

# Polyhedral Cone Invariance Applied to Rendezvous of Multiple Agents

Abhishek Tiwari\* Jimmy Fung†  
Raktim Bhattacharya§ Richard M. Murray¶  
College of Engineering and Applied Science  
California Institute of Technology  
Pasadena, CA-91125

**Abstract**—In this paper, we pose the  $N$ -scalar agent rendezvous as a polyhedral cone invariance problem in the  $N$  dimensional phase space. The underlying dynamics of the agents are assumed to be linear. We derive a condition for positive invariance for polyhedral cones. Based on this condition, we demonstrate that the problem of determining a certificate for rendezvous can be stated as a convex feasibility problem. Under certain rendezvous requirements, we show that there are no robust closed-loop linear solutions that satisfy the invariance conditions. We show that the treatment of the rendezvous problem on the phase plane can be extended to the case where agent dynamics are non-scalar.

## I. INTRODUCTION

The demand for multi-agent coordination and cooperative control (as cited in [1] and [2] for instance) has led to the emergence of interesting control problems such as the *rendezvous problem*. In the rendezvous problem, one desires to have several agents arrive at predefined destination points simultaneously. Real applications of the rendezvous problem include cooperative strike and jamming, ballistic missile interception, spacecraft docking, formation flying, and multi-agent consensus. The rendezvous control problem has been treated in [3], [4], and [5]. However, a systematic theory of rendezvous is still to be explored.

In [3], we pose the two-scalar agent rendezvous problem as a combination of a cone invariance problem and a stability problem in a two-dimensional phase space. We presented a level-set method of constructing control Lyapunov functions. Based on this method, we derived the main result of the paper, a certificate theorem for guaranteeing approximate rendezvous. Using the ideas from [3], we pose the  $N$ -dimensional rendezvous problem on an  $N$ -dimensional phase space where the underlying closed-loop agent dynamics are linear. Because the underlying dynamics are linear, there exist quadratic control Lyapunov functions. Therefore, in this paper we focus our attention on satisfying cone invariance for rendezvous.

Invariance of polyhedral domains is well studied in the literature([6], [7], [8]). Traditionally, polyhedral invariance has been used to study the linear constrained regulation problem ([9], [10]) and problems with control and input

saturation ([11]). Because of the nature of these problems, polyhedral invariance literature is well developed when the polyhedral set is represented in the constraint form (plane representation in [7]). However, in rendezvous applications, we employ a worst case analysis and thus we usually deal with polyhedral sets represented in the generator form (vertex representation in [7]). In this paper, we derive invariance conditions for polyhedral cones represented in the generator form.

In section II we introduce the notation used in the paper and basic results from linear algebra. In section III we represent the  $N$  scalar agent rendezvous problem on the phase plane, and define constraints on the trajectories. In section IV we present a rendezvous certificate theorem. In section V we analyse the implications of the cone invariance conditions on the eigenstructure of the closed loop dynamics. In section VI we demonstrate the applicability of phase-plane concepts to non-scalar agent rendezvous. In section VII we provide a summary of the results in this paper and describe current research thrust.

## II. NOTATIONS AND MATHEMATICAL PRELIMINARIES

*Definition 1:* We will denote the  $2^N$  hyper-octants of the vector space  $\mathbb{R}^N$  as  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_{2^N}$ .

*Definition 2:* We denote the strict interior of a set  $\mathcal{S}$  by  $\text{int}(\mathcal{S})$ . The boundary of the set  $\mathcal{S}$  will be denoted by  $\beta(\mathcal{S})$ .

*Lemma 1:* Let  $v_1, v_2, \dots, v_{2^N}$  be vectors in  $\mathbb{R}^N$  such that

$$v_i \in \text{int}(\mathcal{O}_i).$$

then there is a set of  $N$  linearly independent vectors in the set of  $v_i$ s. In other words, there exist indices  $j_1, j_2, \dots, j_N$  such that

$$v_{j_1}, v_{j_2}, \dots, v_{j_N}$$

is a linearly independent set.

*Proof:* The proof of this lemma is presented in the appendix.

*Definition 3:* The conical hull of the points  $e_1, e_2, \dots, e_m$  in  $\mathbb{R}^N$  is the region defined by

$$\{x \in \mathbb{R}^N : x = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_m e_m, \alpha_i \in \mathbb{R}, \alpha_i \geq 0\}.$$

If  $E \in \mathbb{R}^{N \times m}$  then the conical hull of the columns of  $E$  will be denoted as  $\text{Cone}(E)$ . The points  $e_1, e_2, \dots, e_m$  are called the generators for the cone  $\text{Cone}(E)$ .

\*Graduate Student, atiwari@cds.caltech.edu

† Graduate Student, fung@caltech.edu

§ Postdoctoral Scholar, raktim@cds.caltech.edu

¶ Professor, murray@cds.caltech.edu

*Definition 4:* A polyhedral cone is the one which can be constructed by taking the conical hull of a finite number of generators.

