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Model reduction of interconnected linear systems

H. Sandberg [1] ,* and R. M. Murray [2]

[1] *School of Electrical Engineering, Automatic Control, Royal Institute of Technology (KTH), Osquldas väg 10, SE-100 44 Stockholm, Sweden*

[2] *California Institute of Technology, Control and Dynamical Systems, M/C 107-81, Pasadena, CA 91125, U.S.A.*

SUMMARY

The problem of model reduction of linear systems with certain interconnection structure is considered in this paper. To preserve the interconnection structure between subsystems in the reduction, special care needs to be taken. This problem is important and timely because of the recent focus on complex networked systems in control engineering. Two different model-reduction methods are introduced and compared in the paper. Both methods are extensions to the well-known balanced truncation method. Compared to earlier work in the area these methods use a more general linear fractional transformation framework, and utilize linear matrix inequalities. Furthermore, new approximation error bounds that reduce to classical bounds in special cases are derived. So-called structured Hankel singular values

*Correspondence to: H. Sandberg, School of Electrical Engineering, Automatic Control, Royal Institute of Technology (KTH), Osquldas väg 10, SE-100 44 Stockholm, Sweden

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are used in the methods, and indicate how important states in the subsystems are with respect to a chosen input-output map for the entire interconnected system. It is shown how these structured Hankel singular values can be used to select approximation order. Finally, the two methods are applied to a model of a mechanical device.

KEY WORDS: Model reduction, interconnected systems, networked systems, approximation

1. INTRODUCTION

The motivation for this work on model reduction of interconnected systems is that many models that are of interest to the control community have a network structure [1]. Examples include models of the power grid, formations of vehicles, but also control systems where controllers, actuators, and sensors are distributed over a computer network. In all of these examples there can be many subsystems that are interconnected in one way or another, and the order of the entire system can be very large. It is often desirable to obtain a model with fewer (differential) equations and whose trajectories are provably close to the original model's trajectories. There are standard methods to do this, see for example [2, 3], but unfortunately these methods do not preserve interconnection structures. A standard reference for model reduction in general is the book by Obinata and Anderson [4].

The problem of model reduction of linear systems with a special interconnection structure is considered here. The models consist of interconnected subsystems and the reduced models should retain the original interconnection structure. A naive approach to solve this problem is to approximate subsystems separately in open loop and to interconnect them according to the

original structure. This naive approach does not take the dynamics of the entire system into account when approximating each subsystem, and will only work under special circumstances. A typical case when the naive approach does not work is when the surrounding system excites modes in subsystems that are unimportant in open loop. A more sophisticated method to approximate subsystems is to use frequency-weighted approximation to take the surrounding system into account, see [5]. However, this leads to an iterative procedure. The goal here is to develop more general methods that generate approximation candidates for all subsystems at the same time, in one shot, and to derive error bounds and guidelines for how many states that should be retained in the subsystems.

Two different methods are studied in this paper. The first method is a heuristic that often works, but can fail (generate unstable approximations). The second method is guaranteed to deliver stable approximations and comes with an a priori error bound. However, it requires that there are block-diagonal solutions to two linear matrix inequalities, which is a severe restriction. Hence, we are partly successful in achieving the above goals in this paper. It should be noted, however, that the second method is more widely applicable than one would first think, since it can be applied to meaningful intermediate models. The details of this extension is given in [6].

Earlier work in this area include the work of Li and Paganini [7], and Vandendorpe and Van Dooren [8, 9]. In [7], linear matrix inequalities are used to find structured coordinate transformations, suitable for state truncation. In [8, 9], ideas for frequency-weighted balanced truncation [2] and closed-loop balanced truncation [10] are used to solve the same problem. Just as in [7], we note here the importance of finding block-diagonal solutions to certain linear matrix inequalities. Block-diagonal solutions to linear matrix ineqaulites have long been used

in model reduction of uncertain systems [11], and for controller order reduction [12]. The main problem here is that block-diagonal solutions to the relevant linear matrix inequalities only exist under special circumstances.

The contributions of this paper are that we extend the framework in [8, 9] by using a linear fractional transformation framework and linear matrix inequalities. Furthermore, bounds on the approximation error are stated, and it is shown how structured Hankel singular values can be used to select approximation order for subsystems.

The structure of the paper is as follows. In Section 2, the model framework is introduced and an example illustrates the framework. In Section 3, the two model reduction methods are described, and error bounds are derived. In Section 4, the model reduction methods are applied to the example in the introduction.

Notation

Most notation in the paper is standard. To define transfer function matrices we use the notation $C(sI-A)^{-1}B+D =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] =: [A,B,C,D]$, and $\deg G(s)$ is the McMillan degree of the transfer function matrix $G(s)$. The set RH_∞ is the set of real and rational transfer function matrices in the Hardy space H_∞ [13]. The Laplace transform of a time-domain signal $u(t)$ is denoted by $u(s)$, with some abuse of notation. Let $\|\cdot\|_{[0,\tau]}$ and $\langle\cdot,\cdot\rangle_{[0,\tau]}$ denote the standard norm and scalar product on L_2 over the time interval $[0,\tau]$, and $\|G(s)\|_\infty$ denote the H_∞-norm [13] of $G(s)$. With $\mathrm{diag}\{P_1,P_2\}$ we mean the block-diagonal matrix $\left[\begin{array}{cc} P_1 & 0 \\ 0 & P_2 \end{array}\right]$, with $P\geq 0$ ($P\leq 0$) that P is a positive (negative) semi-definite matrix, and with $|x|_P$ the weighted Euclidean norm $\sqrt{x^TPx}$.

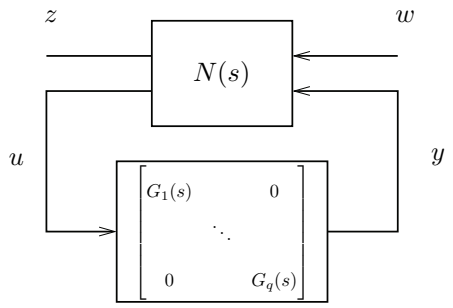

Figure 1. The interconnected system. The subsystem transfer functions to be reduced, $G_1(s), \ldots, G_q(s)$, are stored in a block-diagonal transfer function matrix $G(s)$ and are connected to the transfer function matrix $N(s)$ that models the interconnection topology and excitation and measurement dynamics.

