Stability and Performance of Non-Homogeneous Multi-Agent Systems on a Graph

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Abstract: This paper considers distributed control of interconnected multi-agent systems. The dynamics of the individual agents are not required to be homogeneous and the interaction topology is described by an arbitrary directed graph. We derive the sensitivity transfer functions between every pair of agents and we analyze stability and performance of non-homogeneous systems, showing that the low frequency behavior is influenced not only by topology, but also by static gain and poles of the agents.

1. INTRODUCTION

In numerous mission scenarios, the concept of a group of agents cooperating to achieve a determined goal is very attractive when compared with the solution of one single vehicle. In this class of systems, even if the agents are dynamically decoupled, they are coupled through the common task they have to achieve. When the number of agents grows, centralized control is no longer feasible and distributed control techniques become attractive. Applications of coordinated control of multiple vehicles can be found in many fields, including microsatellite clusters (Burns et al. [2000], Kapilal et al. [1999]), formation flying of unmanned aerial vehicles (Wolfe et al. [1996]), automated highway systems (Swaroop and Hedrick [1999]) and mobile robotics (Yamaguchi et al. [2001]).

The problem of distributed control has been widely studied with tools from graph theory (Corfmat and Morse [1976], Šiljak [1991], Mesbahi and Hadaegh [2001]). We consider agents with non-homogeneous linear dynamics and we model the interconnection topology as a graph, in which the single agents are represented by a vertex, while the interaction links are the arcs.

The distributed control problem has been handled in different ways and with different tools: dissipative theory and linear matrix inequalities in Langbort et al. [2004], edge agreement in Zelazo et al. [2007, 2008], linear quadratic regulator in Borrelli and Keviczky [2008], decomposition and linear matrix inequalities in Massioni and Verhaegen [2009]. In almost all the works mentioned above the control is applied to homogeneous agents interconnected by undirected graphs. If the graph is undirected the problem becomes easier because all the matrices associated with the graph, like the Laplacian, are symmetric.

One approach to distributed control is to use leader-follower arrangement. This approach is well studied and representative papers exploring graph-theoretic ideas in the context of a leader-follower architecture include Mesbahi and Hadaegh [2001] and Jin [2007], where a double-graph control strategy was proposed. This topology represents a particular case, where the leader has a more important role than the other agents and this may not always be desirable.

The importance of cycles in distributed control has already been pointed out in several past works: Zelazo et al. [2007, 2008] investigated the role of cycles and trees in the edge Laplacian for the edge agreement problem, while Fax and Murray [2004] suggested a relation between the presence of cycles and the stability of formation. In Liu et al. [2009] gains over graph cycles are involved in stability conditions for nonlinear network models.

Limits on multi-agent systems performance have already been studied in Barooah and J. [2007] and Bamieh et al. [2009], showing that a global information, such as leader's position or state, is needed to achieve reasonable performance.

In Tonetti and Murray [2010], we have considered only systems with the same identical dynamics $P(s)$ and local controller $C(s)$. However in the most of the distributed physical systems this is only an approximation. We can just think for example to formation of satellites in different orbits, robotics vehicles with different dynamics, internet routers and peer to peer systems. Even if there are agents with different dynamics, the analysis in Tonetti and Murray [2010] still holds, as long as the open loop transfer function $L(s)$ is the same for all the vehicles. This means to shape the controller in order to have $C_i(s) = L(s)/P_i(s)$ for every agent i. But this could not always be feasible. It becomes therefore important to develop results also for non-homogeneous systems. In literature we can find few papers dealing with distributed control of heterogeneous systems: Langbort and D'Andrea [2003], Dullerud and D'Andrea [2004]. In Dunbar and Murray [2006] a distributed receding horizon control for the stabilization of multi-vehicle formation is proposed, where the dynamics is not required to be homogeneous. In Motee and Jadbabaie [2008] the structural properties of optimal control of spatially distributed systems is studied and in Rice and Verhaegen [2009] a distributed control for spatially heterogeneous linear systems is considered.

In the present paper we investigate how our previous results obtained for homogeneous agents can be extended to heterogeneous systems. The contribution of this work is to show a general method to derive the transfer functions between any pair of agents with different dynamics, where the interconnection topology is described by arbitrary directed graphs. We study the stability of special multi-agent systems, where a separation principle is applicable. We analyze mechanisms that rule the behavior of a non-homogeneous multi-agent system and we show intrinsic limits on the controller design due not only to the topology, but also to static gain and poles of the open loop transfer function of each agent.

The current paper is organized as follows. In section 2 we briefly review the principal concepts of graph theory, stability and performance of homogeneous systems. Section 3 presents the sensitivity transfer function for non-homogeneous systems, while in Sections 4 and 5 stability and performance are discussed, respectively. The conclusions of the paper are reported in Section 6.

2. PRELIMINARIES

In this section we summarize some of the key concepts from graph theory, stability and performance of homogeneous interconnected multi-agent systems that will be used in the paper.