*Definition 5:* A real  $m \times m$  matrix  $T$  is said to be nonnegative if all its terms are nonnegative, i.e.

$$T_{ij} \geq 0, \forall i, j.$$

We will denote this by

$$T \geq 0.$$

In order to distinguish from nonnegative matrices, we use the following symbols to denote sign definiteness.

$$\begin{aligned} T \text{ is symmetric positive definite} &\implies T \succ 0 \\ T \text{ is symmetric positive semidefinite} &\implies T \succeq 0 \\ T \text{ is symmetric negative definite} &\implies T \prec 0 \\ T \text{ is symmetric negative semidefinite} &\implies T \preceq 0 \end{aligned}$$

*Definition 6:* A real  $m \times m$  matrix  $T$  is said to be essentially nonnegative if all the off-diagonal terms are nonnegative, i.e.

$$T_{ij} \geq 0, \forall i \neq j.$$

We will denote this by

$$T \stackrel{e}{\geq} 0.$$

*Definition 7:* The spectral abscissa ([12])  $r(A)$  of an  $N \times N$  matrix  $A$  is defined as the maximum real parts of its eigenvalues.

$$r(A) = \max\{Re(\lambda) : \lambda \in spec(A)\}.$$

The following is a well known result in linear algebra and can be derived by extending Perron's results (theorem 8.3.1 in [13]) for nonnegative matrices to essentially nonnegative matrices.

*Lemma 2:* If  $T$  is an  $m \times m$  essentially nonnegative matrix, then  $r(T)$  is an eigenvalue of  $T$  and there is a nonnegative vector  $x \geq 0$ ,  $x \neq 0$ , such that  $Tx = r(T)x$ .

*Proof:* The proof of this lemma is presented in the appendix.

### III. N SCALAR AGENT RENDEZVOUS

In this section we will define the rendezvous problem for  $N$  scalar agents trying to rendezvous at the origin of the real line. We consider  $N$  scalar agents with closed loop linear dynamics.

$$\dot{x} = Ax, \quad (1)$$

$$x = [x_1 \ x_2 \ \cdots \ x_N]^T, \quad (2)$$

$$x_i \in \mathbb{R}.$$

For now, we will assume perfect communication between every pair of agents, therefore in general the matrix  $A$  is full. Rendezvous under lossy communication is an avenue of further ongoing research. The development in this section is very similar to the cone invariance ideas developed for the 2 scalar vehicle case in [3].

Ideally, rendezvous for  $N$  scalar agents  $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_N$  is said to be successful if all the  $N$  agents reach the origin at precisely the same time as each other. Being consistent with [3] we will refer to this as perfect rendezvous. For all practical purposes, it is sufficient that the agents reach the origin with in a small time interval of each other.

We will represent this problem on the  $N$  dimensional phase space. Define regions on the phase space

$$\begin{aligned} U_i &= \{(x_1, x_2, \dots, x_N) \mid -\delta \leq x_i \leq \delta\}, \\ i &\in \{1, 2, \dots, n\} \end{aligned} \quad (3)$$

the rendezvous hypercube is then

$$S = \bigcap_i U_i \quad (4)$$

The arrival times of the agents in the rendezvous region  $\mathcal{R}$  are

$$t_{\mathcal{V}_i} = \min [t \mid x(t) \in U_i] \quad i \in \{1, 2, \dots, n\} \quad (5)$$

We define the earliest arrival time  $t_a$  as

$$t_a = \min(t_{\mathcal{V}_1}, t_{\mathcal{V}_2}, \dots, t_{\mathcal{V}_N}) \quad (6)$$

The approximate rendezvous specification for the  $N$  scalar agents case can be written as

$$\rho = \frac{\max(|x_1(t_a)|, |x_2(t_a)|, \dots, |x_N(t_a)|)}{\delta} \leq \rho_{des} \quad (7)$$

For perfect rendezvous  $\rho_{des} = 1$ . Note that eqn.(7) is an upper bound on the infinity norm of the position of agents at the earliest time of arrival  $t_a$ . In other words eqn.(7) means that at the time of the earliest entry of an agent into the rendezvous region, the rest of the agents should not be farther than  $\delta\rho_{des}$ .

Consider the hypercube  $\mathcal{C}$  of side  $\delta(\rho_{des} - 1)$  in  $\mathbb{R}^n$  whose body diagonal is the line joining the points

$$A = (\delta, \delta, \dots, \delta)$$

and

$$B = (\delta\rho_{des}, \delta\rho_{des}, \dots, \delta\rho_{des})$$

Let  $\Upsilon$  be the set of all the vertices of  $\mathcal{C}$  except  $A$  and  $B$ . Define the polyhedral cone  $\mathcal{I}$  as

$$\mathcal{I} = Cone(x : x \in \Upsilon)$$

Note that  $\Upsilon$  has  $2^N - 2$  points. We will call these points  $e_1^\infty, e_2^\infty, \dots, e_{2^N - 2}^\infty$  as generators. The superscript  $\infty$  is used to denote that these points are the boundary points under  $\infty$  norm specification of approximate rendezvous. Define a matrix  $\mathbb{R}^{N \times (2^N - 2)}$  matrix  $E^\infty$  whose  $i$ th columns is the coordinates of the point  $e_i^\infty$ .

An important observation is that if the polyhedral cone  $\mathcal{I}$  is positively invariant with respect to the system in eqn.(1) and if the system (1) is asymptotically stable then all

trajectories of eqn.(1) that originate in the cone  $\mathcal{I}$  satisfy the approximate rendezvous specification.

In the following example we demonstrate how to identify the cone  $\mathcal{I}$  for  $N = 3$ .

*Example 1:* Using the approximate rendezvous specification given in eqn.(7) for the 3 scalar agent case, the generator points  $e_1^\infty, e_2^\infty, \dots, e_6^\infty$  in the positive orthant are found to be

$$\begin{aligned} e_1^\infty &= (\delta, \delta, \delta\rho_{des}) \\ e_2^\infty &= (\delta, \delta\rho_{des}, \delta) \\ e_3^\infty &= (\delta\rho_{des}, \delta, \delta) \\ e_4^\infty &= (\delta\rho_{des}, \delta\rho_{des}, \delta) \\ e_5^\infty &= (\delta\rho_{des}, \delta, \delta\rho_{des}) \\ e_6^\infty &= (\delta, \delta\rho_{des}, \delta\rho_{des}) \end{aligned}$$

The conical hull of the above points is the outer cone in fig.2.