2. MODEL FRAMEWORK AND PROBLEM STATEMENT

2.1. Model Framework

The model framework that is used throughout the paper is introduced here. An important restriction is that only interconnections of finite-dimensional linear time-invariant systems are considered. The interconnected system is modelled with a linear fractional transformation of two transfer function matrices $G(s)$ and $N(s)$. The interconnection is illustrated in Figure 1. In $G(s)$, the q subsystem transfer function matrices that we want to reduce are stored. It has the block-diagonal structure

$$G(s) = \begin{bmatrix} G_1(s) & & 0 \\ & \ddots & \\ 0 & & G_q(s) \end{bmatrix} =: \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right],$$

where $G_k(s)$, $k = 1 \ldots q$, are the transfer function matrices of the q subsystems, each with m_k inputs and p_k outputs. The block-diagonal transfer function matrix $G(s)$ can always be

realized in the form

$$\begin{aligned} A_G &= \operatorname{diag}\{A_1, \ldots, A_q\}, \quad B_G = \operatorname{diag}\{B_1, \ldots, B_q\}, \\ C_G &= \operatorname{diag}\{C_1, \ldots, C_q\}, \quad D_G = \operatorname{diag}\{D_1, \ldots, D_q\}, \end{aligned} \tag{1}$$

where

$$A_k \in \mathbb{R}^{n_k \times n_k}, \quad B_k \in \mathbb{R}^{n_k \times m_k}, \quad C_k \in \mathbb{R}^{p_k \times n_k}, \quad D_k \in \mathbb{R}^{p_k \times m_k},$$

such that $G_k(s) = D_k + C_k(sI - A_k)^{-1}B_k \in \mathbb{C}^{p_k \times m_k}$. The realization of $G(s)$ is assumed to have the form (1) in the following. The set of all transfer function matrices of the same block-diagonal structure as $G(s)$ is denoted by $\operatorname{Struct} G(s)$,

$$\operatorname{Struct} G(s) = \{F(s) : F(s) = \operatorname{diag}\{F_1(s), \ldots, F_q(s)\} \text{ and } F_k(s) \in \mathbb{C}^{p_k \times m_k},\ k = 1, \ldots, q\}. \tag{2}$$

The interconnection dynamics of the subsystems in $G(s)$ and the external excitation and measurement dynamics are stored in the transfer function matrix $N(s)$, defined by

$$N(s) = \begin{bmatrix} E(s) & F(s) \\ H(s) & K(s) \end{bmatrix} =: \left[\begin{array}{c|cc} A_N & B_{N,1} & B_{N,2} \\ \hline C_{N,1} & D_E & D_F \\ C_{N,2} & D_H & D_K \end{array}\right], \tag{3}$$

where

$$A_N \in \mathbb{R}^{n_N \times n_N}, \quad B_{N,1} \in \mathbb{R}^{n_N \times m_N}, \quad C_{N,1} \in \mathbb{R}^{p_N \times n_N}, \quad D_E \in \mathbb{R}^{p_N \times m_N},$$

and the dimensions of the other matrices are chosen to conform with $G(s)$. The entries of $N(s)$ have the following interpretation: Let the signal $w(t) \in \mathbb{R}^{m_N}$ be the external excitation of the interconnected system, and $z(t) \in \mathbb{R}^{p_N}$ be the external measurement signal. Define the interconnection signals $y(t) \in \mathbb{R}^{\sum_i p_i}$ and $u(t) \in \mathbb{R}^{\sum_i m_i}$. The entry $K(s)$ models how the subsystems in $G(s)$ interact with each other and the entry $H(s)$ models how the external signal $w(t)$ effects the subsystems,

$$u(s) = H(s)w(s) + K(s)y(s), \quad y(s) = G(s)u(s).$$

The Laplace transform of the measurements $z(t)$ are given by the entries $E(s)$ and $F(s)$ as

$$z(s) = E(s)w(s) + F(s)y(s).$$

Hence, the transfer function matrix of the complete interconnected system is given by the lower linear fractional transformation

$$\mathcal{F}_l(N,G) := E(s) + F(s)(I - G(s)K(s))^{-1}G(s)H(s) \tag{4}$$

$$= \left[\begin{array}{cc|c} A_N + B_{N,2}LD_GC_{N,2} & B_{N,2}LC_G & B_{N,1} + B_{N,2}LD_GD_H \\ B_GMC_{N,2} & A_G + B_GMD_KC_G & B_GMD_H \\ \hline C_{N,1} + D_FD_GMC_{N,2} & D_FLC_G & D_E + D_FD_GMD_H \end{array}\right] \tag{5}$$

$$=: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad L := (I - D_GD_K)^{-1}, \quad M := (I - D_KD_G)^{-1}.$$

We assume throughout the paper that the interconnection (4) is well posed and internally stable [13], and $\mathcal{F}_l(N,G) \in RH_\infty$. The following definition is made.

Definition 1. A realization of $\mathcal{F}_l(N,G)$ in the form (5) where A_G, B_G, C_G, D_G are in the block-diagonal form (1) is called a *structured realization* of the interconnected system $\mathcal{F}_l(N,G)$.

Remark 1. The linear fractional transformation is a systematic tool used to handle structure in robust control. Often it is used to model structured model uncertainty ("pulling out the Δ's"), see Chapter 10 in [13], but it can just as well be used to pull out subsystem transfer functions $G_k(s)$, as is done here.

2.2. Problem Statement

The first problem we would like to solve is

$$\hat{G}_{opt}(s) := \arg \min_{\substack{\hat{G}(s) \in \text{Struct } G(s) \\ \deg \hat{G}_k(s) \le r_k,\, k=1,\ldots,q}} \|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\hat{G})\|_\infty, \tag{6}$$

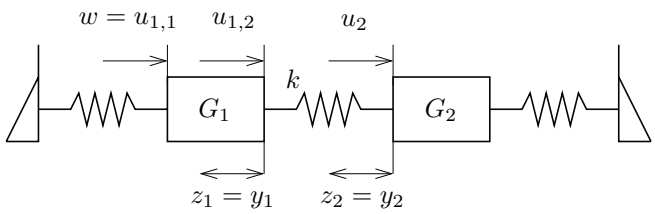

Figure 2. An interconnected mechanical system.

for fixed $r_k \leq n_k$, $k = 1, \ldots, q$, and Struct $G(s)$ is defined in (2). Notice that the dynamics in $N(s)$ is fixed. A solution to this problem would give optimal (in H_∞-sense) reduced-order subsystems $\hat{G}_{opt,k}(s)$ with respect to the entire interconnected system. The second problem we would like solve is to be able to state a simple priniciple on how to choose approximation orders r_k for the subsystems.

Both of these problems are very hard. Even if the structure requirement $\hat{G}(s) \in \text{Struct}\, G(s)$ in (6) is dropped, (6) is a nonconvex optimization problem; see [14]. Instead of trying to solve (6), we will therefore suggest two heuristics in Section 3 that generate candidate approximations $\hat{G}_k(s)$ such that $\deg \hat{G}_k(s) \leq r_k$ and $\hat{G}(s) \in \text{Struct}\, G(s)$ and, in some cases, also provide upper a priori bounds on the error $\|\mathcal{F}_l(N, G) - \mathcal{F}_l(N, \hat{G})\|_\infty$. Structured Hankel singular values will be introduced and used to provide guidelines for selecting approximation orders r_k.

Example 1 (Interconnected mechanical models) Consider Figure 2 where two elastic masses are interconnected by a linear spring. The first mass model, $G_1(s)$, takes as input the two forces $u_{1,1}(t)$ and $u_{1,2}(t)$, and as output the position $y_1(t)$ of one of its edges. The second mass model, $G_2(s)$, takes as input the force $u_2(t)$ and as output the position $y_2(t)$ of one of its edges. These models may be of very high degree, for instance if they are discretized partial

differential equations of elastic bodies. The models are interconnected with a linear spring k, and the mapping of interest could be how the force $w(t) = u_{1,1}(t)$ maps to the two positions $z(t) = [y_1(t) \quad y_2(t)]^T$. The entries of $N(s)$ are

$$E = D_E = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \quad F = D_F = \begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \end{pmatrix}, \quad H = D_H = \begin{pmatrix} 1 \\ \\ 0 \\ \\ 0 \end{pmatrix}, \quad K = D_K = \begin{pmatrix} 0 & 0 \\ \\ -k & k \\ \\ k & -k \end{pmatrix}.$$

Note that there is no dynamics in the interconnection structure in this case. The model reduction problem in this example is to find new models $\hat{G}_1(s)$ and $\hat{G}_2(s)$ of smaller McMillan degree such that the input-output behavior $w(t) \mapsto z(t)$ of the entire system is captured as well as possible.

