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A Framework for Lyapunov Certificates for Multi-Vehicle Rendezvous Problems

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***Abstract*— In this paper we present a dynamical systems representation for multi-agent rendezvous on the phase plane. We restrict our attention to two agents, each with scalar dynamics. The problem of rendezvous is cast as a stabilization problem, with a set of constraints on the trajectories of the agents, defined on the phase plane. We also describe a method to generate control Lyapunov functions that when used in conjunction with a stabilizing control law, such as Sontag's formula, makes sure that the two-agent system attains rendezvous. The main result of this paper is a Lyapunov-like certificate theorem that describes a set of constraints, which when satisfied are sufficient to guarantee rendezvous.**

I. Introduction

Recently there has been considerable interest in multi-agent coordination or cooperative control (as cited in [6] and [2] for instance). This has led to the emergence of several interesting control problems. One such problem is the *rendezvous problem*. In a rendezvous problem, one desires to have several agents arrive at predefined destination points simultaneously. Cooperative strike or cooperative jamming are two examples of the rendezvous problem. In the first scenario, multiple strikes are executed *within an interval*, from different agents firing from different distances and travelling at different speeds. In the second scenario, one or more agents need to start jamming *slightly before* the strike vehicle enters the danger zone and sustain jamming until strike vehicle exits. In both the scenarios, it is imperative that all the agents act simultaneously else the objective is not fulfilled.

The idea of rendezvous extends beyond just convergence to a static set of destination points or the origin. Rendezvous can also entail formation flying or interception problems where the origin is effectively moving. Interception of incoming ballistic missiles is a rendezvous problem where the origin becomes a moving target. Formation flying is a type of rendezvous problem where multiple agents must coordinate position and velocity. The docking of two spacecraft is a rendezvous problem that involves the two spacecraft matching both position and velocity with the proper orientation. Air-to-air refueling is another interception problem. Additional applications arise in submersibles where robotic vehicles must converge upon a set location, either moving or stationary.

As the push towards unmanned vehicles becomes more prevalent in the aerospace industry, methods for guaranteeing rendezvous will be necessary. It will be necessary to answer whether a mission in a cooperative control framework can be accomplished with a high degree of confidence in the presence of uncertainties. The uncertainty set can include differing flight conditions, local parametric variations, component failures on an aircraft, and communications variability such as loss of packets, temporary loss of link, etc.

In the current literature, several researchers have addressed problems related to path planning with timing constraints. In 1963, Meschler in [9] investigated a time optimal rendezvous problem for linear time varying systems. He assumed that both the rendezvous point and rendezvous time are not known a priori and that determining the minimum time at which rendezvous occurred was of interest. In principle, complicated rendezvous problems can be formulated using optimal control theory [3] and solved numerically. However, for many vehicles, obstacles and threats, the resulting optimization problem becomes quite complicated and the computational time increases very rapidly with problem size. In [7], [8] McLain *et al.* have proposed decomposition methods that breaks down the monolithic problem into subproblems that can be solved efficiently in a decentralized manner. Similar decomposition methods have also being proposed in [4], [13] that solve path planning problems with timing constraints in a decentralized manner. Heuristic search based algorithms have also been proposed as an alternative, that approximates single large scale optimization problems into decoupled, partially distributed problems enabling faster computation [1], [11].

In this paper we approach the rendezvous problem from the point of view of Lyapunov stability [5]. Örgen *et al.* in [10] have recently proposed a Lyapunov function approach to multi-agent coordination with application to formation flying. In this paper, we propose Lyapunov function approach to the rendezvous problem.

The paper is organized as follows. The rendezvous problem is defined in Section II along with notions of perfect and approximate rendezvous and with an interpretation of rendezvous on the phase plane. An example is given in Section III of a system of agents which achieve rendezvous under certain conditions with a Lyapunov-function based controller. A level-set method for constructing Lyapunov functions for use in rendezvous control is given in Section IV. The subject of rendezvous certificates is addressed in Section V, and a certificate theorem is given for guaranteeing rendezvous using a certain class of Lyapunov functions. An example illustrating the use of this certificate theorem is given, and remarks are made concerning this and future work.

II. The Rendezvous Problem

In this paper we define the rendezvous problem to be the problem of determining a control algorithm that drives multiple agents to a desired destination point. The trajectories must be such that the agents visit the destination point only once and arrive at the same time. We present results for two agents with scalar dynamics.

Consider two scalar systems or agents $\mathcal{V}_1$ and $\mathcal{V}_2$ defined as

$$\begin{array}{ll} \mathcal{V}_1 : & \dot{x}_1 = f_1(x_1) + g_1(x_1)u_1\,; \quad f_1(0) = 0 \\ \mathcal{V}_2 : & \dot{x}_2 = f_2(x_2) + g_2(x_2)u_2\,; \quad f_2(0) = 0, \end{array} \tag{1}$$

where $x_i \in \mathbb{R}$ for $i \in \{1, 2\}$ and the destination point being the origin. Let x_1 and x_2 in Eqn. (1) be the spatial coordinates of $\mathcal{V}_1$ and $\mathcal{V}_2$ on the real line. It is of interest to design control laws u_1 and u_2 such that $\mathcal{V}_1$ and $\mathcal{V}_2$ reach the origin of the real line at the same time. This is depicted in Fig. 1(a).