Note that the approximate rendezvous can also be specified in the 2-norm or 1-norm sense, our region  $\mathcal{I}$  will be a second order cone or a polyhedral cone with  $N$  generators respectively.

The 2-norm case is dealt with in .For the case of 1-norm the approximate rendezvous specification takes the form

$$|x_1(t_a)| + |x_2(t_a)| + \dots + |x_N(t_a)| \leq \delta\rho_{des} \quad (8)$$

For perfect rendezvous

$$\rho_{des} = N \quad (9)$$

and for feasible approximate rendezvous

$$\rho_{des} \geq N \quad (10)$$

Eqn.(8) will give us  $N$  generator points in each of the hyper-octants. The invariant cones will be defined as the conical hull of the boundary points in each hyper-octant. In the following example we identify the desired invariant cone in  $\mathbb{R}_+^3$ .

*Example 2:* Using the approximate rendezvous specification given in eqn.(8) for the 3 scalar agent case, the generator points  $e_1^1, e_2^1$  and  $e_3^1$  in the positive orthant are found to be

$$\begin{aligned} e_1^1 &= (\delta, \delta, \delta(\rho_{des} - 2)) \\ e_2^1 &= (\delta, \delta(\rho_{des} - 2), \delta) \\ e_3^1 &= (\delta(\rho_{des} - 2), \delta, \delta) \end{aligned}$$

The conical hull of the above points is shown in fig. 1.

#### IV. RENDEZVOUS CERTIFICATE

In this section, we first present a lemma on invariance of polyhedral cones. A similar result appears in [6]. Based on this lemma, we then state state sufficient conditions for rendezvous of  $N$  scalar agents.

*Lemma 3:* Consider a system with closed loop dynamics given by eqn.(1). Let  $e_1, e_2, \dots, e_m$  be points in  $\mathbb{R}^N$  and let  $E$  be a matrix in  $\mathbb{R}^{N \times m}$  constructed by choosing these points as columns. Then the region  $Cone(E)$  is

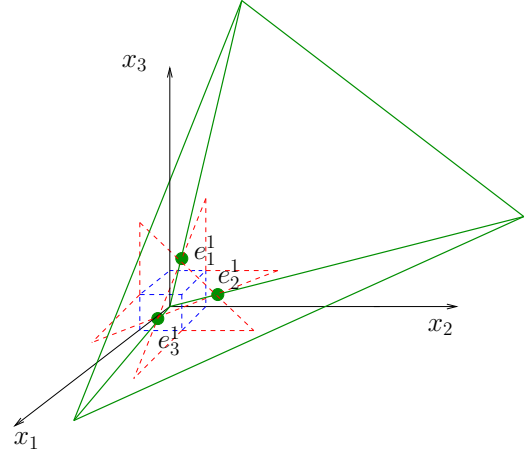


Fig. 1. The desired invariant polyhedral cone in  $\mathbb{R}_+^3$  under 1 norm specification.

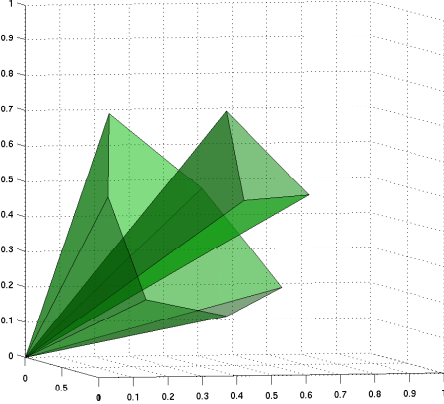


Fig. 2. The desired invariant polyhedral cones under 1 norm and  $\infty$  norm specifications for  $\delta = 0.2$  and  $\rho_{des} = 3.5$  in  $\mathbb{R}_+^3$ .

positively invariant w.r.t. system of eqn.(1) iff there exists an essentially nonnegative  $m \times m$  matrix  $T$  s.t.

$$AE = ET \quad (11)$$

*Proof: (Sufficiency)* Assuming condition (11) holds, we need to prove the following implication:

$$x(0) \in Cone(E) \Rightarrow x(t) \in Cone(E), \forall t > 0 \quad (12)$$

Now

$$AE = ET \quad (13)$$

$$\Rightarrow A^k E = ET^k, \forall k \in \mathbb{N} \quad (14)$$

$$\Rightarrow e^{At} E = Ee^{Tt} \quad (15)$$

Since  $x(0) \in \mathcal{I}$ , therefore there exists a nonnegative vector  $\alpha \in \mathbb{R}^m : \alpha \geq 0$ , s.t.

$$x(0) = E\alpha \quad (16)$$

The expression for  $x(t)$  is given as

$$x(t) = e^{At}x(0), \forall t \geq 0 \quad (17)$$

Substituting (16) in (17) and then using (15) we get

$$x(t) = Ee^{Tt}\alpha \quad (18)$$

Now we will use the following classical result from [14]  
 $T$  essentially nonnegative  $\Leftrightarrow e^{Tt}$  nonnegative:  
 $e^{Tt} \geq 0, \forall t \geq 0$ .

A nonnegative square matrix multiplied by a nonnegative vector will give us some nonnegative vector  $\beta$ . Therefore

$$x(t) = E\beta, \beta \geq 0 \quad (19)$$

Thus  $x(t) \in \text{Cone}(E)$ .

(Necessity) To prove necessity we assume that implication (12) holds.

