamma_{jk} + \gamma_{ik} \leq 0$ whenever w_k is an input signal to subsystem i and an output signal of subsystem j (with $i \neq j$). ◁

III. Region-of-attraction analysis

This section is devoted to a brief discussion on compositional local stability analysis. Building on the following characterization of invariant subsets of the region-of-attraction around the origin, we propose sufficient conditions that enable the construction of a Lyapunov function using local dissipation inequalities for the subsystems.

Lemma 1: COnsider a system governed by $\dot{x} = f(x)$ with $f(0) = 0$ and f locally Lipschitz. Let $R \in \mathcal{R}$ be nonnegative. If there exists a continuously differentiable function $V : \mathcal{R}^n \to \mathcal{R}$ such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0 \quad (8)$$
$$\Omega_{V,R} := \{x \in \mathcal{R}^n \ : \ V(x) \leq R\} \text{ is bounded}, \quad (9)$$
$$\Omega_{V,R} \setminus \{0\} \subset \{ x \in \mathcal{R}^n \ : \ \nabla V(x) f(x) < 0\}, \quad (10)$$

then $\Omega_{V,R}$ is an invariant subset of the region-of-attraction around the origin. ◁

Proposition 2: If there exist continuously differentiable positive definite functions $V_1 : \mathcal{R}^{n_1} \to \mathcal{R}$, $V_2 : \mathcal{R}^{n_2} \to \mathcal{R}$, and $V_3 : \mathcal{R}^{n_3} \to \mathcal{R}$ such that $V_1(0) = 0$, $V_2(0) = 0$, and $V_3(0) = 0$, positive real numbers γ_{11}, γ_{12}, γ_{14}, γ_{21}, γ_{22}, γ_{23}, γ_{33}, and γ_{34}, and nonnegative real numbers R_1, R_1', R_2, R_2', R_3, and R_3' such that Ω_{V_1,R_1}, Ω_{V_2,R_2}, and Ω_{V_3,R_3} are bounded,

$$\begin{aligned} \dot{V}_1(x_1, w_2, w_4) &< -\gamma_{11} w_1^T w_1 + \gamma_{12} w_2^T w_2 + \gamma_{14} w_4^T w_4 \\ &\text{for all } \ w_2, w_4, \text{ and nonzero } x_1 \text{ s.t.} \quad (11a) \\ &V_1(x_1) \leq R_1 \text{ and } \|h_1(x_1)\|_2^2 \leq R_1' \\ \dot{V}_2(x_2, w_1) &< -\gamma_{22} w_2^T w_2 - \gamma_{23} w_3^T w_3 + \gamma_{21} w_1^T w_1 \\ &\text{for all } \ w_1, \text{ and nonzero } x_2 \text{ s.t.} \quad (11b) \\ &V_2(x_2) \leq R_2 \text{ and } \|h_2(x_2)\|_2^2 \leq R_2' \\ \dot{V}_3(x_3, w_3) &< -\gamma_{34} w_4^T w_4 + \gamma_{33} w_3^T w_3 \\ &\text{for all } \ w_3 \text{ and nonzero } \ x_3 \text{ s.t.} \quad (11c) \\ &V_3(x_3) \leq R_3 \text{ and } \|h_3(x_3)\|_2^2 \leq R_3' \end{aligned}$$

and

$$\begin{aligned} -\gamma_{11} + \gamma_{21} &\leq 0 \\ -\gamma_{22} + \gamma_{12} &\leq 0 \\ -\gamma_{23} + \gamma_{33} &\leq 0 \\ -\gamma_{34} + \gamma_{14} &\leq 0 \\ R_2' + R_3' &\leq R_1 \\ R_1' &\leq R_2 \\ R_2' &\leq R_3 \end{aligned} \quad (12)$$

then $\Omega_{V_1+V_2+V_3, \min\{R_1,R_1',R_2,R_2',R_3,R_3'\}}$ is an invariant subset of the region-of-attraction around the origin for the interconnected system shown in Figure 1. ◁

Proof: Let $V := V_1 + V_2 + V_3$ and $R := \min\{R_1, R_1', R_2, R_2', R_3, R_3'\}$. Then, V is positive definite and vanishes at the origin. Moreover, $\Omega_{V,R}$ is bounded and $\dot{V}(x_1, x_2, x_3) < 0$ whenever $x \in \Omega_{V,R}$. Consequently, Proposition 2 follows from Lemma 1 ■