Clearly agents that are exponentially stable will reach the origin as time tends to infinity. Thus comparison of arrival times at the origin, of 2 different agents becomes meaningless. Even with cooperative control in place, if the origin is exponentially stable, rendezvous at origin will occur at infinite time in theory. From a practical standpoint, it is desired that the agents achieve rendezvous in finite time. For this reason we relax the definition of rendezvous to be such that rendezvous is achieved if the agents enter a certain neighborhood around the origin, at the same time. We define this region to be the *rendezvous region* $\mathcal{R}$.

$$\mathcal{R} = \{x \in \mathbb{R} : -\delta \leq x \leq \delta\} \text{ for some } \delta > 0$$

Therefore a valid rendezvous is one in which agents enter $\mathcal{R}$ at the same time. This is illustrated in Fig. 1(b). In Section II-B we

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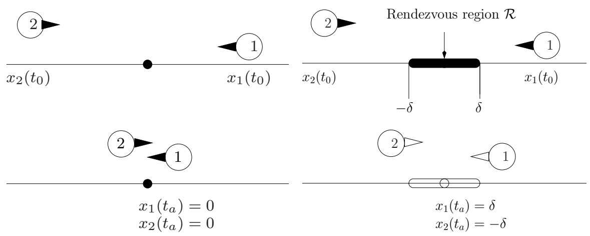


(a) Rendezvous at origin (b) Rendezvous at rendezvous region $\mathcal{R}$

Fig. 1. Rendezvous on the Real Line.

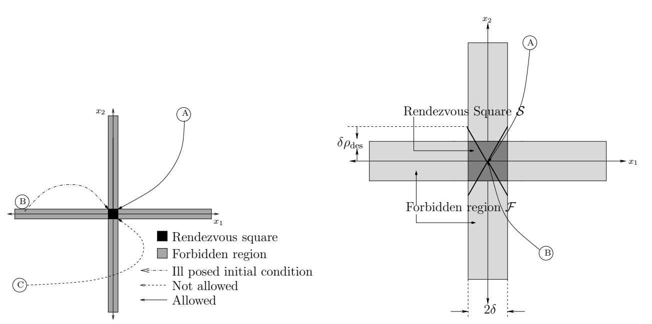
will relax this definition for agents entering $\mathcal{R}$ at approximately the same time.

A. Rendezvous Interpretation on Phase Plane

Rendezvous is best visualized on the phase plane. To interpret rendezvous for the scalar systems in Eqn. (1) in the phase plane, we define the following

$$\begin{array}{rcl} U_1 & = & \{(x_1, x_2) : -\delta \leq x_1 \leq \delta\} \\ U_2 & = & \{(x_1, x_2) : -\delta \leq x_2 \leq \delta\} \\ \mathcal{S} & = & U_1 \cap U_2 \\ \mathcal{F} & = & (U_1 \cup U_2) - (U_1 \cap U_2) \\ \mathcal{W} & = & (\mathbb{R}^2 - (U_1 \cup U_2)). \end{array} \tag{2}$$

We refer to $\mathcal{S}$ as the *rendezvous square* and $\mathcal{F}$ as the *forbidden region*.



(a) Rendezvous in phase plane (b) Approximate rendezvous in phase plane

Fig. 2.

With reference to Fig. 2(a), the strip on x_2-axis is U_1, the strip on x_1-axis is the region U_2 and the rendezvous square is the destination set where the trajectories must converge to. The rendezvous square $\mathcal{S}$ is the set of configurations with both agents in the rendezvous region $\mathcal{R}$. The rendezvous problem is well-posed if the initial conditions of the two agents satisfy

$$(x_1(0), x_2(0)) \in \mathcal{W}, \tag{3}$$

i.e. both the agents start far from the rendezvous region. If the condition in Eqn. (3) is violated either $\mathcal{V}_1$, or $\mathcal{V}_2$, or both start from within the rendezvous region $\mathcal{R}$. In Fig. 2(a) trajectory B starts from an invalid initial point.

The forbidden region is the set of points $\mathcal{F}$ where one agent enters the rendezvous region much before the other. In Fig. 2(a), trajectory C crosses the forbidden region which implies that the agent $\mathcal{V}_1$ with state x_1 comes within the rendezvous region prior to the final entry. Such trajectories are not acceptable, i.e. the trajectories must satisfy

$$(x_1(t), x_2(t)) \notin \mathcal{F} \;\; \forall t. \tag{4}$$

Trajectory A is an example of two agents, with valid initial conditions, achieving rendezvous as desired.

B. Perfect and Approximate Rendezvous

With constraint defined in Eqn. (4), the only way trajectories can enter $\mathcal{S}$ is through the corners of the rendezvous square, i.e. through one of the points

$$(\delta, \delta),\ (\delta, -\delta),\ (-\delta, \delta) \text{ and } (-\delta, -\delta).$$

This implies that the agents are constrained to enter $\mathcal{S}$ at precisely the same time, which is the time the trajectory meets one of the four corners of $\mathcal{S}$ in the phase plane. In reality, agents $\mathcal{V}_1$ and $\mathcal{V}_2$ may reach the rendezvous region within ΔT seconds of each other (through the forbidden region, as is shown below). We now refer to the case when ΔT is zero as *ideal* or *perfect* rendezvous and the case when ΔT is small as *real* or *approximate* rendezvous.