Lets represent  $x(t)$  as

$$x(t) = E\alpha(t), \alpha(t) \geq 0 \forall t \geq 0 \quad (20)$$

Now lets consider an infinitesimal move from the  $i$ th ray of the polyhedral cone. We consider a point  $x_i^0$  on the  $i$ th ray given by

$$x_i^0 = E\alpha, \alpha_j = 0 \forall j \neq i, \alpha_i = 1 \quad (21)$$

Differentiating (20) at  $x_i^0$  gives us

$$\dot{x}(t) \Big|_{x=x_i^0} = E\dot{\alpha}(t) \Big|_{x=x_i^0} \quad (22)$$

$$\Rightarrow Ax_i^0 = E\dot{\alpha}(t) \Big|_{x=x_i^0} \quad (23)$$

$$\Rightarrow AE \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = E\dot{\alpha}(t) \Big|_{x=x_i^0} \quad (24)$$

For a trajectory starting at  $x = x_i^0$ , to stay inside the polyhedral cone  $\mathcal{I}$  we should have

$$\dot{\alpha}_j(t) \geq 0, \forall j \neq i \quad (25)$$

Combining (24) and (25) we can rewrite

$$AE \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = ET \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad (26)$$

Where the  $i$ th column of the matrix  $T$  is given by

$$T_{ji} = \dot{\alpha}_j(t) \Big|_{x=x_i^0} \quad (27)$$

Note that  $T$  is essentially nonnegative by construction

Similarly applying positive invariance for other rays of the polyhedral cone, we can prove that the action of  $AE$  is the same as the action of  $ET$  on a basis of  $\mathbb{R}^m$ .

Therefore there exists a  $\mathbb{R}^{m \times m}$  essentially nonnegative matrix  $T$  such that

$$AE = ET \quad (28)$$

□

The following is a certificate theorem for approximate rendezvous under  $\infty$  norm specification.

*Theorem 1: Consider  $N$  scalar agents with closed loop dynamics*

$$\dot{x} = Ax, \quad x \in \mathbb{R}^N$$

*If there exists a symmetric positive definite matrix  $P \in \mathbb{R}^{N \times N}$  and an essentially nonnegative matrix  $T \in \mathbb{R}^{(2^N-2) \times (2^N-2)}$  s.t.*

$$\left. \begin{array}{l} AE^\infty = E^\infty T \\ T \succeq 0 \end{array} \right\} \text{Positive Invariance and,}$$

$$\left. \begin{array}{l} A^T P + PA \prec 0 \\ P \succ 0 \end{array} \right\} \text{Asymptotic Stability.}$$

*where  $E^\infty \in \mathbb{R}^N \times 2^{N-2}$  is the matrix whose columns are the points  $e_1^\infty, e_2^\infty, \dots, e_{2^N-2}^\infty$ , then the agents will achieve rendezvous with  $\infty$  norm specification  $\rho_{des}$ , for all initial conditions lying in the region  $\text{Cone}(E^\infty)$ .*

*Proof:* The Proof of Theorem 1 directly follows from Lemma 3 and a well known result on asymptotic stability of linear systems.

□

*Notes:*

- 1) A similar result can be written down for the case when approximate rendezvous is specified in terms of 1 norm.
- 2) The conditions in the theorem are linear in  $T$  and  $P$ . Checking whether the conditions are satisfied is a convex feasibility problem.

*Example 3:* Consider the a closed loop system of 3 scalar agents described by

$$\dot{x} = \begin{bmatrix} -3.5 & 1.0607 & 1.0607 \\ 1.0607 & -4.25 & 0.75 \\ 1.0607 & 0.75 & -4.25 \end{bmatrix} x, \quad x \in \mathbb{R}^3. \quad (29)$$

Suppose we want to attain rendezvous in the  $\infty$  norm as well as the 1 norm sense for  $\rho_{des} = 3.5$  and  $\delta = 0.2$ . The corresponding  $E$  matrices in the first quadrant are found to be.

$$E_1^\infty = \begin{bmatrix} 0.2 & 0.2 & 0.7 & 0.7 & 0.7 & 0.2 \\ 0.2 & 0.7 & 0.2 & 0.7 & 0.2 & 0.7 \\ 0.7 & 0.2 & 0.2 & 0.2 & 0.7 & 0.7 \end{bmatrix}$$

$$E_1^1 = \begin{bmatrix} 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.2 \end{bmatrix}$$

The eigenvalues of the closed loop system are all negative so it is asymptotically stable. Solving the convex invariance

conditions of Theorem 1 gives us the following certificates of invariance for the  $\infty$  norm and 1 norm case.

$$T^\infty = \begin{bmatrix} -5.1204 & 0.27406 & 0.32365 & 0.40817 & 0.38185 & 0.68189 \\ 0.27406 & -5.1204 & 0.32365 & 0.38185 & 0.40817 & 0.68189 \\ 0.59777 & 0.59777 & -3.967 & 1.1164 & 1.1164 & 0.75311 \\ 0.33292 & 0.72742 & 0.40784 & -4.2645 & 0.70919 & 0.73746 \\ 0.72742 & 0.33292 & 0.40784 & 0.70919 & -4.2645 & 0.73746 \\ 0.31615 & 0.31615 & 0.25364 & 0.26942 & 0.26942 & -5.237 \end{bmatrix}$$

$$T^1 = \begin{bmatrix} -4.8561 & 0.14389 & 0.15153 \\ 0.14389 & -4.8561 & 0.15153 \\ 2.576 & 2.576 & -2.2878 \end{bmatrix}$$

Thus the system in eqn.(29) achieves rendezvous in both  $\infty$  and 1 norm sense for the given set of specifications.