3. STRUCTURED BALANCED TRUNCATION

As mentioned in Section 2, it is not known how to solve the optimal approximation problem (6) effectively even in the unstructured case. To do unstructured model reduction it is customary to use suboptimal approaches (in H_∞-sense), such as Hankel norm approximation and balanced truncation; see for example [2, 3, 4]. In this paper, structured extensions of balanced truncation are used.

3.1. Approximation by State Projection

Many methods for model reduction, such as balanced truncation, use state projection [4]. State projections work as follows. For a given n-th order transfer function matrix $[A, B, C, D]$, two projection matrices $S_L, S_R \in \mathbb{R}^{n\times r}$ satisfying $S_L^T S_R = I_r$ are constructed (method dependent). The matrix S_L^T projects the n-dimensional original state space down to the r-dimensional

reduced-order state space: $\hat{x}(t) = S_L^T x(t)$. The matrix S_R is the inverse mapping that gives the coordinates of $\hat{x}(t)$ in the original state space. A reduced-order transfer function matrix of McMillan degree less or equal to r is then given by $[S_L^T A S_R, S_L^T B, C S_R, D]$. State projection is used also in this paper, but special care needs to be taken in the construction of S_L and S_R. The reason is the structure $\mathcal{F}_l(N, G) = [A, B, C, D]$ and it is not true that

$$[S_L^T A S_R, S_L^T B, C S_R, D] = \mathcal{F}_l(N, \hat{G}) \quad \text{and} \quad \hat{G}(s) \in \text{Struct}\, G(s), \tag{7}$$

(Struct $G(s)$ defined in (2)) for just any projections S_L and S_R. In this section, conditions are given on S_L and S_R such that (7) is true. The first step is to define structured coordinate transformations that do not mix the states of different subsystems.

Definition 2. A *structured coordinate transformation* $\tilde{x} = Tx$ on $\mathcal{F}_l(N, G)$ with structured realization (5) has the form

$$T = \text{diag}\{T_N, T_1, \ldots, T_q\}, \quad T_N \in \mathbb{R}^{n_N \times n_N}, \quad T_k \in \mathbb{R}^{n_k \times n_k}, \quad k = 1, \ldots, q, \tag{8}$$

where T is invertible.

Observation 1. *If a structured coordinate transformation is applied to a structured realization of $\mathcal{F}_l(N, G)$, then the realization remains structured.*

The above structured coordinate transformation defines a coordinate transformation T_k on each realization of the subsystems G_k. Assuming this coordinate transformation is well chosen, some of the states of G_k may be less important and can be truncated. In order for the error analysis in the following sections to apply, we choose to first do a *coordinate permutation* P_c that collects all states to be truncated in the lower end of the state vector x of $\mathcal{F}_l(N, G)$. That is, let $x_k = \begin{bmatrix} x_{k1}^T & x_{k2}^T \end{bmatrix}^T \in \mathbb{R}^{n_k}$ be the state of G_k, and only $x_{k1} \in \mathbb{R}^{r_k}$ is to be retained. Then

the permutation matrix P_c is defined by

$$P_c x = \begin{bmatrix} x_N^T & x_{11}^T & \dots & x_{q1}^T & x_{12}^T & \dots & x_{q2}^T \end{bmatrix}^T. \tag{9}$$

Let us also introduce the *canonical projection* C_p by

$$C_p = \begin{bmatrix} I_r & 0_{r\times\bar{r}} \end{bmatrix} \in \mathbb{R}^{r\times n}, \quad r = n_N + \sum_{k=1}^{q} r_i, \quad \bar{r} = \sum_{k=1}^{q} n_k - r_k, \quad n = r + \bar{r}, \tag{10}$$

that satisfies $C_p C_p^T = I_r$. Using these definitions, the following observation follows almost immediately.

Observation 2. *Suppose the realization of $\mathcal{F}_l(N,G) = [A,B,C,D]$ is structured (5), that T is a structured coordinate transformation (8), and that P_c, C_p are defined as in (9) and (10). If the projections S_L and S_R have the structure*

$$S_L^T = C_p P_c T \in \mathbb{R}^{r\times n}, \quad S_R = T^{-1} P_c^T C_p^T \in \mathbb{R}^{n\times r},$$

then $[S_L^T A S_R, S_L^T B, C S_R, D]$ is a structured realization of $\mathcal{F}_l(N,\hat{G})$ where $\hat{G}(s) = [\hat{A}_G, \hat{B}_G, \hat{C}_G, \hat{D}_G] \in$ Struct $G(s)$ and

$$\hat{A}_G = diag\{\hat{A}_1, \dots, \hat{A}_q\}, \qquad \hat{B}_G = diag\{\hat{B}_1, \dots, \hat{B}_q\},$$
$$\hat{C}_G = diag\{\hat{C}_1, \dots, \hat{C}_q\}, \qquad \hat{D}_G = diag\{D_1, \dots, D_q\},$$

and

$$\hat{A}_k \in \mathbb{R}^{r_k\times r_k}, \quad \hat{B}_k \in \mathbb{R}^{r_k\times m_k}, \quad \hat{C}_k \in \mathbb{R}^{p_k\times r_k}, \quad k = 1 \dots q.$$

Observation 2 shows how coordinate projections that preserve the inherent structure of the interconnected system can be constructed. Obviously great care must be taken in the choice of structured coordinate transformation T and approximation orders r_k to make $\|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\hat{G})\|_\infty$ small. How this can be done is analyzed next.

3.2. Choosing Coordinate Projections: Method 1

The controllability Gramian P and the observability Gramian Q for the interconnected system $\mathcal{F}_l(N,G) = [A,B,C,D]$ are solutions to the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0. \tag{11}$$

If A is a Hurwitz matrix, which it can be chosen to be when $\mathcal{F}_l(N,G) \in RH_\infty$, then there are always solutions $P \geq 0$ and $Q \geq 0$ to (11). The Gramians contain quantitative information about how controllable and observable the states are [13]. A common approach to model reduction via state projection is to make the Gramians diagonal and equal by a coordinate transformation [15]. This in general requires a full coordinate transformation T that is not structured as in Definition 2. Hence, we will introduce a weaker notion of subsystem balancing.