Since the phase plane does not reveal time explicitly, we use a related measure ρ to characterize rendezvous. We will first define ρ, its relation to ΔT will be explained thereafter. To define ρ, we first introduce $t_{\mathcal{V}_1}$ and $t_{\mathcal{V}_2}$ to be the arrival times of agents $\mathcal{V}_1$ and $\mathcal{V}_2$ at the boundary of the rendezvous region $\mathcal{R}$, i.e.

$$\begin{array}{rcl} t_{\mathcal{V}_1} & = & \min\ [\ t \mid x_1(t) \in U_1\ ] \\ t_{\mathcal{V}_2} & = & \min\ [\ t \mid x_2(t) \in U_2\ ]. \end{array}$$

Clearly, ΔT is given by

$$\Delta T = |t_{\mathcal{V}_1} - t_{\mathcal{V}_2}|\,. \tag{5}$$

Therefore the time t_a at which the trajectory enters the region $U_1 \cup U_2$ in the phase plane is given by

$$t_a = \min(t_{a_1}, t_{a_2}).$$

For a given trajectory ($x(t) = [x_1(t)\ x_2(t)]^T$), ρ is the maximum ratio of the distance from the origin of the two agents, after one of them has reached the rendezvous region $\mathcal{R}$. It can be expressed as

$$\rho = \frac{\max(|x_1(t_a)|, |x_2(t_a)|)}{\min(|x_1(t_a)|, |x_2(t_a)|)} = \frac{\max(|x_1(t_a)|, |x_2(t_a)|)}{\delta}. \tag{6}$$

For the rest of the paper, rendezvous will always be specified by δ and a design measure of approximate rendezvous, ρ_{des}. In other words we will call a given rendezvous to be successful, if all the trajectories satisfy

$$\rho \ \leq\ \rho_{\text{des}}. \tag{7}$$

This notion of approximate rendezvous is illustrated in Fig. 2(b). Whenever a trajectory starting in the first quadrant enters the region $U_1 \cup U_2$ it is constrained to lie within the angle generated by joining the points

$$(\delta, \delta\rho_{\text{des}}),\ (0, 0),\ \text{and}\ \ (\delta\rho_{\text{des}}, \delta)\,.$$

There exists similar constraints for trajectories originating in the other quadrants. The introduction of ρ in the definition of rendezvous allows trajectories to enter the forbidden region $\mathcal{F}$ as long as they remain within the above mentioned angle set by the design constraint.

By the definition of ρ in Eqn. (6) it is clear that for a given trajectory $\rho \ \geq\ 1$. Therefore a specification of rendezvous is meaningful if and only if

$$\rho_{\text{des}} \geq 1. \tag{8}$$

Note that for perfect rendezvous the specification becomes $\rho_{\text{des}} = 1$.

In the worst case, at the time of entry of the first agent, t_a, the distances of the 2 agents from the origin can differ by $\delta(\rho_{\text{des}} - 1)$. By ensuring that the trajectories remain within the bold lines in Fig. 2(b), upon entry in the region $U_1 \cup U_2$ we can make sure that the 2 agents enter the rendezvous region $\mathcal{R}$ within a small time ΔT of each other. Thus the constraint in Eqn. (7) helps keep ΔT small.

In Fig. 2(b) both trajectories A and B fail to achieve perfect rendezvous as they do not enter the rendezvous square $\mathcal{S}$ from its four corners. On the basis of Eqn. (7), trajectory B is unacceptable. Trajectory A is acceptable since it lies within the angle defined by the bold lines.

III. Lyapunov Functions

In this section we motivate the use of control Lyapunov functions (CLFs) to solve the rendezvous problem. Consider the Lyapunov function candidate

$$V(x_1, x_2) = x_1^2 + x_2^2 + (x_1^2 - x_2^2)^2. \quad (9)$$

Ensuring $\dot{V} < 0$ guarantees that all the three terms in Eqn. (9) goes to zero as time tends to infinity. If x_1 and x_2 denote the spatial coordinates of agents $\mathcal{V}_1$ and $\mathcal{V}_2$ and the origin is the rendezvous point, the first two terms ensure that they converge to the origin and the third term ensures that the agents reach the origin simultaneously. This is demonstrated by the following example.

Let the dynamics of the agents be given by

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2. \end{aligned} \quad (10)$$

It is easy to verify that $V(x)$ in Eqn. (9) is a CLF. Sontag in [12] proposed a formula for producing a stabilizing controller based on the existence of a CLF $V(x)$. Because of its guarantee of stabilization and of providing a convenient relationship between closed-loop trajectories and CLF level sets, Sontag's formula is used here. For nonlinear systems with affine input such as

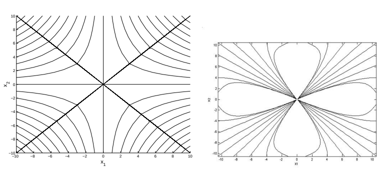
$$\dot{x} = f(x) + g(x)u,$$

Sontag's formula can be written as

$$u_s = \begin{cases} -\frac{V_x f + \sqrt{(V_x f)^2 + q(x) V_x g g^T V_x^T}}{V_x g g^T V_x^T} g^T V_x^T & V_x g \neq 0 \\ 0 & V_x g = 0 \end{cases} \quad (11)$$

where $V_x = \frac{\partial V(x)}{\partial x}$.