2.1 Graph theory

A *directed graph* $\mathcal{G}$ is a set of vertices or nodes V and a set of arcs $A \subset V^2$ whose elements $a = (u, v) \in A$ characterize the relation between distinct pairs of vertices $u, v \in V$. For an arc (u, v) we call u the *tail* and v the *head*. The *in(out)degree* of a vertex v is the number of arcs with v as its head (tail). A *directed path* in a graph is a sequence of vertices such that from each of its vertices there is an arc to the next vertex in the sequence. A directed path with no repeated vertices is called a *simple directed path*. A directed graph is called *strongly connected* if there is a directed path from each vertex in the graph to every other vertex. A directed graph is *weakly connected* if every vertex can be reached from every other but not necessarily following the directions of the arcs. A *complete directed graph* is a graph where each pair of vertices has an arc connecting them. A *simple cycle* is a closed path that is self-avoiding (does not revisit nodes, other than the first). A *acyclic directed graph* is a directed graph without cycles.

The structure of a graph can be described by appropriate matrices. The *normalized Laplacian matrix* $\mathcal{L}$ of a directed graph $\mathcal{G}$ is a square matrix of size $|V|$, defined by $\mathcal{L}_{ij} = 1$ if $i = j$, $\mathcal{L}_{ij} = 1/d_i$ if $(i, j) \in A$, where d_i is the outdegree of the ith vertex, $\mathcal{L}_{ij} = 0$ otherwise.

A more detailed presentation of graph theory can be found in Tutte [2005].

2.2 Stability of homogeneous systems

We consider a formation of N agents with identical linear dynamics. The normalized Laplacian matrix $\mathcal{L}$ of the graph is used to represent the interaction topology. Suppose each individual agent is a SISO system with local loop composed of a local controller $C(s)$ and a plant model $P(s)$. According to Fax and Murray [2004], the multi-agent system is stable if and only if the net encirclement of the critical points $-\lambda_i^{-1}(\mathcal{L})$ by the Nyquist plot of $P(s)C(s)$ is zero for all nonzero $\lambda_i(\mathcal{L})$, where $\lambda_i(\mathcal{L})$ are the eigenvalues of the normalized Laplacian matrix $\mathcal{L}$ of the graph.

2.3 Performance of homogeneous systems

The *Laplacian weight* of a simple directed path of length k from i to j, where $i = i_0,\ i_1, \ldots, i_k = j$, is the product of the negative inverse of the outdegrees d of all the nodes in the path besides the last one:

$$\mathcal{L}w^k_{i_0 i_k} := \operatorname{sgn}(k) \prod_{t=0}^{t=k-1} \left(-\frac{1}{d_{i_t}}\right), \qquad (1)$$

where $\operatorname{sgn}(k) = -1$ if k is odd, $\operatorname{sgn}(k) = +1$ if k is even. A path is degenerate if it is a path of length zero between a node and itself and we define its Laplacian weight as one: $\mathcal{L}w^0_{ii} = 1$. The Laplacian weight of a cycle of length k is

$$\mathcal{L}w^k_o := \operatorname{sgn}(k-1) \prod_{t=0}^{t=k-1} \left(-\frac{1}{d_{i_t}}\right),\ i_0 = i_k, \qquad (2)$$

Disjoint cycles in $\mathcal{G}$ are a set of non-adjacent simple cycles, that is, two simple cycles that do not share any common nodes. The length of disjoint cycles is given by the sum of the lengths of the composing simple cycles, while the Laplacian weight of disjoint cycles is given by the product of the Laplacian weights of the composing simple cycles. The subgraph $\mathcal{G}^k_{ij}$ is the subgraph of $\mathcal{G}$ obtained from $\mathcal{G}$ by removing all the nodes and all the arcs touching the simple directed path from node i to node j of length k. The subgraph $\mathcal{G}_i$ is the subgraph of $\mathcal{G}$ obtained from $\mathcal{G}$ by removing node i.

According to Tonetti and Murray [2010], the transfer function between every pair of nodes i and j of a generic graph $\mathcal{G}$ can be derived using a version of *Mason's Direct Rule* (Mason [1953, 1956]). It is studied the low frequency behavior of the network sensitivity functions and it is proved that no matter how the controller is designed, there are fundamental limitations to performance. The analysis demonstrated that the presence of cycles in the interaction topology degenerates the system's performance.

3. SENSITIVITY TRANSFER FUNCTION FOR NON-HOMOGENEOUS SYSTEMS

In this section we show how to derive the non-homogeneous networked sensitivity transfer functions between any pair of agents for a given topology, extending the results obtained in Tonetti and Murray [2010].