Let us use the following partition for the Gramians,

$$\begin{aligned} Q &= \begin{bmatrix} Q_N & Q_{NG} \\ Q_{NG}^T & Q_G \end{bmatrix}, \quad Q_G = \begin{bmatrix} Q_1 & \dots & Q_{1q} \\ \vdots & \ddots & \vdots \\ Q_{1q}^T & \dots & Q_q \end{bmatrix}, \\ P &= \begin{bmatrix} P_N & P_{NG} \\ P_{NG}^T & P_G \end{bmatrix}, \quad P_G = \begin{bmatrix} P_1 & \dots & P_{1q} \\ \vdots & \ddots & \vdots \\ P_{1q}^T & \dots & P_q \end{bmatrix}, \end{aligned} \tag{12}$$

conformal to the structured realization of $\mathcal{F}_l(N,G)$ with state vector

$$x = \begin{bmatrix} x_N^T & x_1^T & \dots & x_q^T \end{bmatrix}^T. \tag{13}$$

The block-diagonal elements of the Gramians contain information about the importance of the states in the subsystems. For example, if all states except the ones in subsystem k are zero

at time zero, then

$$\|z(t)\|^2_{[0,\infty]} = x_k(0)^T Q_k x_k(0). \tag{14}$$

That is, Q_k determines how observable the states in the subsystem k are in the output $z(t)$. Assume next that all states of the interconnected system are zero at $t = -\infty$, and that we would like to control the states of subsystem k to the specific state $x_k(0) = x_k^*$. Then the minimum control signal satisfies

$$\min_{\substack{u\in L_2(-\infty,0)\\ x(0)\in X_k^*}} \|u(t)\|^2_{[-\infty,0]} = (x_k^*)^T P_k^{-1} x_k^*, \quad X_k^* = \{x : x \text{ has structure (13) and } x_k = x_k^*\}. \tag{15}$$

In (15), the states $x_N(0)$ and $x_i(0)$, $i \neq k$, are free variables. The result (15) is an application of Lemma 3 in [9]. The block-diagonal element P_k determines how controllable the states in subsystem k are, if all other states are free.

Since we are interested in structured coordinate transformations, the following definition is made.

Definition 3. The structured realization of the interconnected system $\mathcal{F}_l(N, G)$ is *subsystem balanced* and has *subsystem balanced Gramians* if the block-diagonal elements of the Gramians satisfy

$$P_k = Q_k = \Sigma_k = \text{diag}\{\sigma_{k,1}, \ldots, \sigma_{k,n_k}\}, \quad \sigma_{k,1} \geq \ldots \geq \sigma_{k,n_k}, \quad k = 1 \ldots q,$$

where $\sigma_{k,\cdot}$ are called *structured Hankel singular values* of $G_k(s)$.

In a subsystem balanced system, the states of the subsystems are ordered in decreasing order of controllability and observability, in the sense of (14) and (15). That is, the first state of $G_k(s)$ is the state that is easiest to control in $G_k(s)$ if all other states in the interconnected system are free. The first state is also the most observable state in $G_k(s)$, if all other states

in the interconnected system are at zero. Also note that the structured Hankel singular values of $G_k(s)$ generally depend on $G_i(s)$, $i \neq k$, and $N(s)$. The regular Hankel singular values of $G_k(s)$ [2, 3, 13] depend only on $G_k(s)$.

The following theorem shows that subsystem balanced realizations can always be obtained. It is a generalization of the results in [8].

Theorem 1. *Assume that $\mathcal{F}_l(N, G)$ has a structured realization and Gramians P and Q as in (12). Then there is a structured coordinate transformation $T = diag\{T_N, T_1, \ldots, T_q\}$ that makes P and Q subsystem balanced. Furthermore, the coordinate transformation and the subsystem balanced Gramians satisfy*

$$T_k P_k T_k^T = T_k^{-T} Q_k T_k^{-1} = \Sigma_k, \quad k = 1 \ldots q,$$

and the structured Hankel singular values can be computed as

$$\sigma_{k,i} = \sqrt{\lambda_i(P_k Q_k)},$$

and are invariant under structured coordinate transformations.

Proof. Under coordinate transformations T, the Gramians transform as $\tilde{P} = TPT^T$ and $\tilde{Q} = T^{-T} Q T^{-1}$; see [13]. Note that for structured coordinate transformations, T_k effects the block-diagonal elements P_k, Q_k, and not P_i, Q_i, $i \neq k$. Hence, each T_k can be constructed independently by considering only P_k, Q_k using normal balancing techniques such that $\tilde{P}_k = \tilde{Q}_k = \Sigma_k$; see [13]. ■

Theorem 1 gives a method to construct structured coordinate transformations T. These can be used directly together with Observation 2 to construct structured coordinate projections S_L, S_R.

Remark 2. The idea of balancing block-diagonal elements of the Gramians P and Q goes back to Enns [2] where it was used for frequency-weighted model reduction. The idea has been extended to closed-loop model reduction in [10] and to interconnected systems without dynamical interconnections in [8]. Here the idea is extended to the linear fractional transformation framework.

3.2.1. Choosing Approximation Orders r_k. Suppose that we would like to have an r-th order approximation $\mathcal{F}_l(N, \hat{G})$. The question then is how to choose $r_1, \ldots, r_q$ such that $n_N + r_1 + \ldots + r_q = r$ and $\|\mathcal{F}_l(N, G) - \mathcal{F}_l(N, \hat{G})\|_\infty$ is small. Since the structured Hankel singular values measure the controllability and observability of the states in subsystem balanced coordinates, these are reasonable to use to choose r_k. We propose that all structured Hankel singular values $\{\sigma_{k,i}\}_{k,i}$ are sorted together in decreasing order, and that the $r - n_N$ largest values are retained in the construction of $\hat{G}_1(s), \ldots, \hat{G}_q(s)$. This gives a unique selection of $r_1, \ldots, r_q$, if the structured Hankel singular values are distinct. The principle is tested in Section 4 on the example.

3.2.2. Error Analysis. Balanced truncation is often used because there is an a priori error bound [2, 3]. This bound does generally not hold for truncated subsystem balanced systems as is seen next. Still we can say something about the approximation error.

For simplicity, the case where only a single state is truncated is studied, and the error bound is given in the following theorem. Let us partition the realization of $\mathcal{F}_l(N, G)$ as

$$\mathcal{F}_l(N,G) \left\{ \begin{array}{l} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} w(t), \quad x(0) = 0, \\ \qquad z(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} x(t) + Dw(t), \end{array} \right. \tag{16}$$

where $x(t) \in \mathbb{R}^n$, and $x_1(t) \in \mathbb{R}^{n-1}$. Assume that the realization (16) is subsystem balanced and that a state permutation has been applied, see (9), Observation 2, and Theorem 1, so that it only remains to apply the canonical projection $C_p = [I_{n-1}\ 0]$ to obtain the reduced model $\mathcal{F}_l(N, \hat{G})$ with one state less.

Suppose the system has subsystem balanced Gramians $P > 0$ and $Q > 0$ given by

$$P^{-1} = \begin{bmatrix} P_{11} & P_{21}^T \\ P_{21} & \sigma \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{21}^T \\ \tilde{P}_{21} & (\sigma - p)^{-1} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{21}^T \\ Q_{21} & \sigma \end{bmatrix}, \tag{17}$$

where $p = P_{21}P_{11}^{-1}P_{21}^T < \sigma \in \mathbb{R}$. The structured realization of $\mathcal{F}_l(N, \hat{G})$ is given by

$$\mathcal{F}_l(N, \hat{G}) \begin{cases} \dot{\hat{x}}_1(t) = A_{11}\hat{x}_1(t) + B_1 w(t), \quad \hat{x}_1(0) = 0, \\ \\ \hat{z}(t) = C_1\hat{x}_1(t) + Dw(t), \end{cases} \tag{18}$$

where $\hat{x}_1(t) \in \mathbb{R}^{n-1}$. The following truncation error bound then holds.