For the system in Eqn. (10) and control derived from $V(x)$ in Eqn. (9) using Sontag's formula, the phase portrait is shown in Fig. 3(a). The term $(x_1^2 - x_2^2)^2$ in Eqn. (9) ensures that the agents



(a) (b)

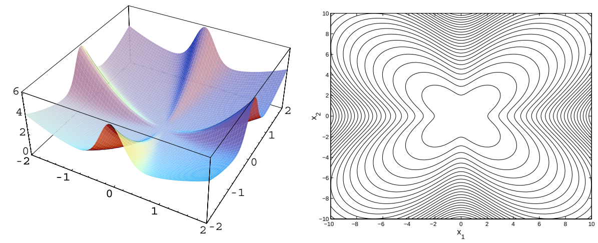
Fig. 3. Rendezvous using control Lyapunov functions.

become equidistant from the origin by converging them to the lines $x_1 = \pm x_2$ prior to their arrival at the origin. In this sense, rendezvous is achieved for any ρ_{des} for any δ. Fig. 3(b) shows the phase portrait for the same system but with Lyapunov function defined as

$$V(x_1, x_2) = (x_1^2 + x_2^2)\left[a + b e^{-8x_1^2 x_2^2 / d^2 (x_1^2 + x_2^2)^2}\right]. \quad (12)$$

Rendezvous is achieved by $\mathcal{V}_1$ and $\mathcal{V}_2$ in Fig. 3(b) only under restricted values of ρ_{des} for a given δ. In one sense, however, rendezvous achieved by $\mathcal{V}_1$ and $\mathcal{V}_2$ in Fig. 3(b) is "better" than that in Fig. 3(a) because the agents are equidistant from the origin only locally. Rendezvous in Fig. 3(a) forces the agents to be equidistant from the origin even at large distances, which may not be necessary.

Thus, it is possible to implicitly satisfy the constraints on ρ, as defined in Eqn. (7), if the Lyapunov function has a certain form. For valid rendezvous, trajectories in phase plane should not cross either axes. If $\dot{V}$ is negative definite for all points in the phase plane and trajectories are constrained to be within the quadrant they start from, outside $\mathcal{S}$, the level sets are expected to have clover leaf appearance as shown in Fig. 4(b). Figure 4(b) shows the level sets of the Lyapunov function defined in Eqn. (12). The level set of these control Lyapunov functions provide insight into why rendezvous is achieved for these cases. With control using Sontag's formula for the system in Eqn. (10), rendezvous is achievable because trajectories are constrained to be normal to the



(a) (b)

Fig. 4. Desired Lyapunov surface and its level sets

level set contours. Controllers based on CLF's, whose level sets are similar to those in Fig. 4(b), should drive agents for system Eqn. (10) to a successful rendezvous. The next section describes a level set method for constructing control Lyapunov functions, and a certificate theorem for testing whether rendezvous is achievable for a system is given in Section V.

IV. Generating Lyapunov Surfaces Using Level Sets

In this section we will present a method to design Lyapunov surfaces by first designing their level sets. As already demonstrated in the previous section, the level sets for all the cases we are interested in, look similar to that shown in Fig. 4(b)

The main idea is to first write down an equation for a curve in $\mathbb{R}^2$ using polar coordinates $r^n = h(\theta), n \in \mathbb{Z}^+$. Then we try to find a positive definite function $V(r,\theta)$ such that for some $c_0 > 0$, the following two equations are equivalent.

$$V(r,\theta) = c_0 \text{ and } r^n = h(\theta), n \in \mathbb{Z}^+ \quad (13)$$

i.e., they describe the same curve in $\mathbb{R}^2$.

Definition 1: We define a family $\mathcal{T}$ of real valued functions $h : [0, 2\pi] \to \mathbb{R}$ with the following properties:

1) the function h is continuous and strictly differentiable;
2) the function h is strictly positive:
$$h(\theta) > 0, \quad \forall \quad \theta;$$
3) in the interval $\theta \in [0, \pi/2)$, h attains a minimum value at $\theta = 0$;
4) In the interval $\theta \in [0, \pi/2)$, h attains a maximum value at $\theta = \pi/4$;
5) the function h is symmetric about $\theta = \pi/4$:
$$h(\theta) = h(\pi/2 - \theta); \text{ and}$$
6) the function h is periodic with period $\pi/2$:
$$h(\theta) = h(\pi/2 + \theta).$$

Example 1: The function

$$h(\theta) = \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\cos(4\theta), \forall \alpha, \beta > 0 \text{ and } \alpha > \beta$$

satisfies all the properties in Defn. (1). The Fig. 5(a) shows a plot of $h(\theta)$ vs θ for α = 5 and β = 1.

Example 2: The function

$$h(\theta) = \frac{1}{a + b e^{-\frac{1-\cos 4\theta}{d^2}}}$$

where a, b, $d \in \mathbb{R}$ and

$$a + b > 0$$

is also a member of $\mathcal{T}$.