We consider a formation of N agents. Each individual agent i is a SISO system with local controller $C_i(s)$ and plant model $P_i(s)$. The normalized Laplacian matrix $\mathcal{L}$ of the graph is used to represent the interaction topology. A representation of the feedback control scheme is shown in Fig. 1, where $\mathbf{r} \in R^N$ is the vector of the reference signals of each agent, $\mathbf{e} \in R^N$ are the errors between $\mathbf{r}$ and the process outputs $\mathbf{y} \in R^N$, $\mathbf{u} \in R^N$ is the control signal vector and $\mathbf{d} \in R^N$ and $\mathbf{n} \in R^N$ are the load disturbances and the measurement noises respectively. The open loop

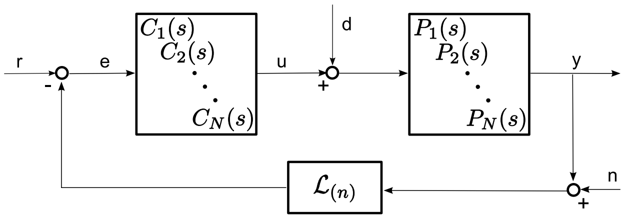


Fig. 1. Block diagram of a non-homogeneous multi-agent feedback system

transfer function of each agent is $L_i(s) = P_i(s)C_i(s)$. We define the *networked sensitivity function matrix* $\widetilde{S}(s)$ as

$$\widetilde{S}(s) = \left(I + \mathcal{L}_{(n)}\overline{L}(s)\right)^{-1},$$

where $\overline{L}(s) = \operatorname{diag}(L_1(s), L_2(s), \cdots, L_N(s))$. From now on, without loss of generality, we will consider $n = 1$ so that each agent has a single output variable that is being controlled. In analogy with the single agent case, in order to guarantee stability, robustness and good performance, we want to have $|\widetilde{S}(j\omega)| \ll 1$ for $\omega \ll \omega_c$, and $|\widetilde{S}(j\omega)| \approx 1$ for $\omega \gg \omega_c$, where ω_c is the cutoff frequency.

We define $\mathcal{O}(o)$ the set of nodes belonging to the simple cycle o, $\mathcal{P}(p)$ the set of nodes belonging to the directed simple path p besides the starting node.

Theorem 1. The sensitivity transfer function between every pair of nodes i and j of a generic graph $\mathcal{G}$ with arbitrary dynamics and local controller, can be still expressed through a version of Mason's Direct Rule:

$$\widetilde{S}_{ij} = \frac{1}{\Delta} \sum_{\text{paths } p \,\in\, \mathcal{G}} T_p \Delta_p, \tag{3}$$

where now the determinant of $(I + \mathcal{L}\overline{L}(s))$ is

$$\begin{aligned}\Delta &= \prod_{f=1}^{N} (1 + L_f) \\ &+ \sum_{\text{cycles } o \,\in\, \mathcal{G}} \left(\mathcal{L}w_o \prod_{z \,\in\, \mathcal{O}(o)} (L_z) \prod_{m \,\notin\, \mathcal{O}(o)} (1 + L_m) \right),\end{aligned} \tag{4}$$

the 'gain' of the p^{th} simple directed path from node i to node j of length k is

$$T_p = \mathcal{L}w_{ij} \prod_{z \,\in\, \mathcal{P}(p)} (L_z), \tag{5}$$

and the value of Δ for the subgraph $\mathcal{G}_{ij}^k$ not touching the p^{th} simple directed path from node i to node j of length k is

$$\begin{aligned}\Delta_p &= \prod_{f \,\in\, \mathcal{G}_{ij}^k} (1 + L_f) \\ &+ \sum_{\text{cycles } o \,\in\, \mathcal{G}_{ij}^k} \left(\mathcal{L}w_o \prod_{z \,\in\, \mathcal{O}(o)} (L_z) \prod_{m \,\notin\, \mathcal{O}(o)} (1 + L_m) \right),\end{aligned} \tag{6}$$

where $\overline{k}$ represents the length of the cycles in $\mathcal{G}_{ij}^k$.

Proof. For a signal flow graph $\mathcal{G}$, the gain matrix M (Mason and Zimmermann [1960]) is

$$M = (I - \bar{\mathcal{A}})^{-1}, \tag{7}$$

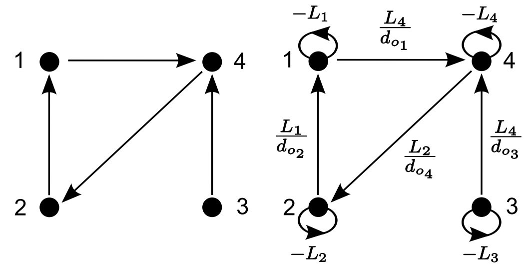
where $\bar{\mathcal{A}}$ is the weighted adjacency matrix associated with the signal flow. Suppose now instead of having $\mathcal{G}$ we have a transformed graph $\widetilde{\mathcal{G}}$ (as the example in Figure 2), with the same topology of $\mathcal{G}$ but with the weight of each arc equal to

$$w_{ij} = \frac{1}{d_{o_i}} L_j, \ \forall (i,j) \in \widetilde{\mathcal{G}},$$

and self-loops in each node with weight

$$w_{ii} = -L_i, \ \forall i \in \widetilde{\mathcal{G}}.$$

We take the generic case of complete directed graph. In



(a) $\mathcal{G}$ (b) $\widetilde{\mathcal{G}}$

Fig. 2. Example of transformation from a graph $\mathcal{G}$ to the signal flow graph $\widetilde{\mathcal{G}}$.