Now consider the system

$$\dot{x} = \begin{bmatrix} -3.5 & 1.299 & 0.75 \\ 1.299 & -3.875 & 0.6495 \\ 0.75 & 0.6495 & -4.625 \end{bmatrix} x, x \in \mathbb{R}^3. \quad (30)$$

The eigenvalues of the closed loop matrix have negative real parts as before. The invariance conditions result in a feasible solution for the  $\infty$  norm case. But for 1 norm the problem is infeasible. The certificate  $T^\infty$  is found to be:

$$T^\infty = \begin{bmatrix} -5.6439 & 0.11616 & 0.11318 & 0.20068 & 0.36674 & 0.47836 \\ 0.38285 & -4.7636 & 0.47993 & 0.45272 & 0.49437 & 1.1309 \\ 0.51386 & 0.58304 & -4.1351 & 0.89419 & 1.4422 & 0.66432 \\ 0.37492 & 1.0447 & 0.8554 & -3.4123 & 1.0348 & 1.0556 \\ 0.60755 & 0.16985 & 0.15355 & 0.29562 & -5.1806 & 0.43858 \\ 0.44786 & 0.1523 & 0.14734 & 0.17189 & 0.2755 & -5.4927 \end{bmatrix}$$

This example shows that for given values of  $\rho_{des}$  and  $\delta$ , 1 norm specifications impose stricter constraints on the trajectories than  $\infty$  norm specifications. Fig. 3 shows some trajectories of the system in eqn.(30). Notice that while the trajectories are invariant w.r.t. to the  $\infty$  norm cone, they move in and out of the 1 norm cone.

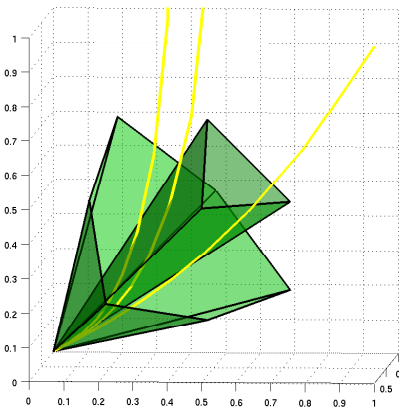


Fig. 3. Trajectories invariant w.r.t. the outer cone but not the inner cone

## V. IMPLICATION OF THE INVARIANCE RESULT ON THE EIGENSTRUCTURE OF $A$

Theorem 1 only guarantees rendezvous for initial conditions lying in  $Cone(E^\infty)$  which lies completely inside the positive hyper-orthant. If rendezvous has to be guaranteed for initial conditions lying in all equivalent cones in all other hyper-orthants (for example the region  $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3 \cup \mathcal{I}_4$

in fig. (4)), then the required sufficient conditions are the collection of all invariance conditions for each of these cones. But as we will find out in this section, satisfying all such conditions imposes restrictions on the eigenstructure of the closed loop  $A$  matrix and results in solutions which are non-robust.

In this section we will present results for approximate rendezvous under both the  $\infty$  norm and 1 norm specifications. Due to the nonsingular nature of the  $E$  matrices in the case of 1 norm, the result is much stronger, as compared to the  $\infty$  norm case. We first present results for the  $\infty$  norm case.

Recall from Section III, that for the  $\infty$  norm case the  $E^\infty$  matrix in the positive hyper-orthant belongs to  $\mathbb{R}^{N \times 2^N - 2}$ . If  $\rho_{des} > 1$  then  $E^\infty$  is full rank. From now on we will call the  $E^\infty$  matrix in the positive hyper-orthant to be  $E_1^\infty$ . The equivalent matrices in the other hyper-orthants will be numbered  $E_2^\infty, E_3^\infty, \dots, E_{2^N}^\infty$ .

Note that  $E_2^\infty, E_3^\infty, \dots, E_{2^N}^\infty$  can be obtained from  $E_1^\infty$  by multiplying it with a nonsingular rotation matrix. Therefore  $E_2^\infty, E_3^\infty, \dots, E_{2^N}^\infty$  are also full rank. We will now state and prove the following lemma.

*Theorem 2: Let  $Cone(E_1^\infty)$  be the desired invariant cone in the positive hyper-orthant for  $\infty$  norm approximate rendezvous specification  $\rho_{des} > 1$  and let  $Cone(E_2^\infty), Cone(E_3^\infty), \dots$  and  $Cone(E_{2^N}^\infty)$  be the symmetric rotations of  $Cone(E_1^\infty)$  in the other hyper-orthants. Now if all the above cones  $Cone(E_1^\infty), Cone(E_2^\infty), \dots$  and  $Cone(E_{2^N}^\infty)$  are positively invariant with respect to the system*

$$\dot{x} = Ax, x \in \mathbb{R}^N$$

*then all eigenvalues of  $A$  are real.*

*Proof:  $Cone(E_1^\infty), Cone(E_2^\infty), \dots$  and  $Cone(E_{2^N}^\infty)$  are positively invariant with respect to the above system, therefore by Lemma 3 there exist essentially nonnegative matrices  $T_1, T_2, \dots, T_{2^N}$ , such that*

$$AE_i^\infty = E_i^\infty T_i \quad (31)$$

$$T_i \succeq 0 \quad (32)$$

$$T_i \in \mathbb{R}^{(2^N - 2) \times (2^N - 2)} \quad (33)$$

Now by lemma 2,  $r(T_i)$  is an eigenvalue of  $T_i$  and there exists  $x_i \geq 0, x_i \neq 0$  such that

$$T_i x_i = r(T_i) x_i \quad (34)$$

Multiplying both sides by  $E_i^\infty$  we get

$$E_i^\infty T_i x_i = r(T_i) E_i^\infty x_i \quad (35)$$

$$AE_i^\infty x_i = r(T_i) E_i^\infty x_i \quad (36)$$

Now  $E_i^\infty x_i \neq 0$  and  $E_i^\infty x_i \in Cone(E_i^\infty)$ , therefore  $r(T_i)$  is an eigenvalue of  $A$  and there exists a corresponding eigenvector in  $Cone(E_i^\infty)$ .