Definition 2: We define a family $\mathcal{C}$ of closed curves $c(r,\theta) = 0$ in $\mathbb{R}^2$ where

$$c(r,\theta) = 0 \text{ and } r^n = h(\theta)$$

describe the same closed curve in $\mathbb{R}^2$ for h(θ) $\in \mathcal{T}$ and a real number $n > 1$, with $\mathcal{T}$ as defined above.

Example 3: The closed curve described by

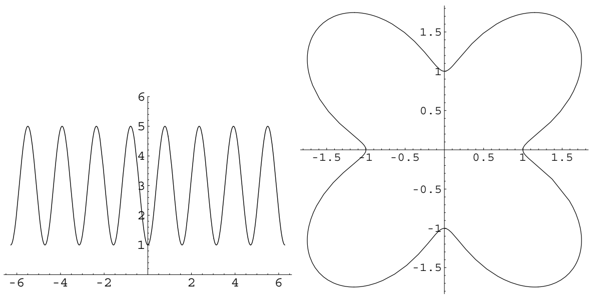
$$r^2 = 3 - 2\cos(4\theta)$$

is a member of $\mathcal{C}$ as defined above. See Fig. 5(b)

Example 4: The closed curve described by

$$r^2 = \frac{1}{a + be^{-\frac{1-cos4\theta}{d^2}}}$$

belongs to the family $\mathcal{C}$ of closed curves as defined above.



(a) (b)

Fig. 5. Examples

a) Constructing Lyapunov surface: We can now construct a Lyapunov surface as

$$V(r,\theta) = \frac{c_0 r^n}{h(\theta)}, \quad c_0 > 0, \quad h(\theta) \in \mathcal{T}, \quad n \in \mathbb{R}, \quad n > 1. \quad (14)$$

The following is a lemma for the properties of the associated Lyapunov function to the surface mentioned above. A proof is listed only for part 5 of the Lemma.

Lemma 1: The Lyapunov surface $V(r,\theta)$ of Eqn. (14) has the following properties:

1) *$V(r,\theta)$ is continuous and differentiable everywhere on $\mathbb{R}^2$;*
2) *at the origin of $\mathbb{R}^2$*
$$V(0,\theta) = 0;$$
3) *$V(r,\theta)$ is positive definite:*
$$V(r,\theta) > 0, \quad \forall\theta, \quad r > 0;$$
4) *all level curves of $V(r,\theta)$ belong to the family $\mathcal{C}$ of curves as defined above; and*
5) *all the level curves $V(r,\theta) = \xi$ have the same slope $\frac{dy}{dx}$ at the point of intersection with any line $\theta = \theta_0$ irrespective of the value of ξ.*

Proof of property 5 of Lemma 1: *Consider a level curve of $V(r,\theta)$*

$$\frac{c_0 r^n}{h(\theta)} = \xi, \quad \xi > 0 \quad (15)$$

Now for any curve in $\mathbb{R}^2$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}, \quad (16)$$

the point of intersection of the curve (15) with the line $\theta = \theta_0$ is given by

$$\left(\left[\frac{\xi h(\theta_0)}{c_0}\right]^{1/n}, \theta_0\right). \quad (17)$$

The quantity $\frac{dr}{d\theta}$ can be evaluated as

$$\frac{dr}{d\theta} = \frac{\xi}{c_0}\frac{h'(\theta)}{nr^{n-1}}. \quad (18)$$

Therefore the slope evaluated at the point of intersection is given by

$$\left.\frac{dy}{dx}\right|_{\left(\left[\frac{\xi h(\theta_0)}{c_0}\right]^{1/n}, \theta_0\right)} = \frac{h'(\theta_0)\sin\theta_0 + nh(\theta_0)\cos\theta_0}{h'(\theta_0)\cos\theta_0 - nh(\theta_0)\sin\theta_0} \quad (19)$$

which is independent of ξ. □

Example 5: The Lyapunov surface in Eqn. (12) in Section III can be generated by using the function $h(\theta)$ as given in example 2 and eqn.(14) with $n = 2$ and $c_0 = 1$ and then converting to Cartesian coordinates.

V. Rendezvous Certificates

In Section III we listed an example of a controller for achieving rendezvous. In this section we present a Lyapunov certificate theorem for rendezvous. Schemes for guaranteeing rendezvous are absolutely necessary to answer whether a mission in a cooperative control framework can be accomplished with a high degree of confidence in the presence of uncertainties. The uncertainty set can include differing flight conditions, local parametric variations, component failures on an aircraft, and communications variability such as loss of packets, temporary loss of link, etc. The result presented here is only a sufficient condition and we are currently working on finding the necessary conditions.

Consider the following system of two agents:

$$\begin{array}{lll} \mathcal{V}_1: & \dot{x}_1 = f_1(x_1,x_2); & f_1(0,0) = 0 \\ \mathcal{V}_2: & \dot{x}_2 = f_2(x_1,x_2); & f_2(0,0) = 0 \end{array} \quad (20)$$

where x_1 and $x_2 \in \mathbb{R}$. The problem is to determine whether or not $\mathcal{V}_1$ and $\mathcal{V}_2$ achieve rendezvous in the region $\mathcal{R}$ around the origin given a specification ρ_{des} as defined in Section II-B. Before we state our main result we give a few definitions and a lemma.