this way the transformed weighted adjacency matrix $\widetilde{\mathcal{A}}$ for the graph $\widetilde{\mathcal{G}}$ will be:

$$\widetilde{\mathcal{A}} = \begin{bmatrix} -L_1 & \frac{1}{d_1}L_2 & \cdots & \frac{1}{d_1}L_N \\ \frac{1}{d_2}L_1 & -L_2 & \cdots & \frac{1}{d_2}L_N \\ \vdots & & \ddots & \vdots \\ \frac{1}{d_N}L_1 & \frac{1}{d_N}L_2 & \cdots & -L_N \end{bmatrix} = -\mathcal{L}\overline{L}.$$

Applying the equation (7), we get the transformed gain matrix $\widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{G}}$:

$$\widetilde{\mathcal{M}} = (I - \widetilde{\mathcal{A}})^{-1} = (I + \mathcal{L}\overline{L})^{-1}.$$

This is exactly what we need to solve in order to compute the matrix sensitivity transfer function. Applying the Mason's direct rule to $\widetilde{\mathcal{G}}$ we obtain exactly the denominator and numerators in equations (4), (5) and (6).

Of course we can observe that if $L_i = L_j$ for all i, j, equations (4)–(6) became exactly like equations for homogeneous multi-agent systems in Tonetti and Murray [2010]. Even if polynomials of the network sensitivity functions include different plant models and local controllers, paths and cycles structures influence the performances in the same way as homogeneous systems.

4. STABILITY OF NON-HOMOGENEOUS SYSTEMS

In the following we analyze the formation stability of non-homogeneous systems for some special class of interaction topologies.

In order to study the closed loop stability of the relative formation dynamics we have to look at det $\left[I + \mathcal{L}\overline{L}\right]$. In the case of non-homogeneous systems we can no longer use the Schur transformation of $\mathcal{L}$ to find a separation principle as in Fax and Murray [2004]. We can no longer transform $(I + \mathcal{L}\overline{L})$ in a triangular form. Therefore we are forced to study the entire expression of det $\left[I + \mathcal{L}\overline{L}\right] = \Delta$, with Δ defined in equation (4). The only thing we can say is that the controllers C_i stabilize the non-homogeneous interconnected system if and only if $\det(\Delta) \neq 0$ and the Nyquist plot of $\det(\Delta)$ encircles the origin n_u times in the counterclockwise sense, where n_u denotes the total number of open loop system unstable poles, counting multiplicity.

Because of this limitation, in the following sections will be presented special topologies for which a separation principle is applicable.

4.1 Stability on a directed acyclic graph

If the interconnection topology is represented by an acyclic directed graph, the denominator in (4) simplifies in Δ_a and the stability is given by

$$\Delta_\mathrm{a} = \prod_{f=1}^{N} \det[1 + L_f]. \tag{8}$$

The critical point becomes -1 as in a single-agent system. Therefore the interconnected non-homogeneous system is stable if and only if every single-agent system is stable.

4.2 Stability of a condensable multi-agent system

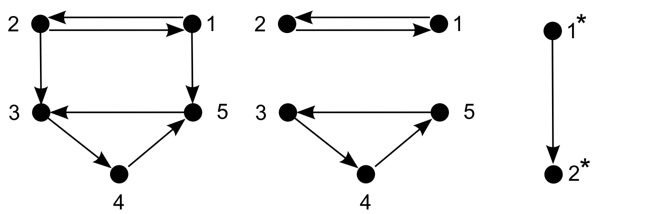
In the following we will recall some definitions about graph condensation which can be found in Šiljak [1991].

Definition 2. A subgraph $\mathcal{G}_k^* = [V_k, (V_k \times V_k) \cap A]$ is a strong component of $\mathcal{G}$ if it is strongly connected and there are no two nodes $v_i \in \mathcal{G}_k^*$ and $v_j \notin \mathcal{G}_k^*$ which lie on the same cycle in $\mathcal{G}$. Given a digraph $\mathcal{G} = (V, A)$, define

$$\begin{aligned} V^* &= \{V_k : V_k \text{ equivalence class of } \mathcal{G}\}, \\ A^* &= \{(V_j, V_i) : v_j \in V_j, v_i \in V_i, (v_j, v_i) \in A, V_j \neq V_i\} \end{aligned}$$

The digraph $\mathcal{G}^* = (V^*, A^*)$ is the condensation of $\mathcal{G}$. Given any digraph $\mathcal{G}$, its condensation $\mathcal{G}^*$ is acyclic.

An example of graph condensation is shown in Fig. 3. In



(a) Digraph $\mathcal{G}$ (b) Strong components $\mathcal{G}_1^*$ and $\mathcal{G}_2^*$ (c) Condensation $\mathcal{G}^*$

Fig. 3. How to condensate a digraph: an example

order to deal with non trivial condensation, from now on we will consider only not strongly connected graphs with cycles. Matrices related to not strongly connected graphs have a nice property:

Definition 3. A matrix is reducible if and only if its associated digraph is not strongly connected. In addition, a matrix is reducible if and only if it can be placed into block upper-triangular form by simultaneous row/column permutations.

We will call a multi-agent system a *condensable multi-agent system* (CMAS) satisfying the following property:

(i) agents belonging to the same strong component $\mathcal{G}_i^*$ have the same open loop transfer function L_i^*.