Theorem 2. *The difference between the outputs of (16) and (18) is bounded by*

$$\|z(t) - \hat{z}(t)\|_{[0,\tau]} \leq 2\sigma\sqrt{(1 - p/\sigma)(1 + \rho\alpha(\tau))}\|w(t)\|_{[0,\tau]} \tag{19}$$

where σ is the truncated structured Hankel singular value, and

$$0 < p = P_{21}P_{11}^{-1}P_{21}^T < \sigma, \quad \rho = \max\{|Q_{21}/(\sigma(\sigma - p)) + \tilde{P}_{21}|, |Q_{21}/(\sigma(\sigma - p)) - \tilde{P}_{21}|\},$$

$$\alpha(\tau) = \frac{1}{2}K_\eta(\tau)(K_{x_1}(\tau) + K_{\hat{x}_1}(\tau)), \quad K_{(\cdot)}(\tau) = \sup_{w \in L_2[0,\tau]} \frac{\|(\cdot)(t)\|_{[0,\tau]}}{\|w(t)\|_{[0,\tau]}},$$

and the entries of P and Q are defined in (17), and where the functions $(\cdot)$ in $K_{(\cdot)}$ are given by $\eta(t) = A_{21}\hat{x}_1(t) + B_2 w(t)$, and $x_1(t)$ and $\hat{x}_1(t)$ are solutions to (16) and (18).

Proof. The result is an extension of Lemmas 3 and 4 in [16]. The extension lies in that off-diagonal elements $\tilde{P}_{21}$ and Q_{21} are allowed in the Gramians. By multiplying the observability Lyapunov equation $QA + A^TQ + C^TC = 0$ with $\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 \end{bmatrix}$ from the right, and with $\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 \end{bmatrix}^T$

from the left,

$$\frac{d}{dt}\left\|\begin{bmatrix} x_1(t)-\hat{x}_1(t) \\ x_2(t) \end{bmatrix}\right\|_Q^2 + |z(t)-\hat{z}(t)|^2 = 2\eta(t)\sigma x_2(t) + 2\eta(t)Q_{21}[x_1(t)-\hat{x}_1(t)], \tag{20}$$

is obtained, if we use that

$$C\begin{bmatrix} x_1(t)-\hat{x}_1(t) \\ x_2(t) \end{bmatrix} = z(t)-\hat{z}(t), \quad A\begin{bmatrix} x_1(t)-\hat{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t)-\dot{\hat{x}}_1(t) \\ \dot{x}_2(t)-\eta(t) \end{bmatrix},$$

and $\eta(t)$ is defined by $\eta(t) = A_{21}\hat{x}_1(t) + B_2 w(t)$.

By transforming the Lyapunov controllability equation $PA^T + AP + BB^T = 0$ into a linear matrix inequality, see equation (30) in [16], and multiplying it with $\begin{bmatrix} x_1+\hat{x}_1 \\ x_2 \end{bmatrix}$ from the right and with the transpose from the left, the following *inequality* is obtained in a similar way,

$$\frac{d}{dt}\left\|\begin{bmatrix} x_1(t)+\hat{x}_1(t) \\ x_2(t) \end{bmatrix}\right\|_{\tilde{P}}^2 + 2\eta(t)(\sigma-p)^{-1}x_2(t) + 2\eta(t)\tilde{P}_{21}[x_1(t)+\hat{x}_1(t)] \le 4|w(t)|^2. \tag{21}$$

By multiplying (21) with $\sigma(\sigma-p)$, and then adding the inequality to (20) and integrating over the time interval $[0,\tau]$, we obtain

$$\begin{aligned}\|z-\hat{z}\|_{[0,\tau]}^2 &\le 4\sigma(\sigma-p)\|w\|_{[0,\tau]}^2 + 2\langle \eta, (Q_{21}-\sigma(\sigma-p)\tilde{P}_{21})x_1\rangle_{[0,\tau]} - 2\langle \eta, (Q_{21}+\sigma(\sigma-p)\bar{P}_{21})\hat{x}_1\rangle_{[0,\tau]} \\ &\le 4\sigma(\sigma-p)\|w\|_{[0,\tau]}^2 + 2\sigma(\sigma-p)\rho\|\eta\|_{[0,\tau]}(\|x_1\|_{[0,\tau]} + \|\hat{x}_1\|_{[0,\tau]})\end{aligned}$$

where the constant ρ is defined in the theorem statement. By introducing the bounds $K_{x_1}(\tau), K_{\hat{x}_1}(\tau)$, and $K_\eta(\tau)$ on $\|x_1\|_{[0,\tau]}, \|\hat{x}_1\|_{[0,\tau]}$, and $\|\eta\|_{[0,\tau]}$ the statement follows. ■

Remark 3. Notice that p/σ and ρ measure how "unstructured" the Gramians are. In the case of block-diagonal Gramians, then $p = 0$ and $\rho = 0$. When the system is balanced in the classical sense, i.e., when $P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}$, then $\sigma(\sigma-p) = \sigma_n^2$, $Q_{21} = \tilde{P}_{21} = 0$, and (19) reduces to the classical bound $\|z(t)-\hat{z}(t)\|_{[0,\infty]} \le 2\sigma_n\|w(t)\|_{[0,\infty]}$; see [2, 3].

The error bound in Theorem 2 is not an a priori error bound if $\rho > 0$. This is because the function $\alpha(\tau)$ is needed, and it can only be computed *after* the model $\mathcal{F}_l(N, \hat{G})$ has been constructed. The computation of the induced norms in $K_{x_1}(\tau)$ and $K_{\hat{x}_1}(\tau)$ is a non-trivial task, since it requires the solution of time-varying differential Riccati equations; see [17]. Thus the main value of the theorem is to show that the error scales with the truncated Hankel singular value σ, and that the bound transitions into an a priori bound as the Gramians become block diagonal ("become structured"), see Remark 3. Method 2 in Section 3.3 is based on this observation.

Remark 4. There is no guarantee that Method 1 gives a stable approximation $\mathcal{F}_l(N, \hat{G})$, even if $\mathcal{F}_l(N, G)$ is stable. Notice that the other methods mentioned in Remark 2 suffer from the same problem. Nevertheless, Method 1 often performs well as is seen in Section 4, and Theorem 2 is always valid because it is a statement over the *finite* time interval $[0, \tau]$. If $\mathcal{F}_l(N, \hat{G})$ is unstable, it means that $K_{\hat{x}_1}(\tau) \to \infty$ as $\tau \to \infty$.

3.3. Choosing Coordinate Projections: Method 2

The bound in Theorem 2 shows that block-diagonal Gramians yield simple a priori error bounds. However, Gramians satisfying (11) are typically not block diagonal. Therefore we use more flexible generalized Gramians, see [11], in Method 2. The generalized block-diagonal Gramians should satisfy the linear matrix inequalities

$$\begin{array}{ll} \min \operatorname{trace} P & \min \operatorname{trace} Q \\ AP + PA^T + BB^T \leq 0 & A^TQ + QA + C^TC \leq 0 \\ P = \operatorname{diag}\{P_N, P_1, \ldots, P_q\} & Q = \operatorname{diag}\{Q_N, Q_1, \ldots, Q_q\}. \end{array} \tag{22}$$

The convex optimization problem (22) can be solved using standard linear matrix inequality solvers such as [18]. If there is a solution then $P \geq 0$ and $Q \geq 0$, since A is a Hurwitz matrix. The following definition is made.