Definition 3: Coverage Angle: We define the *coverage angle* θ_0 as

$$\theta_0 = \tan^{-1}\left(\frac{1}{\rho_{\text{des}}}\right). \quad (21)$$

Since we know from Eqn. (8)

$$\rho_{\text{des}} \geq 1$$

therefore

$$\theta_0 \in [0, \pi/4]. \quad (22)$$

Definition 4: We define the region $\mathcal{I} \subset \mathbb{R}^2$ in polar coordinates as

$$\mathcal{I} = \{(r,\theta)| \quad \frac{n\pi}{2} + \theta_0 \leq \theta \leq \frac{(n+1)\pi}{2} - \theta_0, \quad n \in \mathbb{Z}\}. \quad (23)$$

The region $\mathcal{I}$ is shown in Fig. 6.

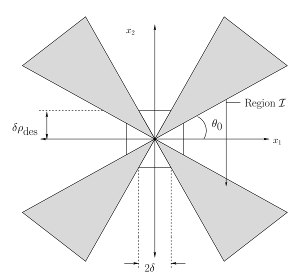


Fig. 6. The region $\mathcal{I}$

Definition 5: Define region $\mathcal{Z} \subset \mathbb{R}^2$ as

$$\mathcal{Z} = \mathcal{I} \cap \mathcal{W} \quad (24)$$

where $\mathcal{W}$ is given by Eqn. (2).

Note that by this definition, trajectories in $\mathcal{Z}$ also fall within the specification ρ_{des} and are thus considered valid for rendezvous. However, trajectories with initial conditions in $\mathcal{Z}$ may not stay in $\mathcal{Z}$. The following is a lemma for the invariance of $\mathcal{I}$ (and thus $\mathcal{Z}$) given a Lyapunov function of the form described in the previous section.

Lemma 2 (Invariance of region $\mathcal{I}$): Consider a system of two agents

$$\begin{array}{lll} \mathcal{V}_1: & \dot{x}_1 = f_1(x_1,x_2); & f_1(0,0) = 0 \\ \mathcal{V}_2: & \dot{x}_2 = f_2(x_1,x_2); & f_2(0,0) = 0 \end{array}$$

where x_1 and $x_2 \in \mathbb{R}$, and suppose that the origin is shown to *be asymptotically stable under a Lyapunov function of the form*

$$V(x_1, x_2) = \frac{c_0({x_1}^2 + {x_2}^2)^{n/2}}{h\left(\tan^{-1}\left(\frac{x_2}{x_1}\right)\right)} \tag{25}$$

with positive real constants c_0 *and* $n \geq 1$*, and with* $h \in \mathcal{T}$*. Furthermore, consider a coverage angle* θ_0 *corresponding to a design specification* ρ_{des} *and identified with regions* $\mathcal{I}$ *and* $\mathcal{Z}$*. The region* $\mathcal{I}$ *is an invariant region for the system if*

$$\left.\frac{(\frac{\partial V}{\partial x})^T . f(x_1, x_2)}{\|\frac{\partial V}{\partial x}\| \, \|f(x_1, x_2)\|}\right|_{x_2 = x_1 \tan\theta_0} \leq \cos(\pi + \theta_1 - \theta_0) \tag{26}$$

where

$$f(x_1, x_2)^T = [f_1(x_1, x_2) \quad f_1(x_1, x_2)]$$
$$\frac{\partial V}{\partial x}^T = \begin{bmatrix}\frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2}\end{bmatrix}$$

with

$$\theta_1 = \tan^{-1}\left(\frac{h'(\theta_0)\sin\theta_0 + nh(\theta_0)\cos\theta_0}{h'(\theta_0)\cos\theta_0 - nh(\theta_0)\sin\theta_0}\right), \text{ and } \theta_1 \geq \theta_0.$$

Note that θ_1 is defined along the boundary of $\mathcal{I}$ with $\theta_1 \geq \theta_0$. Conceptually, invariance is determined from the inner product of the closed-loop vector field f along the boundary of $\mathcal{I}$ and the boundary itself. This lemma follows from examining the geometry of the boundary of the region $\mathcal{I}$, the level set curves of the Lyapunov function V, and the trajectory; a proof is offered in the Appendix. Similar lemmas may follow from considering cases other than $\theta_1 \geq \theta_0$ and for other forms of the invariant region $\mathcal{I}$; we will explore those cases as this research is ongoing. Now we present the main result of the paper, a *Rendezvous Certificate Theorem*.

Theorem 1 (Rendezvous Certificate Theorem): Consider a system of two agents

$$\mathcal{V}_1 : \quad \dot{x}_1 = f_1(x_1, x_2); \quad f_1(0,0) = 0$$
$$\mathcal{V}_2 : \quad \dot{x}_2 = f_2(x_1, x_2); \quad f_2(0,0) = 0$$

where x_1 *and* $x_2 \in \mathbb{R}$*, and suppose that the origin is shown to be asymptotically stable under a Lyapunov function of the form*

$$V(x_1, x_2) = \frac{c_0({x_1}^2 + {x_2}^2)^{n/2}}{h\left(\tan^{-1}\left(\frac{x_2}{x_1}\right)\right)}$$

with positive real constants c_0 *and* $n \geq 1$*, and with* $h \in \mathcal{T}$*. Consider a coverage angle* θ_0 *corresponding to a design specification* ρ_{des} *and identified with regions* $\mathcal{I}$ *and* $\mathcal{Z}$*. If a region* $\mathcal{I}$ *is an invariant region for the system, then the agents attain rendezvous in the region* $\mathcal{R}$ *around the origin within the design specification for all initial conditions lying in the region* $\mathcal{Z}$*.*

Proof of Theorem 1: Follows from asymptotic stability of the origin and invariance of $\mathcal{I}$ with the associated Lyapunov function.