For example, the system represented in Fig. 3 is a CMAS if and only if $L_1 = L_2 = L_1^*$ and $L_3 = L_4 = L_5 = L_2^*$.

Theorem 4. The stability of a condensable multi-agent system is given by

$$\Delta = \prod_{i=1}^{m} \Delta_i^*, \tag{9}$$

where

(i) m is the number of strong components in $\mathcal{G}$;
(ii) Δ_i^* is the determinant of the strong component $\mathcal{G}_i^*$,

$$\begin{aligned} \Delta_i^* = &\ (1 + L_i^*)^{N_i^*} \\ &+ \sum_{\text{cycles } o \,\in\, \mathcal{G}_i^*} \big(\mathcal{L}w_o^k\big)(1+L_i^*)^{(N_i^*-k)}(L_i^*)^k; \end{aligned} \tag{10}$$

(iii) N_i^* is the number of nodes in $\mathcal{G}_i^*$.

Therefore the global system is stable if and only if all the strong components are stable.

Proof. Since we are considering not strongly connected graphs, by Definition 3, the normalized Laplacian matrix associated to $\mathcal{G}$ is reducible. It means that choosing an appropriate ordering for the nodes, $\mathcal{L}$ can be placed into block upper-triangular form, where each diagonal block $\mathcal{L}_i^*$ represents a strong component, while the off diagonal blocks $\mathcal{L}_{i^*j^*}$ represent how the strong components are connected:

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1^* & \mathcal{L}_{1^*2^*} & \cdots & \mathcal{L}_{1^*m^*} \\ & \mathcal{L}_2^* & \cdots & \mathcal{L}_{2^*m^*} \\ & & \ddots & \vdots \\ & & & \mathcal{L}_m^* \end{bmatrix}.$$

We already know that the stability of a multi-agent system depends on $\det(I + \mathcal{L}\widehat{P}\widehat{C}) = \Delta$, where $\widehat{M} = I_N \otimes M$ represents a matrix M repeated N times along the diagonal. If $\mathcal{L}$ is block upper-triangular, $(I + \mathcal{L}\widehat{P}\widehat{C})$ has the same property with blocks equal to $(I + \mathcal{L}_i^*\widehat{L_i^*})$ on the main diagonal, where $\widehat{L_i^*} = I_{N_i^*} \otimes L_i^*$. Equation (9) follows from the multiplicative property of the determinant of block triangular matrices.

It has to be noticed that the Laplacian weights in (10) are computed with the outdegrees of nodes in $\mathcal{G}$, as in the definition (2) and not with outdegrees of nodes in $\mathcal{G}_i^*$.

For a CMAS we can have three cases:

(1) if $\mathcal{G}$ is acyclic, it has no strong components and its condensation $\mathcal{G}^*$ coincides with the graph itself $\mathcal{G}^* \equiv \mathcal{G}$. The stability of the system depends on the stability of each single agent (8): $\Delta = \Delta_\mathrm{a}$;

(2) if $\mathcal{G}$ is strongly connected, it has only one strong component and its condensation $\mathcal{G}^*$ will be a single node $\mathcal{G}^* \equiv i^*$. The stability is given by the stability of the entire homogeneous system (Fax and Murray [2004]);
(3) if $\mathcal{G}$ has cycles but it is not strongly connected, its condensation $\mathcal{G}^*$ is an acyclic graph. The stability is given by (9).

4.3 Example

Consider the CMAS with graph topology as in Fig. 3 and open loop transfer functions equal to $L_1^* = (800s + 2000)/(s^3 + 41s^2 + 44s)$ and $L_2^* = (50s + 100)/(0.0475s^3 + 2.375s^2 + s)$. We want to investigate the stability of the system. The normalized Laplacian matrix is block upper-triangular:

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1^* & \mathcal{L}_{1^*2^*} \\ & \mathcal{L}_2^* \end{bmatrix},$$

where the diagonal blocks are

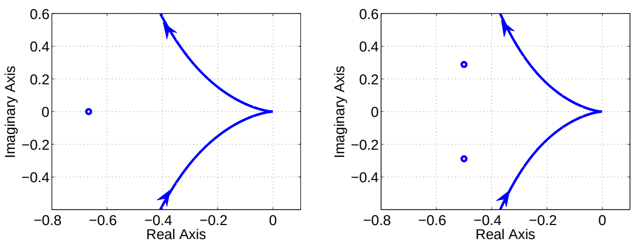
$$\mathcal{L}_1^* = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix},$$

$$\mathcal{L}_2^* = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

and the off diagonal block is

$$\mathcal{L}_{1^*2^*} = \begin{bmatrix} 0 & 0 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix}.$$

The stability of the two strong components can be checked applying the stability criterion for homogeneous multi-agent systems in Fax and Murray [2004], where the critical points are the nonzero eigenvalues of $\mathcal{L}_1^*$ for $\mathcal{G}_1^*$ and of $\mathcal{L}_2^*$ for $\mathcal{G}_2^*$. The Nyquist plots in Fig. 4 show that the global system is stable because the strong components are separately stable.