Definition 4. The structured realization of the interconnected system $\mathcal{F}_l(N, G)$ has *structured Gramians* if there are generalized Gramians

$$P = \text{diag}\{P_N, P_1, \ldots, P_q\}, \quad Q = \text{diag}\{Q_N, Q_1, \ldots, Q_q\}, \tag{23}$$

that satisfy (22).

There is no simple result that states when an interconnected system has structured Gramians. It is simpler to state necessary conditions for the existence of structured Gramians, however: The diagonal blocks of A corresponding to the subsystems generally need to be stable.

Observation 3. Suppose $[A, B, C, D]$ is a minimal realization of the stable interconnected system $\mathcal{F}_l(N, G)$, and that $\mathcal{F}_l(N, G)$ has structured Gramians. Denote the i-th diagonal block of A, corresponding to P_i, Q_i, by A_i. Then A_i, $i = N, 1, \ldots, q$, have no unstable modes.

Proof. Since $[A, B, C, D]$ is a minimal stable realization, it holds that $P > 0$ and $Q > 0$. The proof then follows by contradiction. Assume that there is an unstable mode in A_i with eigenvalue λ, Re $\lambda > 0$, and eigenvector v, $A_i v = \lambda v$. By (22) and the minimality assumption it holds that

$$A_i^T Q_i + Q_i A_i + C_i^T C_i \leq 0, \quad Q_i > 0,$$

where C_i is the i-th block of C. If we multiply from the left with v' (complex conjugate transpose of v) and from the right with v, we obtain

$$(2\text{Re } \lambda) v' Q_i v + v' C_i^T C_i v \leq 0, \quad v' Q_i v > 0, \quad v' C_i^T C_i v \geq 0,$$

which is a contradiction when Re $\lambda > 0$. ■

The practical consequence of Observation 3 is that there can be no unstable (open loop) modes in the subsystems in $\mathcal{F}_l(N, G)$. Such system can be called "super stable".

Remark 5. It seems restrictive to require structured Gramians and that only few systems will have such Gramians. However, Method 2 is the foundation for the continued work reported in [6], where it is shown that meaningful intermediate models that always have structured Gramians can be constructed.

Structured projections can be constructed just as in Method 1, but using the structured generalized Gramains instead of the regular Gramians. Then the following a priori bound holds.

Theorem 3. *Assume that the interconnected system $\mathcal{F}_l(N, G)$ has a structured realization $[A, B, C, D]$ and has structured Gramians (23). Then it holds that*

$$\|\mathcal{F}_l(N, G) - \mathcal{F}_l(N, \hat{G})\|_\infty \leq 2 \sum_{k=1}^{q} \sum_{i=r_k+1}^{n_k} \sigma_{k,i},$$

where $\sigma_{k,i} = \sqrt{\lambda_i(P_k Q_k)}$ are the structured Hankel singular values, and $\mathcal{F}_l(N, \hat{G}) = [S_L^T A S_R, S_L^T B, C S_R, D]$ as in Observation 2, and the structured coordinate transformation T in S_L^T and S_R is subsystem balancing as in Theorem 1.

Proof. The calculations leading to the error bound in Theorem 2 also hold for generalized Gramians, just as in [16]. Notice that for each truncated state, $p = 0$ and $\rho = 0$, since the Gramians are structured. The result then follows by iteratively applying the bound for each truncated state, and using the triangle inequality. ■

Remark 6. An argument for choosing to minimize the trace of the generalized Gramians

in (22) is that $\sum_{k=1}^{q}\sum_{i=1}^{n_k}\sigma_{k,i} \leq (\text{trace } P)(\text{trace } Q)$ (use Proposition 1 in [19]). Thus these criteria tend to make the error bound small. See also discussion in Section 4.3.

3.3.1. Choosing Approximation Orders r_k. For the selection of approximation orders r_k for the subsystems in Method 2, we suggest the same principle as in Method 1, see Section 3.2.1. In fact, that principle minimizes the a priori error bound in Theorem 3, for fixed total approximation order $r = n_N + r_1 + \ldots + r_q$.

4. EXAMPLE REVISITED

Let us consider Example 1 again. The models of the elastic masses are chosen as 8-th and 10-th order transfer function matrices $G_1(s)$ and $G_2(s)$, respectively. Their Bode magnitude plots are shown in Figure 2. Indeed, these are not very high-order subsystems, but they are complex enough to illustrate the points of the paper. The interconnected system $\mathcal{F}_l(N, G)$ has a force disturbance w acting as input, and the positions of two of the edges of the masses, z_1, and z_2, acting as outputs. The system is studied as the spring constant k of the spring that interconnects the subsystems is 0.1 or 10.

Three different model reduction techniques are applied to the example: regular balanced truncation [2, 3], and Method 1 and Method 2 from the previous section. The block diagram structure of the models coming in and out of the different techniques are illustrated in Figure 4. Regular balanced truncation does not preserve structure and the reduced model $\hat{G}(s)$ is just a black box model approximating the input-output behavior $w \mapsto z$. In addition to approximating the input-output behavior $w \mapsto z$, Method 1 and 2 also preserve the interconnection structure.

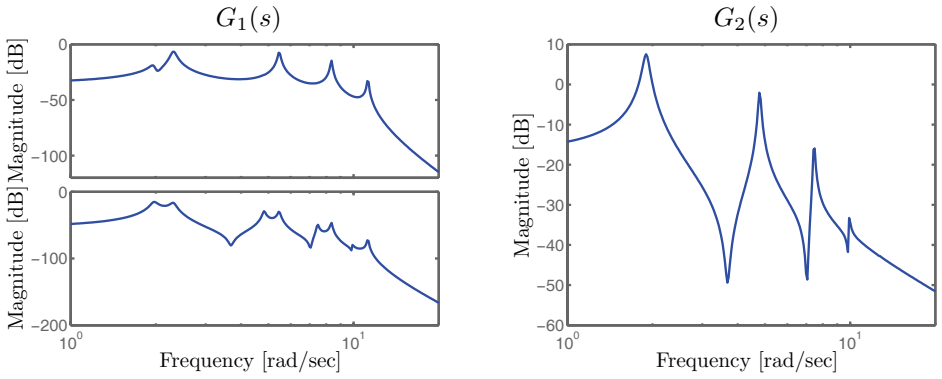

Figure 3. Bode magnitude plots of the transfer function matrices of the mass models $G_1(s)$ and $G_2(s)$ in Example 1.

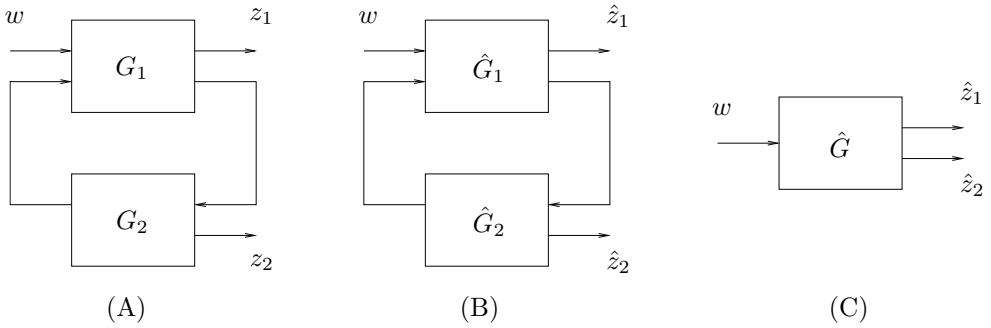

Figure 4. Block diagrams of the different models in Example 1. (A) is the structure of the original model. (B) is the structure of the model generated by Method 1 and Method 2. (C) is the structure of the model generated by regular balanced truncation.