□

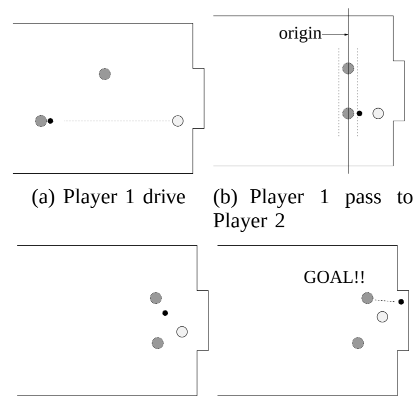
Note that the equation of any level set of the Lyapunov function $V(x_1, x_2)$, Eqn. (25) in polar coordinates is given by

$$r^n = \frac{\xi}{c_0} h(\theta)$$

This describes a many-one mapping from θ to r. In other words for a given value of θ there are several values of real positive r. Thus invariance of $\mathcal{I}$, and hence rendezvous, can be examined unambiguously. In other words, since we know from Lemma 1 property 5, that all level sets cut the line $\theta = \theta_0$ with the same slope at the point of intersection, the right hand side of Eqn. (26) is a constant.

Example 6: Consider the following scenario from soccer. Suppose two members from one team are driving the soccer ball towards their opponent's goal. These two members are traveling along the edges of the field, with one member in possession of the ball. If the team member with the ball, identified as Player 1, is too close to the opponent goal keeper, the opponent goal keeper is capable of either intercepting Player 1 or intercepting a pass from Player 1 to Player 2. If a pass is made too early, the goal keeper is capable of intercepting Player 2 after a pass is made to him.

Suppose these two players decide on the following strategy. Player 1 chooses to drive toward the goal, drawing the goal keeper toward him. In the meantime, Player 2 is also running toward the goal. Just before the goal keeper can intercept Player 1, Player 1 makes a pass to Player 2. The pass must be made out of the reach of the goal keeper. Finally, before the goal keeper can intercept Player 2, Player 2 scores a goal. This is illustrated in the following figures.



(a) Player 1 drive (b) Player 1 pass to Player 2

(c) Player 2 reception (d) Player 2 goal

Fig. 7. Soccer strategy.

This scenario may be cast as a rendezvous problem. The trajectories of the two players are linear and so the dynamics of the two players may be represented by a system of two scalar agents. The combined events of the pass from Player 1 to Player 2 and the final attempt at the goal is representative of rendezvous. The constraint of avoiding (player or ball) interception by the goal keeper may be posed as a rendezvous performance problem. Suppose that the dynamics of the two players about the origin, as shown in Fig. 7(b), are represented by

$$\mathcal{V}_1 : \quad \dot{x}_1 = -2x_1 - 4x_1(x_1^2 - x_2^2)$$
$$\mathcal{V}_2 : \quad \dot{x}_2 = -2x_2 - 4x_2(x_2^2 - x_1^2),$$

and suppose that the design specification given is $\rho_{\text{des}} = 1/(2 - \sqrt{3})$. Then with the Lyapunov function from Eqn. (12) (repeated here for clarity),

$$V(x_1, x_2) = (x_1^2 + x_2^2)\left[a + be^{-8x_1^2 x_2^2/d^2(x_1^2 + x_2^2)^2}\right],$$

and according to the theorem, the agents attain rendezvous for any initial condition lying in the region $\mathcal{Z}$ as defined according to the specification. Note that the corresponding coverage angle is $\theta_0 = 15°$.

VI. Concluding Remarks

We have approached the rendezvous problem from the point of view of dynamics on the phase plane and of Lyapunov stability and invariance. On the phase plane, rendezvous can be realized in a rigorous fashion through the introduction of the rendezvous region $\mathcal{R}$ and coverage region $\mathcal{Z}$ with the respective design specifications δ and ρ_{des}. Because of this phase plane interpretation, Lyapunov stability theory can be directly applied to both the construction of controllers for rendezvous and the certification of achieving rendezvous. Lyapunov-function based controller design is practical and intuitive for the rendezvous problem, because achieving rendezvous bears a connection to achieving asymptotic stability, and because level sets of the control Lyapunov function are related to the system trajectories. A level set method was introduced for constructing Lyapunov functions for the purpose of rendezvous control. Trajectories which begin in certain invariant regions of phase space achieve rendezvous, which can be used to motivate Lyapunov function and controller design. Finally, a certificate theorem was given as a sufficient condition for rendezvous for a system, given the existence of invariant regions of phase space corresponding to a Lyapunov function which guarantees asymptotic stability of the rendezvous point.