(a) $\mathcal{G}_1^*$ (b) $\mathcal{G}_2^*$

Fig. 4. Strong components Nyquist Plot and critical points

5. PERFORMANCE OF NON-HOMOGENEOUS SYSTEMS

We are going now to analyze the system low frequency behavior, in order to see where considerations done for homogeneous agents can be extended to non-homogeneous systems.

Theorem 5. In a non-homogeneous interconnected multi-agent system all the asymptotic values of $\widetilde{S}_{ii}$, for graph with $d_o > 0$, $\forall i \in \mathcal{G}$, sum up to the unity:

$$\sum_{i=1}^{N} \lim_{|L_i| \to \infty} |\widetilde{S}_{ii}| = 1. \quad (11)$$

Proof. Even if now polynomials of $\widetilde{S}_{ii}$ have more than one variable, if $d_o > 0$ the coefficient of $\prod_{f=1}^{N} L_f$ at the denominator is always zero, as proved in Theorem 3 (Tonetti and Murray [2010]). So we will consider only terms given by $\prod_{f \in \mathcal{G}_i} L_f$. Starting from Δ_p/Δ, with Δ_p expressed in (6) and Δ in (4), with some algebraic considerations, as $|L_i| \to \infty$ the asymptotic value for a diagonal interconnected sensitivity function is

$$\lim_{|L_i| \to \infty} \widetilde{S}_{ii} = \lim_{|L_i| \to \infty} \frac{\left(1 + \sum_{\mathcal{G}_i} (\mathcal{L}w_o)\right) \prod_{f \in \mathcal{G}_i} L_f}{\sum_{i=1}^{N} \left[\left(1 + \sum_{\mathcal{G}_i} (\mathcal{L}w_o)\right) \prod_{f \in \mathcal{G}_i} L_f \right]}, \quad (12)$$

and it is clear that the sum over all the nodes is equal to one, despite the expression of L_i.

From Theorem 5 we can see that even in non-homogeneous multi-agent systems there are fundamental limitations to what can be achieved by control, and control design is a redistribution of disturbance rejection at low frequencies among agents.

5.1 Example

Consider the strongly connected graph of Fig. 5. Agents

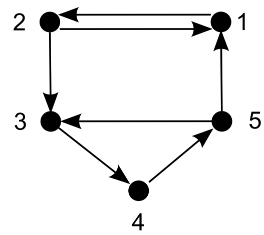


Fig. 5. Interconnection topology

1 and 2 have the same stable open loop transfer function equal to $L_1 = (800s + 2000)/(s^3 + 41s^2 + 44s)$, while agents 3, 4 and 5 have $L_2 = (50s + 100)/(0.0475s^3 + 2.375s^2 + s)$. The network sensitivity functions are shown in Fig. 6.

Suppose now we want to improve low frequency behavior of agents 1 and 2 choosing a higher gain for L_1. As we can see in Fig. 7, a lower asymptotic value for agents 1 and 2 implies a higher value for the other agents, as predicted in Theorem 5.

Corollary 6. In a non-homogeneous multi-agent system, interconnected by an arbitrary directed graph with $d_o > 0$ $\forall i \in \mathcal{G}$, the low frequency asymptotic value of the diagonal sensitivity function is

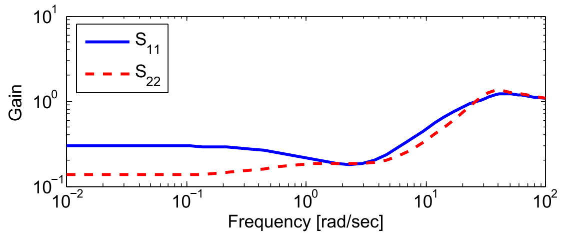


Fig. 6. Non-homogeneous network sensitivity functions

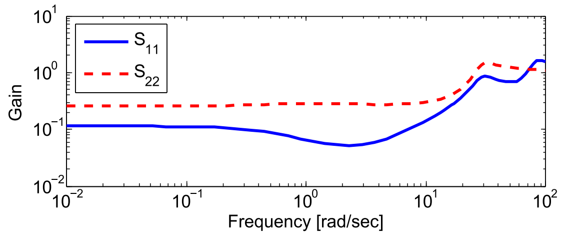


Fig. 7. Non-homogeneous network sensitivity functions with higher gain on L_1

$$\lim_{|L_i|\to\infty} \widetilde{S}_{ii} =$$

$$\lim_{s\to 0} \frac{\left(1+\sum_{\mathcal{G}_i}(\mathcal{L}w_o)\right)\prod_{f\in\,\mathcal{G}_i}\frac{\mu_f}{s^{g_f}}}{\sum_{i=1}^{N}\left[\left(1+\sum_{\mathcal{G}_i}(\mathcal{L}w_o)\right)\prod_{f\in\,\mathcal{G}_i}\frac{\mu_f}{s^{g_f}}\right]}, \tag{13}$$

where μ_f is the zero frequency gain, which will be called static gain, and g_f is the number of poles in the origin of the open loop transfer function of agent $f \neq i$.