The dual decomposition procedure from the previous section can be applied to decouple the constraints in (11) by penalizing the violations of the constraints in (12). Let us introduce the dual variables λ_1, λ_2, λ_3, and λ_4, as before, to dualize the constraints on γ_{ij} in (12) and the extra dual variables μ_1, μ_2, and μ_3 for the remaining constraints in (12). Then, the master problem is written as

$$\max_{\lambda_1,\ldots,\lambda_4,\mu_1,\mu_2,\mu_3 \geq 0} \quad \phi_1(\lambda,\mu) + \phi_2(\lambda,\mu) + \phi_3(\lambda,\mu), \quad (13)$$

where ϕ_1, ϕ_2, ϕ_3 are solutions to the following three problems respectively.

$$\begin{array}{cc} \min\limits_{\gamma_{11},\gamma_{12},\gamma_{14},R_1,R_1',\ V_1} & \begin{array}{r} -\lambda_1\gamma_{11} + \lambda_2\gamma_{12} + \lambda_4\gamma_{14} \\ -\mu_1 R_1 + \mu_2 R_1' \end{array} \\ \text{subject to} & \text{conditions in Prop 2 excluding (12)} \\ \min\limits_{\gamma_{21},\gamma_{22},\gamma_{23},R_2,R_2',\ V_2} & \begin{array}{r} \lambda_1\gamma_{21} - \lambda_2\gamma_{22} - \lambda_3\gamma_{23} \\ +\mu_1 R_2' - \mu_2 R_2 + \mu_3 R_2' \end{array} \\ \text{subject to} & \text{conditions in Prop 2 excluding (12)} \\ \min\limits_{\gamma_{33},\gamma_{34},R_3,R_3',\ V_3} & \lambda_3\gamma_{33} - \lambda_4\gamma_{34} + \mu_1 R_3' - \mu R_3 \\ \text{subject to} & \text{conditions in Prop 2 excluding (12).} \end{array}$$

The rules to update $\lambda_1, \ldots, \lambda_4, \mu_1, \ldots, \mu_3$ can be adapted similar to the development in section II.

Conditions in (11) are local dissipation inequalities, i.e., they hold in certain subsets (Ω_{V_i,R_i}) of the state space. If the subsystem i starts from rest, then these conditions ensure that the corresponding weighted $\mathcal{L}_2$-gain relations hold between the inputs and outputs if the inputs to subsystem i have an $\mathcal{L}_2$-norm less than or equal to $\sqrt{R_i}$. On the other hand, recall that every signal w_k is an input to subsystem i and at the same time an output of subsystem j. Therefore, the constraint on $\|w_k\|_2 = \|h_j(x_j)\|_2$ in (2) ensures that w_k does not violate the assumptions made on the input norm at subsystem i. For example, the $\mathcal{L}_2$-norm of w_1 as an output of subsystem 1 cannot exceed that as an input to subsystem 2 (i.e., $R_1' \leq R_2$). These conditions are motivated by *assume-guarantee* type compositional analysis ideas [4], [5] and the dual decomposition based hierarchical scheme automates the adjustment of assumptions.

IV. Example

For $i = 1, 2, 3$, let the state evolution and outputs of the subsystems be governed by

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} x_1 + I_2 \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} \\ w_1 &= 0.5 \begin{bmatrix} 1 & 1 \end{bmatrix} x_1 \\ \dot{x}_2 &= \begin{bmatrix} -8 & 0 \\ 12 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_1 \\ \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} &= 0.5 I_2 x_2 \\ \dot{x}_3 &= \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_3 \\ w_4 &= 0.4 \begin{bmatrix} 1 & 1 \end{bmatrix} x_3 \end{aligned}$$

We apply the compositional analysis to verify the global asymptotic stability of the interconnected system. Let the step size in the subgradient algorithm be

$$\alpha^k = \frac{0.01}{10+k}.$$

In this case, the optimization problems in (6a)-(6c) are LMIs with the decision variables γ_{ij}'s and those in V_i's. The dual variables are initialized at $\lambda_1^0 = 0$, $\lambda_2^0 = 0$, $\lambda_3^0 = 0$, and $\lambda_4^0 = 0$. Figure 2 shows the violations in the constraints on the γ_{ij}'s (negative value means that the corresponding constraint is not violated). The subgradient iterations are terminated when all constraints are satisfied. Figure 3 shows the values of the dual variables λ_i's versus the the iteration number and Figure 4 shows the value of the objective function in (4) versus the iteration number.