The phase plane interpretation for the rendezvous problem has applications in many areas and this research is ongoing. For instance, it would be interesting to explore the notions of rendezvous region and coverage region for different geometries than those discussed here.

In addition, similar certificate theorems can be constructed for other families of Lyapunov functions. The rendezvous problem may be recast include systems of larger numbers of agents with general dynamics in phase space of higher dimension. Necessary conditions could be explored for the rendezvous problem, and certificate theorems could be constructed for other types of rendezvous such as interception and avoidance.

VII. Acknowledgements

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VIII. Appendix

Proof of Lemma 2: Suppose that the origin of the system

$$\begin{array}{lll} \mathcal{V}_1: & \dot{x}_1 = f_1(x_1, x_2); & f_1(0,0) = 0 \\ \mathcal{V}_2: & \dot{x}_2 = f_2(x_1, x_2); & f_2(0,0) = 0 \end{array}$$

is asymptotically stable under the Lyapunov function

$$V(x_1, x_2) = \frac{c_0({x_1}^2 + {x_2}^2)^{n/2}}{h\left(\tan^{-1}\left(\frac{x_2}{x_1}\right)\right)}.$$

A proof will be constructed by contradiction.

We first assume the contradictory and say that a particular trajectory of the system (20)

$$x(t): \quad t \in [t_i, t_f] \tag{27}$$

with $x(t_i) \in \mathcal{Z}$ goes out of the region $\mathcal{I}$, i.e.

$$x(t_f) \notin \mathcal{I}. \tag{28}$$

Now since the trajectory is continuous there exists $t_c > t_i$ such that

$$x(t_c) \in \sigma(\mathcal{I}) \tag{29}$$

where $\sigma(\mathcal{I})$ denotes the boundary of the region $\mathcal{I}$.

Since $h(\theta) \in \mathcal{T}$ it is periodic with period $\pi/2$ and is symmetric about $\theta = \pi/4$, we can without loss of generality assume that $x(t_c)$ lies on the line $\theta = \theta_0$. Because of the periodic and symmetric nature of $h(\theta)$ a similar proof, like the one about to be presented will hold if $x(t_c)$ lies on any other line bounding region $\mathcal{I}$.

Since t_c is the time, the trajectory crosses over from region $\mathcal{I}$ to region $\mathbb{R}^2 - \mathcal{I}$, therefore

$$\begin{array}{lll} x({t_c}^-) & \in & \mathcal{I} \\ x({t_c}^+) & \in & \mathbb{R}^2 - \mathcal{I}. \end{array} \tag{30}$$

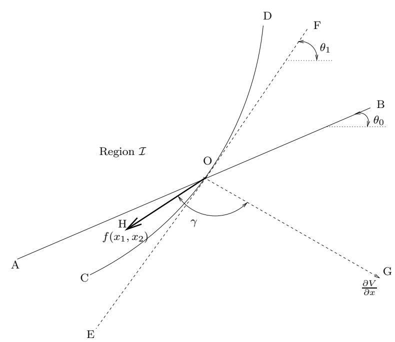

Fig. 8. Intersection of a level curve with the line $\theta = \theta_0$.

$$\dot{x}(t_c) = \begin{bmatrix} \dot{x_1}(t_c) \\ \dot{x_2}(t_c) \end{bmatrix} = f(x_1(t_c), x_2(t_c))$$

points out of the region $\mathcal{I}$.

Now refer to Fig. 8. O denotes the point $(x_1(t_c), x_2(t_c))$. AOB is the line $\theta = \theta_0$ or $x_2 = \tan(\theta_0)x_1$ in Cartesian coordinates. COD is the level curve of V that passes through the point O. EOF is the tangent and OG is the outward normal to the level curve COD at the point O. Thus OG represents the vector $\partial V/\partial x$. OH is the vector $f(x_1, x_2)$ and as already explained it points out of region $\mathcal{I}$.

Let γ be the angle between the vectors $\vec{OH}$ and $\vec{OG}$. But since the vector $\vec{OG}$ points away from the region $\mathcal{I}$ we have

$$\gamma < \pi/2 + \theta_1 - \theta_0 \tag{31}$$

and in light of Eqn. (22),

$$\gamma, \; [\pi/2 + \theta_1 - \theta_0] \in [0, \pi]. \tag{32}$$

Cosine is decreasing in the interval $[0, \pi]$, therefore

$$\cos\gamma > \cos[\pi/2 + \theta_1 - \theta_0]. \tag{33}$$

Note that

$$\cos\gamma = \left.\frac{(\frac{\partial V}{\partial x})^T \cdot f(x_1, x_2)}{\left\|\frac{\partial V}{\partial x}\right\| \|f(x_1, x_2)\|}\right|_{x_2 = x_1 \tan\theta_0}; \tag{34}$$

this implies that

$$\left.\frac{(\frac{\partial V}{\partial x})^T . f(x_1, x_2)}{\left\|\frac{\partial V}{\partial x}\right\| \|f(x_1, x_2)\|}\right|_{x_2 = x_1 \tan\theta_0} > \cos[\pi/2 + \theta_1 - \theta_0] \tag{35}$$

which contradicts Eqn. (26). Hence all trajectories of the system in Eqn. (20) that originate in the region $\mathcal{Z}$ remain in the region $\mathcal{I}$ for all time. $\square$