This means that  $A$  has  $2^N$  eigenvectors in the strict interior of each orthant. Therefore by Lemma 1,  $N$  of these vectors are linearly independent. Therefore all  $N$

eigenvalues of  $A$  are real and are given by the spectral abscissa of the  $T_i$  matrices.

□

We are able to prove a stronger under 1 norm specification, which we present now,

*Theorem 3: Let  $Cone(E_1^1)$  be the desired invariant cone in the positive hyper-octant for 1 norm approximate rendezvous specification  $\rho_{des} > N$  and let  $Cone(E_2^1)$ ,  $Cone(E_3^1)$ ,  $\dots$  and  $Cone(E_{2N}^1)$  be the symmetric rotations of  $Cone(E_1^1)$  in the other hyper-octants. now if all are all the above cones  $Cone(E_1^1)$ ,  $Cone(E_2^1)$ ,  $\dots$  and  $Cone(E_{2N}^1)$  are positively invariant with respect to the system*

$$\dot{x} = Ax, \quad x \in \mathbb{R}^N$$

then  $A$  has a single real eigenvalue with algebraic multiplicity = geometric multiplicity =  $N$ .

*Proof:* All steps of the proof for Theorem 2 hold by replacing the matrices  $E_i^\infty$  by the matrices  $E_i^1$ . In addition now we know that  $E_i^1 \in \mathbb{R}^{N \times N}$ , therefore the matrices  $A$ ,  $T_1$ ,  $T_2$ ,  $\dots$  and  $T_{2N}$  are similar. Thus we have

$$r(T_1) = r(T_2) = \dots = r(T_{2N}) \quad (37)$$

So all eigenvalues of  $A$  are the same with  $N$  linearly independent eigenvectors.

□

*Example 4 (2 scalar agent rendezvous):* In this example we demonstrate Theorem 3 in the 2 dimensional phase space. In 2 dimensions the desired invariant cones are the same for both  $\infty$  norm and 1 norm cases. The cone  $\mathcal{I}_1$  in fig. 4 can be represented as

$$\mathcal{I}_1 = Cone(E_1^1) \quad (38)$$

$$E_1^1 = \begin{bmatrix} \delta \rho_{des} & \delta \\ \delta & \delta \rho_{des} \end{bmatrix} \quad (39)$$

The cones  $\mathcal{I}_2$ ,  $\mathcal{I}_3$  and  $\mathcal{I}_4$  can be generated by rotating  $\mathcal{I}_1$  by  $\pi/2$ ,  $\pi$ , and  $3\pi/2$  radians respectively therefore

$$\mathcal{I}_i = Cone(E_i^1), \quad (40)$$

$$E_i^1 = R^{i-1} E_1^1, \quad i \in \{1, 2, 3, 4\} \text{ and } (41)$$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (42)$$

Now from Lemma 3, if all the cones  $\mathcal{I}_1$ ,  $\mathcal{I}_2$ ,  $\mathcal{I}_3$  and  $\mathcal{I}_4$  are positively invariant w.r.t. the system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^2,$$

then  $A$  has a unique real eigenvalue with algebraic multiplicity = geometric multiplicity = 2. In other words the system  $\dot{x} = Ax$  has radial vector fields as shown in fig. (5).

Note:

- If  $\rho_{des} > N$  the polyhedral cone corresponding to the 1norm specification is fully contained in the corresponding polyhedral cone for the  $\infty$  norm case. We

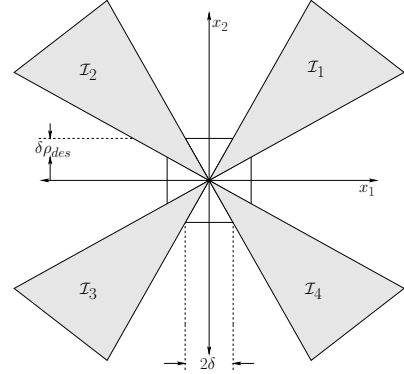


Fig. 4. The desired invariant cones for 2 scalar agent rendezvous.

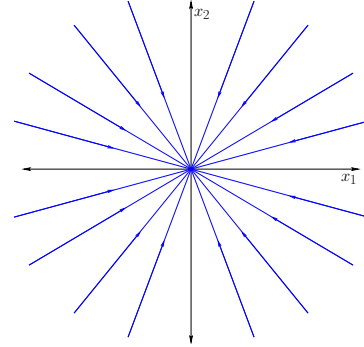


Fig. 5. Radial vector fields.

conjecture that the only solution possible in the *inf*ty norm specification is the one that results in radial fields (all eigenvalues same and real), however we are still working on the proof of this stronger result.

- The radial fields solutions thus obtained are non-robust to disturbance and uncertainty. The trajectories live on the boundary of the polyhedral cone and can easily deviate out of the cone under uncertainty.
- If rendezvous is desired for initial conditions lying in all the hyperoctants, nonlinear control design along the lines of [3] is likely to give a robust solution.