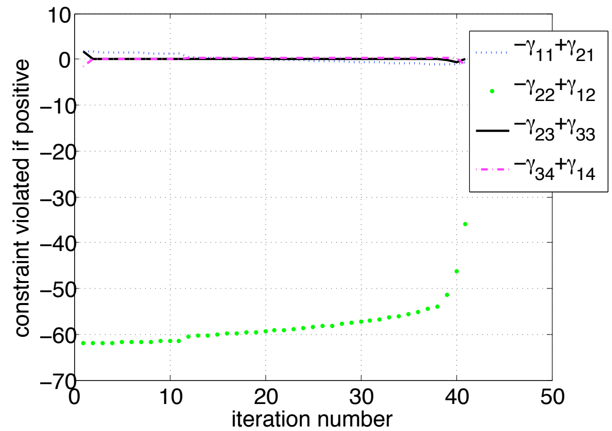

Fig. 2. Violations of the constraints on γ_{ij}'s. Negative value indicates that the corresponding constraint is satisfied.

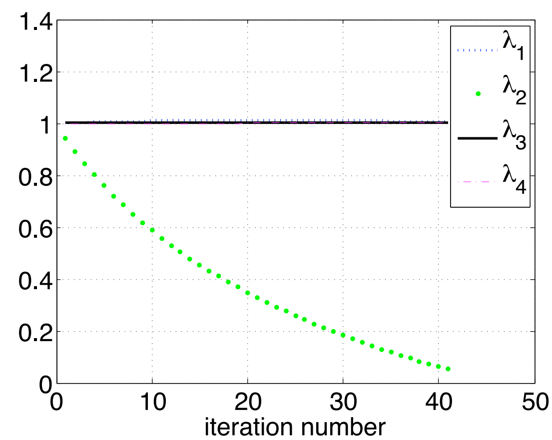

Fig. 3. The dual variable (λ_i'ss) versus the iteration number. The correspondence between the dual variables and the constraints is as in (4).

V. Conclusions and Critique

We proposed a compositional stability analysis methodology for verifying properties of systems that are interconnections of multiple subsystems. The proposed method assembles stability certificates for the interconnected system based on the certificates for the input-output properties of the subsystems. The hierarchy in the analysis is achieved by utilizing dual decomposition ideas in optimization. Decoupled subproblems establish subsystem level input-output properties whereas the "master" problem imposes and updates the conditions on the subproblems toward ensuring interconnected system level stability properties. Both global stability analysis and region-of-attraction analysis were dis-

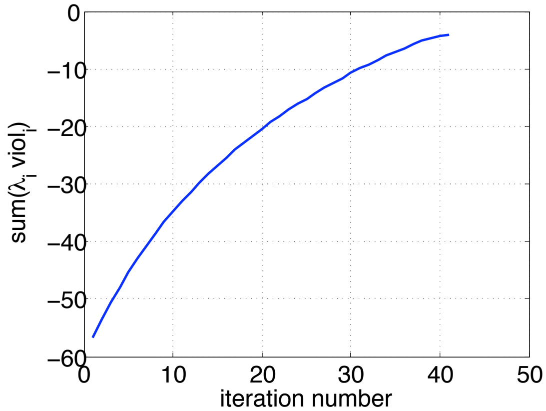

Fig. 4. The value of the objective function in (4) versus the iteration number.

cussed.

There are a series of limitations and also possible extensions of the method proposed here. First of all, the decomposition in analysis is achieved through a fixed decomposition of the sufficient conditions for (local and global) stability based on the specific choice of supply rates, i.e., $\mathcal{L}_2$-gain relations, (in (2) and (11)) for the subsystems. It may be possible to reduce the conservatism associated with these specific choices by exploring optimization over the choice of supply rates. Also note that the proposed method is inherently more conservative than a centralized search (for the overall closed-loop dynamics) for a general Lyapunov functions and even a Lyapunov function of the form $V(x) = \sum_{i=1}^{N} V_i(x_i)$ (with N being the number of subsystems).