4.1. Comparing Method 1 with Regular Balanced Truncation

Here regular balanced truncation is compared to Method 1 when the spring constant is $k = 10$. Note that regular balanced truncation generally *does not* preserve the interconnection structure in the reduced model $\hat{G}(s)$, see Figure 4 (C). Hence, it will generally not be possible to extract reduced models of the individual masses, $\hat{G}_1(s)$ and $\hat{G}_2(s)$, from $\hat{G}(s)$. The reason for using regular balanced truncation here is to see how good approximations can be done without the

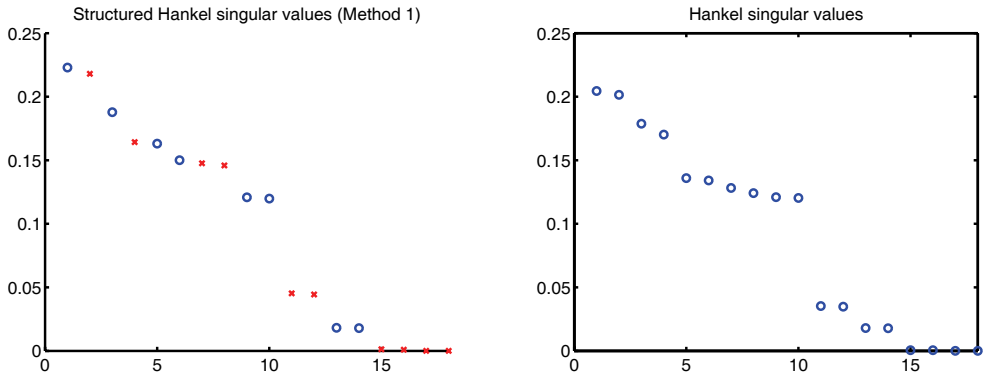

Figure 5. (Left) The structured Hankel singular values of Method 1 for the 18-th order interconnected model $\mathcal{F}_l(N, G)$ when $k = 10$. '$\circ$' belong to $G_1(s)$, $\{\sigma_{1,k}\}_{k=1}^{8}$, and '$\times$' belong to $G_2(s)$, $\{\sigma_{2,k}\}_{k=1}^{10}$. (Right) The regular Hankel singular values for the same model. The structured Hankel singular values can be associated with different subsystems, whereas the regular singular values are associated with the entire interconnected system.

structure constraint. Thus it is expected that Method 1 gives a larger approximation error.

In Figure 5, the regular and structured Hankel singular values are plotted. The structured singular values from $G_1(s)$ and $G_2(s)$ are plotted together to show the relative importance of states in $G_1(s)$ and $G_2(s)$ with respect to the mapping $w \mapsto z$. There is a significant drop after 10 singular values in both plots. This indicates that around 10 states are needed to approximate the model (choose $r \approx 10$). In Table I and II in the appendix, the approximation errors of regular balanced truncation and Method 1 are shown for various approximation orders r, r_1, and r_2. The orders r, r_1, and r_2 are reduced in steps of two, since this reduces the number of cases to tabulate.

In Table I, it is seen that the principle in Section 3.2.1 actually picks out the best choices of r_1 and r_2 when the total approximation order $r = r_1 + r_2$ is fixed and larger or equal to 8. (This is also true if r_1 and r_2 are allowed to be odd numbers.) Remember that the principle

simply means that states corresponding to the smallest structured singular values in Figure 5 are truncated. For $r = r_1 + r_2$ smaller than 8, the approximation error is as least as large as the system norm itself for all choices r_1 and r_2, and Method 1 does not work. It is also noted that Method 1 gives unstable approximations for $r_1 = 2$. In Table II, the approximation errors of Method 1 and regular balanced truncation are compared. As expected, Method 1 always yields a larger approximation error, but around $r = 10$, the error is only a factor $1 - 3$ larger.

In summary, Method 1 works well for the example down to approximation order $r = 8$. This corresponds to a reduction of 10 states. The principle in Section 3.2.1 gives the right selection of r_1 and r_2 in these cases. It should be noted that regular balanced truncation yields a relative error of 60% for $r = 8$, and unstructured model reduction also does a relatively bad job for $r \leq 8$.

4.2. Comparing Method 1 with Method 2

Here Method 1 is compared to Method 2. There must exist structured Gramians for Method 2 to work. This is the case when the spring constant k is less than 0.4. We choose $k = 0.1$ since this is a factor 100 smaller than in Section 4.1. The structured singular values are shown in Figure 6. The first thing to notice is that when k decreases from 10 to 0.1 the relative importance of the model $G_2(s)$ in the mapping $w \mapsto z$ decreases. This can be seen by comparing the relative size of the structured singular values $\{\sigma_{2,k}\}_{k=1}^{10}$ and $\{\sigma_{1,k}\}_{k=1}^{8}$. $\{\sigma_{2,k}\}_{k=1}^{10}$ are smaller in Figure 6 (left) than in Figure 5 (left). This is natural since the interaction between the models has been decreased.

In Table III and IV, the approximation errors are shown for various approximation orders r_1 and r_2. Notice that the errors of Method 1 and Method 2 are almost identical for all r_1

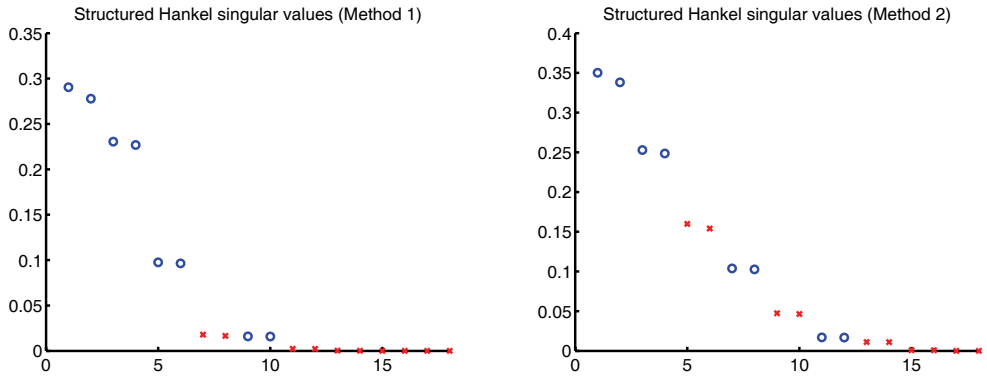

Figure 6. Comparison of structured Hankel singular values for Method 1 and Method 2 when $k = 0.1$. '$\circ$' are structured Hankel singular values belonging to $G_1(s)$, and '$\times$' are values belonging to $G_2(s)$.

and r_2. The structured Hankel singular values are slightly different, however, as can be seen in Figure 6. In particular, if the principles in Section 3.2.1 and 3.3.1 are applied, the choices of r_1 and r_2 are different when $r_1 + r_2 = 10$ and 6. It is seen that the principle works better for Method 1: The principle then gives the best choices of r_1 and r_2 for all cases except when $r = 8$. The advantage with Method 2 compared to Method 1 is the existence of the a priori error bound in Theorem 3. The bound is studied in the next section.