## VI. NON-SCALAR AGENT RENDEZVOUS

In this section we will demonstrate the applicability of the theory developed in sections III and IV to the case of non-scalar agents. We will demonstrate the simple case of two planar agents trying to rendezvous at the origin of  $\mathcal{R}^2$ . Let us consider two planar vehicles with combined closed loop dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} \quad (43)$$

where  $x_i, y_i \in \mathcal{R}$ .

In fig. 6, the rendezvous task for agents 1 and 2 is to reach the inner square of side  $\delta$  around origin within a small time interval of each other. To accomplish this task,

we require that at the first instant when one agent enters the inner square, the other agent should be at least inside the outer square of side  $\delta\rho_{des}$ . To state this condition formally we define:

$$\begin{aligned} t_{\nu_i} &= \min [ t : \max\{|x_i(t)|, |y_i(t)|\} \leq \delta ], \\ i &\in \{1, 2\}, \\ t_a &= \min \{t_{\nu_1}, t_{\nu_2}\}. \end{aligned}$$

In this sense,  $t_{\nu_i}$  is the arrival time of the  $i$ -th agent to the inner square, Therefore for successful rendezvous,

$$\max_i [\max\{|x_i(t_a)|, |y_i(t_a)|\}] \leq \delta\rho_{des}. \quad (44)$$

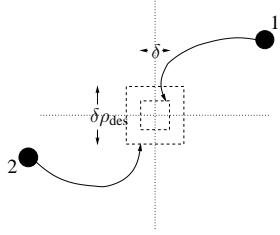


Fig. 6. Rendezvous on a plane.

Define the region  $\mathcal{W} \in \mathbb{R}^4$  as

$$\mathcal{W} = \left( \begin{array}{l} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} : \begin{array}{l} \frac{1}{\rho_{des}} \leq \left| \frac{x_1}{x_2} \right| \leq \rho_{des} \\ \frac{1}{\rho_{des}} \leq \left| \frac{y_1}{y_2} \right| \leq \rho_{des} \end{array} \end{array} \right). \quad (45)$$

In order to guarantee rendezvous for all initial conditions lying in  $\mathcal{W}$ , we will break down the planar rendezvous problem into two scalar rendezvous problems. The idea being, that if  $x_1, x_2$  and  $y_1, y_2$  attain rendezvous in their respective two dimensional phase spaces, then rendezvous will be successful on the plane.

We define the matrix  $E_1^{\text{planar}}$  as

$$E_1^{\text{planar}} = \begin{bmatrix} \delta & \rho_{des} & 0 & 0 \\ 0 & 0 & \delta & \rho_{des} \\ \rho_{des} & \delta & 0 & 0 \\ 0 & 0 & \rho_{des} & \delta \end{bmatrix}. \quad (46)$$

Its is easy to verify the following result:

*Lemma 4:*

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} \in \text{Cone}(E_1^{\text{planar}}) \quad (47)$$

if and only if

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Cone} \left( \begin{bmatrix} \delta & \delta\rho_{des} \\ \delta\rho_{des} & \delta \end{bmatrix} \right) \quad (48)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \text{Cone} \left( \begin{bmatrix} \delta & \delta\rho_{des} \\ \delta\rho_{des} & \delta \end{bmatrix} \right). \quad (49)$$

Now we can state the rendezvous certificate theorem for planar rendezvous,

*Theorem 4:* If the system in eqn.(43) is asymptotically stable and there exists a  $4 \times 4$  matrix  $T$  such that

$$AE_1^{\text{planar}} = E_1^{\text{planar}}T \quad (50)$$

$$T \stackrel{e}{\geq} 0, \quad (51)$$

then we can guarantee rendezvous for all initial conditions such that

$$\begin{aligned} x_i(0), y_i(0) &\geq 0 \\ \begin{bmatrix} x_1(0) \\ y_1(0) \\ x_2(0) \\ y_2(0) \end{bmatrix} &\in \mathcal{W}. \end{aligned}$$

*Proof:* The proof of the theorem follows from Lemma 3 and Lemma 4. □

It is important to note that this theorem only provides sufficient conditions for rendezvous, and yields certificates for trajectories where an agent never crosses from one quadrant to another on the plane. For instance, although the trajectories shown in fig.7 achieve successful planar rendezvous, the trajectories violate the invariance conditions required by the theorem. Deriving a more general certificate theorem which covers these cases is the subject of ongoing research.

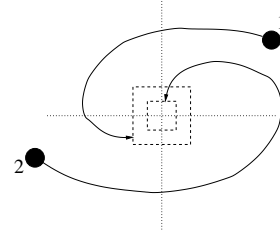


Fig. 7. Limitations of Theorem 4.

## VII. CONCLUSIONS AND FUTURE WORK

We have extended the concepts outlined in [3] to the case of  $N$  scalar agents and have demonstrated their utility for non-scalar agents. While in [3] we considered 2 scalar agents with non-linear dynamics, in this paper the underlying dynamics are always assumed to be linear. We have employed the theory of invariance of polyhedral regions to derive a set of convex conditions, which when feasible yield a certificate of successful rendezvous. We have also shown that if rendezvous certificates are desired for initial conditions lying in a much larger symmetric set around the origin, the problems is over-constrained. The only feasible closed-loop linear dynamics that satisfy this over-constrained problem are the ones with radially decaying vector fields. All such solutions are non-robust to uncertainties. This suggests that for robustness in the over-constrained case, we need to use non-linear synthesis.



Currently we focus on the problem of designing linear state feedback controller for rendezvous of  $N$  scalar agents. A first attempt at the synthesis problem lead us to a set of conditions which are bilinear. We are surveying mathematics literature ([15], [16]) for a method to minimize the spectral abscissa of an essentially nonnegative matrix under cone invariance constraints. Future directions also include introducing uncertainty and communication link failure between the cooperating agents.