Proof. A generic open loop transfer function $L(s)$ can be expressed as

$$L(s) = \frac{\mu}{s^g}\frac{\prod_m(1+s\tau_m)}{\prod_n(1+sT_n)}, \tag{14}$$

where μ is the static gain, $g \geq 0$ is the number of poles in the origin, τ_m and T_n are zeros and poles time constants, respectively.

Equation (13) comes from equation (12), where each L_f has been substituted by $\lim_{s\to 0} L(s) = \mu/s^g$ of equation (14).

In the following some special cases of Corollary 6 will be considered, showing examples.

5.2 Regular graph and same low frequency behavior

We start first considering directed regular graphs, where there is a full symmetry and in a homogeneous case the asymptotic value of the sensitivity functions would be the same for all the nodes and equal to $1/N$. We moreover consider open loop transfer functions with the same number of poles in the origin, $g_i = g_j \; \forall i, j \in \mathcal{G}$. This means they approach $s = 0$ with the same speed and therefore they have the same low frequency behavior.

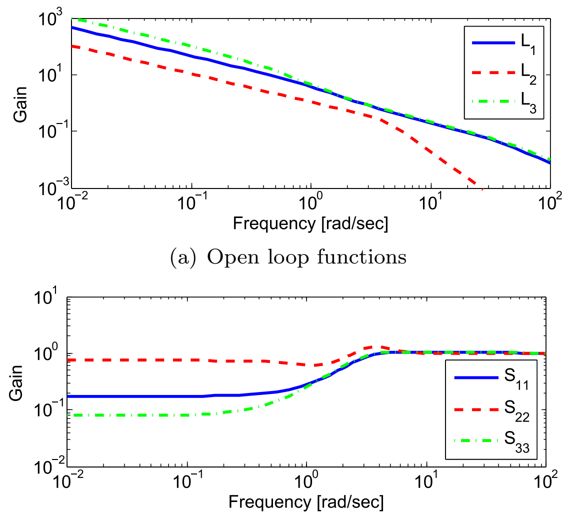
Starting from equation (13) and following our hypothesis: if the graph is regular, $1 + \sum_{\mathcal{G}_i}(\mathcal{L}w_o)$ is the same for all the nodes and it can first be taken out of the sum at the denominator and then simplified with the one at the numerator; if all the open loop functions have the same number of poles we can simplify all the g_f. So the asymptotic value becomes

$$\lim_{s\to 0}\widetilde{S}_{ii} = \frac{\prod_{f\in\,\mathcal{G}_i}\mu_f}{\sum_{i=1}^{N}\prod_{f\in\,\mathcal{G}_i}\mu_f}, \tag{15}$$

and it is evident that, since μ_i does not appear in the numerator of $\widetilde{S}_{ii}$, if agent i has the highest μ_i with respect to the other agents, the numerator of $\widetilde{S}_{ii}$ will have the lowest low frequency magnitude. An agent with the lowest μ_i, will have the highest $\widetilde{S}_{ii}$ low frequency magnitude and therefore the poorest disturbance rejection behavior. Same considerations can be done for all the other agents.

In a non-homogeneous interconnected multi-agent system, connected by a directed regular graph and with $L_i(s)$ with the same number of poles in the origin for all the nodes, the highest is the static gain μ_i of an agent with respect to the others, the better disturbance rejection properties has that agent in the formation. This tell us that even if the interconnection topology is symmetric, the agents behavior is different and it is ruled by the static gain of the open loop function.

Consider a formation of 3 agents connected by a complete graph and with the following stable open loop transfer functions with one pole in the origin: $L_1 = (80s + 200)/(s^3 + 41s^2 + 44s)$, $L_2 = 18/(s^3 + 5s^2 + 17s)$ and $L_3 = (5s + 10)/(0.0475s^3 + 2.375s^2 + s)$. Static gains are: $\mu_1 = 4.54$, $\mu_2 = 1.06$, $\mu_3 = 10$. The magnitude of $L_i(s)$ is depicted in Fig. 8(a) and it is clear that all the functions approach $s = 0$ with the same speed, but $|L_2(s \to 0)| < |L_1(s \to 0)| < |L_3(s \to 0)|$. We expect $\widetilde{S}_{33}$ to have the lowest value at low frequencies, while agent 2 to have the worst disturbance rejection behavior, as shown in Fig. 8(b). We can also verify that the sum of the asymptotic values is equal to one: $|\widetilde{S}_{11}(s \to 0)| = 0.174$, $|\widetilde{S}_{22}(s \to 0)| = 0.747$, $|\widetilde{S}_{33}(s \to 0)| = 0.079$.