The convergence of the subgradient based optimization schemes is known to be slow. Therefore, it may be of interest to both theoretically and practically investigate the convergence properties of the proposed scheme. Note that we have presented the compositional analysis methodology for a specific interconnection structure. Although the extensions to larger number of subsystems with a general interconnection structure are straightforward, the effect of these extensions on convergence is to be examined. Nevertheless, we emphasize that one of the main difficulties in solving large-scale semidefinite programs is the memory requirements of the interior-point type algorithms [11]. Therefore, compositional analysis may be the sole option (even if it converges slowly) for certain systems for which solving large-scale semidefinite programming problems corresponding to system level certificates is not practical.

We have only considered stability analysis (i.e., no exogenous signals). It may be possible to extend the framework to identify system level input-output properties (in the presence of exogenous inputs and outputs).

Finally, we note that the current manuscript aims at a simplified exposition of and a proof of concept for a compositional stability analysis framework and it is far from being complete. The final submission will include more realistic examples including those for the local analysis as well as more detailed discussions on the limitations and possible extensions pointed above.

References

[1] J. C. Willems, "Dissipative dynamical systems I: Genaral theory," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–343, 1972.

[2] W. Tan, A. Packard, and T. Wheeler, "Local gain analysis of nonlinear systems," in *Proc. American Control Conf.*, Minneapolis, MN, 2006, pp. 92–96.

[3] U. Topcu, A. Packard, P. Seiler, and G. Balas, "Stability region estimation for systems with unmodeled dynamics," in *Proc. European Control Conf.*, 2009, under review.

[4] J. M. Cobleigh, D. Giannakopoulou, and C. S. Pasareanu, "Learning assumptions for compositional verification." Springer-Verlag, 2003, pp. 331–346.

[5] L. Alfaro and T. A. Henzinger, "Interface theories for component-based design," in *EMSOFT '01: Proceedings of the First International Workshop on Embedded Software*. Springer-Verlag, 2001, pp. 148–165.

[6] P. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming Series B*, vol. 96, no. 2, pp. 293–320, 2003.

[7] W. Tan and A. Packard, "Stability region analysis using polynomial and composite polynomial Lyapunov functions and sum-of-squares programming," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 565–571, 2008.

[8] U. Topcu and A. Packard, "Local stability analysis for uncertain nonlinear systems," 2009, to appear in the IEEE Transactions on Automatic Control.

[9] U. Topcu, A. Packard, and P. Seiler, "Local stability analysis using simulations and sum-of-squares programming," *Automatica*, vol. 44, pp. 2669 – 2675, 2008.

[10] U. Topcu, A. Packard, P. Seiler, and G. Balas, "Robust region-of-attraction estimation," 2008, submitted to IEEE Transaction on Automatica Control.

[11] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. MPS-SIAM Series on Optimization, 2001.

[12] J. N. R. Sandell, P. Varaiya, M. Athans, and M. G. Safonov, "Survey of decentralized control methods for large scale systems," *IEEE Transactions on Automatic Control*, vol. 23, no. 2, pp. 108–128, 1978.

[13] D. D. Siljak, *Decentralized Control of Complex Systems*. Academic Press, 1991.

[14] C. L. ad X. Lin, R. D'Andrea, and S. Boyd, "A decomposition approach to distributed analysis of networked systems," in *Proc. Conf. on Decision and Control*, 2004, pp. 3980–3985.

[15] A. Rantzer, "Dynamic dual decomposition for distributed control," in *Proceedings of American Control Conference*, 2009, to appear.

[16] H. K. Khalil, *Nonlinear Systems*, 3^{rd} ed. Prentice Hall, 2002.

[17] M. Vidyasagar, *Nonlinear Systems Analysis*, 2^{nd} ed. Prentice Hall, 1993.

[18] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia: SIAM, 1994.

[19] D. P. Bertsekas, *Nonlinear Programming*, 2nd ed. Athena Scientific, 1999.

[20] F. Kelly, A. Maulloo, and D. Tan, "Rate control for communication networks: shadow prices, proportional fairness, and stability," *Journal of the Operational Research Society*, vol. 49, pp. 237–252, 1997.

[21] S. Boyd, L. Xiao, A. Mutapcic, and J. Mattingley, "Notes on decomposition methods," 2007, available at http://www.stanford.edu/class/ee364b/notes/decomposition_notes.pdf.