#### REFERENCES

- [1] T. McLain. Coordinated control of unmanned air vehicles. Technical Report ASC-99-2426, Air Vehicles Directorate of the Air Force Research Laboratory, 1999.
- [2] Inc. Alphatech. Alphatech Technologies and Products: Dynamic Control of Agent-Based Systems. <http://www.alphatech.com/secondary/techpro/task.html>.
- [3] A. Tiwari, J. Fung, J. Carson, R. Bhattacharya, and R. M. Murray. A Framework for Lyapunov Certificates for Multi-Vehicle Rendezvous Problems. *American Control Conference*, 2004.
- [4] P. Chandler and M. Pachter. Hierarchical Control for Autonomous Teams. *AIAA Guidance Navigation and Control Conference*, Aug 2001.
- [5] P. R. Chandler, M. Pachter, D. Swaroop, J. M. Fowler, J. K. Howlett, S. Rasmussen, C. Schumacher, and K. Nygard. Complexity in UAV Cooperative Control. *Proceedings of American Control Conference*, May 2002.
- [6] E. B. Castelan and J. C. Hennes. On Invariant Polyhedra of Continuous-Time Linear Systems. *IEEE Transactions on Automatic Control*, 38(11):1680–1685, Nov 1993.
- [7] F. Blanchini. Set Invariance in Control. *Automatica*, 35(11):1747–1767, Nov 1999.
- [8] K. Yoshida, H. Kawabe, and Y. Nishimura. Simple LP-Type Criteria for Positively Invariant Polyhedral Sets. *IEEE Transactions on Automatic Control*, 45(1):98–101, Jan 2000.
- [9] M. Vassilaki and G. Bitsoris. Constrained Regulation of Linear Continuous-Time Dynamical Systems. *Systems and Control Letters*, 13:247–252, 1989.
- [10] C. E. T. Dórea and B. E. A. Milani. Design of L-Q Regulators for State Constrained Continuous-Time Systems. *IEEE Transactions on Automatic Control*, 40(3):544–548, Mar 1995.
- [11] J. M. Gomes da Silva Jr. and S. Tarbouriech. Invariance and Contractivity of Polyhedra for Linear Continuous-Time Systems with Saturating Controls. *Revista Controle & Automação*, 10(3):149–156, Dec 1999.
- [12] A. Berman, M. Neumann, and R. J. Stern. *Nonnegative Matrices in Dynamic Systems*. Wiley-Interscience, 1989.
- [13] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1999.
- [14] H. Schneider and M. Vidyasagar. Cross-positive matrices. *SIAM Journal of Numerical Analysis*, 7(4), Nov 1970.
- [15] J. E. Cohen. Convexity of the Dominant Eigenvalue of an Essentially Nonnegative Matrix. *Proceedings of the American Mathematical Society*, 81(4):657–658, Apr 1981.
- [16] C. R. Johnson and D. P. Stanford. Dominant Eigenvalues Under Trace-Preserving Diagonal Perturbations. *Linear Algebra and its Applications*, 212/213:415–435, 1994.

#### VIII. APPENDIX

*Proof of Lemma 1:* We will need the following Lemma for this proof

*Lemma 5:* Let  $v_1, v_2, \dots, v_{2N}$  be vectors in  $\mathbb{R}^N$  such that  $v_i \in \text{int}(\mathcal{O}_i)$  then there exist  $\alpha_i \in \mathbb{R}$ ,  $\alpha_i \geq 0$  such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{2N} v_{2N} = 0 \quad (52)$$

We skip the proof of Lemma 5. Continuing with the proof of Lemma 1, let  $e_i = [0, 0, \dots, 1, 0, \dots, 0]$  with

1 in the  $i$ th coordinate, then we shall show that  $e_i \in \text{span}\{v_1, v_2, \dots, v_{2N}\}$ . Renumber  $v_j$ 's if necessary so that the  $i$ th coordinate of the first  $2^{N-1}$  vectors is positive. Let  $u_j$  be the vector  $v_j$  without the  $i$ th coordinate. Then  $u_1, \dots, u_{2^{N-1}}$  are in different hyper-octants of  $\mathbb{R}^{2^{N-1}}$ . Therefore by Lemma 5 there exist nonnegative reals  $\alpha_1, \dots, \alpha_{2^{N-1}}$  such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_{2^{N-1}} u_{2^{N-1}} = 0$$

Hence

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{2^{N-1}} v_{2^{N-1}} = \alpha e_i$$

where  $\alpha > 0$ . Hence  $e_i \in \text{span}\{v_1, v_2, \dots, v_{2N}\}$ . Hence proved.  $\square$

*Proof of Lemma 2:* If  $T \stackrel{e}{\geq} 0$  then there is some  $\psi > 0$  such that  $\psi I + T \geq 0$ .

Note that  $x_i$  is an eigenvector of  $\psi I + T$  with corresponding eigenvalue  $\lambda_i$  iff  $x_i$  is an eigenvector of  $T$  with the corresponding eigenvalue  $\lambda_i - \psi$ . Now from Theorem 8.3.1 in [13] we know that the spectral radius  $\rho(\psi I + T)$  is an eigenvalue of  $\psi I + T$  with a nonnegative eigenvector. Hence  $\rho(\psi I + T) - \psi$  is an eigenvalue of  $T$  with a nonnegative eigenvector. Now if  $\lambda_i$  are the set of eigenvalues of  $\psi I + T$  then we have

$$\text{Re}(\rho(\psi I + T) - \psi) \geq \text{Re}(\lambda_i - \psi)$$

Hence proved.  $\square$