(a) Open loop functions

(b) Diagonal networked sensitivity functions

Fig. 8. Non-homogeneous system transfer functions

5.3 Arbitrary graph and same low frequency behavior

What does it happen if the graph is not regular but the number of poles is still the same for all the $L_i(s)$? Starting from equation (13) we will have that $1 + \sum_{\mathcal{G}_i} (\mathcal{L}w_o)$ will be different from agent to agent, while all the g_f can be simplified and the asymptotic value will be

$$\lim_{s\to 0} \widetilde{S}_{ii} = \frac{\left(1 + \sum_{\mathcal{G}_i} (\mathcal{L}w_o)\right) \prod_{f\in\ \mathcal{G}_i} \mu_f}{\sum_{i=1}^{N} \left[\left(1 + \sum_{\mathcal{G}_i} (\mathcal{L}w_o)\right) \prod_{f\in\ \mathcal{G}_i} \mu_f\right]}. \quad (16)$$

Here the product of the static gains of all the other agents has to be weighted by a value depending on the cycles not passing through that agent. The fewer cycles pass through a node, the lower is $1 + \sum_{\mathcal{G}_i} (\mathcal{L}w_o)$ and the better is the agent low frequency behavior.

Consider the formation of Example 5.1 but with arc 35 added. The open loop transfer functions have the same number of poles in the origin. With this topology on nodes 3 and 5 pass more cycles than on node 4. Therefore, even if agents 3, 4 and 5 have the same open loop transfer function, we can see in Fig. 9 that agent 4 has a better low frequency behavior with respect to 3 and 5, while in Fig. 6 the behavior was the same.

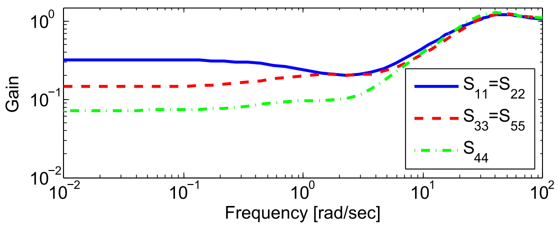


Fig. 9. Non-homogeneous network sensitivity functions affected by topology

5.4 Regular graph and different low frequency behavior

Consider now a formation connected by a regular graph but with at least one vehicle with different number of poles with respect to the others $g_i \neq g_j$. The asymptotic value is

$$\lim_{s\to 0} \widetilde{S}_{ii} = \frac{\prod_{f\in\ \mathcal{G}_i} \frac{\mu_f}{s^{g_f}}}{\sum_{i=1}^{N} \prod_{f\in\ \mathcal{G}_i} \frac{\mu_f}{s^{g_f}}}. \quad (17)$$

The agent with the highest g, which will be called g_{max}, since it will appear only at the denominator, will lead its asymptotic value to zero. From Theorem 5 we know that the asymptotic values sum up to the unity, so the redistribution of disturbance rejection will affect only the agents with $g < g_{max}$. For example if there is only one agent with $g < g_{max}$, the disturbance entering in it will not be attenuated at all because its asymptotic value will be equal to one.

Consider the same formation of the example in Section 5.2, but with L_2 with two poles in the origin, instead of one. In Section 5.2 the low frequency behavior of agent 2 was the poorest, but in Fig. 10, because of the pole added, the disturbance rejection is very good. The asymptotic values of $|\widetilde{S}_{11}| = 0.687$ and $|\widetilde{S}_{33}| = 0.312$ get higher if compared to the example in Section 5.2 because they have to sum up to the unity.

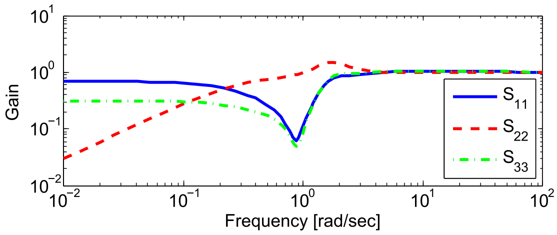


Fig. 10. Non-homogeneous network sensitivity functions with different low frequency behavior

5.5 Bode's integral formula

In Tonetti and Murray [2010] is proved that in a homogeneous multi-agent system, Bode's integral formula for stable open loop systems still holds for each diagonal interconnected sensitivity function $\widetilde{S}_{ii}$, no matter what the interconnection topology is:

$$\int_0^\infty \log(|\widetilde{S}_{ii}(j\omega)|)d\omega = 0.$$

This theorem can easily be extended to non-homogeneous multi-agent systems because numerator and denominator of $\widetilde{S}_{ii}$ have the same constant term. We have that Δ_p expressed in (6) and Δ in (4) are both equal to one for $|L(s)| = 0$.

6. CONCLUSION

In this paper we have explored stability and performance of non-homogeneous systems, extending results obtained by Fax and Murray [2004] and Tonetti and Murray [2010] for homogeneous systems.

We have presented a class of multi-agent systems for which a separation principle is possible, in order to relate formation stability to interaction topology. Future research must investigate this interaction for more arbitrary formation graphs.

We have shown that cycles and paths are still involved in the network sensitivity functions, but if the agent dynamics is different, topology is not the only player in determining system's performance. The low frequency behavior is also influenced by static gain and poles of the open loop function of each agent. If every single agent has the same number of poles, the larger the low frequency gain, the better the formation disturbance rejection. If an agent has a higher number of poles with respect to the others, it will behave like a single agent in the formation.

We can conclude that there are fundamental limitations to what can be achieved by distributed control of non-homogeneous systems. If the behavior of one agent improves, the behavior of the others get worse. Control design is a redistribution of disturbances at low frequency among agents.

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