In summary, Method 1 and Method 2 give roughly the same approximation errors, but the order selection principle works better for Method 1. Both Method 1 and Method 2 work (relative error less than 100%) for all choices of r_1 and r_2. Method 2 is theoretically better motivated, but it is only applicable when ($k \leq 0.4$).

4.3. Comparing the Error Bounds of Regular Balanced Truncation and Method 2

Here the error bounds in Theorem 3 and [2, 3] are compared. In Table V, the approximation errors and bounds are given for various r, when $k = 0.1$. The models for Method 2 are the

same as in Section 4.2.

For regular balanced truncation, the error bound is uniformly a factor $2-5$ too conservative. For Method 2, the conservatism depends heavily on the approximation order r. For large r, the bounds are very conservative. An explanation is that the structured Gramians are obtained by minimization of their trace. The trace criterium tends to emphasize the size of the largest eigenvalues of the Gramians. Hence, this criterium does not value an absolute decrease in the small eigenvalues as much. More elaborate criteria could be considered to fix this, at the expense of higher computational complexity. Since the actual approximation errors become quite small using the trace criterium, we do not consider such fixes here. For r around 8 (which is a reasonable order for the approximation) the conservatism of the bound for Method 2 is a factor $3-9$, which is not far worse than the bound for regular balanced truncation.

4.4. Summary of Example

Method 1 performs well in both cases $k = 0.1$ and $k = 10$, even though the dynamics is very different. Method 1 is also relatively computationally cheap to use, compared to Method 2, since no convex optimization is needed. The drawback with Method 1 is that there are only weak guarantees on its performance (Theorem 2), and it can even generate unstable approximations, see $r_1 = 2$ when $k = 10$. It is seen that the principle in Section 3.2.1 generally works well. This is helpful since we then do not need to try all possible combinations of r_1 and r_2. Finally, the error bound Theorem 3 performs reasonably well when the approximation order is not large.

5. CONCLUSIONS

In this paper, two methods for structure-preserving model reduction of interconnected linear systems have been presented. Both methods are based on ideas from classical balanced truncation. The methods differ from the method in [8, 9] by that they use a more general linear fractional transformation framework, and the second method also uses linear matrix inequalities. Approximation error bounds were derived for the methods. When structured (block-diagonal) Gramians are available, the error bound is an a priori bound. When the Gramians are unstructured, the error bound is only an a posteriori bound. A model of a mechanical device was also presented and the methods were successfully applied to it. In particular, it was shown how the methods pick out relevant modes in the subsystems, based on a global approximation criteria. Both approximation methods are valuable by themselves, but they also form the foundation for the more advanced method presented in [6].

An interesting problem for future work is how to deal with uncertain network models $N(s)$. Other interesting problems are structured approximation in gap metrics, and to provide lower error bounds on the approximation error.

REFERENCES

1. R.M. Murray, editor. *Control in an Information Rich World - Report of the Panel on Future Directions in Control, Dynamics, and Systems*. Society for Industrial and Applied Mathematics, Philadelphia, 2003.
2. D. Enns. Model reduction with balanced realizations: an error bound and a frequency weighted generalization. In *Proceedings of the IEEE Conference on Decision and Control*, Las Vegas, Nevada, 1984.
3. K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds. *International Journal of Control*, 39:1115–1193, 1984.

4. G. Obinata and B.D.O. Anderson. *Model Reduction for Control System Design*. Springer-Verlag, London, UK, 2001.
5. H. Sandberg and R. M. Murray. Frequency-weighted model reduction with applications to structured models. In *Proceedings of the 2007 American Control Conference*, pages 941–946, New York City, New York, July 2007.
6. H. Sandberg and R. M. Murray. Model reduction of networked systems. In *In Proccedings of the IFAC World Congess*, 2008. To appear. Available from `http://www.ee.kth.se/~hsan/ifac.pdf`.
7. L. Li and F. Paganini. Structured coprime factor model reduction based on LMIs. *Automatica*, 41(1):145–151, January 2005.
8. A. Vandendorpe and P. Van Dooren. On model reduction of interconnected systems. In *Proceedings International Symposium Math. Th. Netw. Syst.*, Belgium, 2004.
9. A. Vandendorpe and P. Van Dooren. *Model Order Reduction: Theory, Research Aspects and Applications*, chapter Model reduction of interconnected systems. Mathematics in Industry Series. Springer Verlag, 2007.
10. G. Schelfhout and B. De Moor. A note on closed-loop balanced truncation. *IEEE Transactions on Automatic Control*, 41:1498–1500, 1996.
11. C. Beck, J.C. Doyle, and K. Glover. Model reduction of multidimensional and uncertain systems. *IEEE Transactions on Automatic Control*, 41(10):1466–1477, October 1996.
12. K. Zhou, C. D'Souza, and J.R. Cloutier. Structurally balanced controller order reduction with guaranteed closed loop performance. *Systems and Control Letters*, 24:235–242, 1995.
13. K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, Upper Saddle River, New Jersey, 1996.
14. G.E. Dullerud and F. Paganini. *A course in robust control theory — a convex approach*. Springer-Verlag, 2000.
15. B.C. Moore. Principal component analysis in linear systems: controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, 26(1):17–32, February 1981.
16. H. Sandberg and A. Rantzer. Balanced truncation of linear time-varying systems. *IEEE Transactions on Automatic Control*, 49(2):217–229, February 2004.
17. M. Green and D.J.N. Limebeer. *Linear robust control*. Information and system sciences series. Prentice Hall, Englewood Cliffs, New Jersey, 1995.
18. J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1999. Available from

http://fewcal.kub.nl/sturm/software/sedumi.html.

19. P. Bendotti and C.L. Beck. On the role of LFT model reduction methods in robust controller synthesis for a pressurized water reactor. *IEEE Transactions on Control Systems Technology*, 7(2):248–257, March 1999.

APPENDIX

A. Approximation Errors in Example

Table I. The approximation error $\|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\hat{G})\|_\infty$, where $\hat{G}$ is computed with Method 1, and $k = 10$. The orders of $\hat{G}_1$ and $\hat{G}_2$ are r_1 and r_2, respectively. The boxed values represent the choices of r_1 and r_2 using the policy in Section 3.2.1. The norm of the system is $\|\mathcal{F}_l(N,G)\|_\infty = 0.4016$.

$r_2 \setminus r_1$	8	6	4	2	0
10	$\boxed{0.00}$	$3.58 \cdot 10^{-2}$	$2.40 \cdot 10^{-1}$	∞	$4.02 \cdot 10^{-1}$
8	$\boxed{1.23 \cdot 10^{-5}}$	$3.58 \cdot 10^{-2}$	$2.40 \cdot 10^{-1}$	∞	$4.02 \cdot 10^{-1}$
6	$\boxed{7.63 \cdot 10^{-3}}$	$\boxed{3.58 \cdot 10^{-2}}$	$2.40 \cdot 10^{-1}$	∞	$4.02 \cdot 10^{-1}$
4	$2.08 \cdot 10^{-1}$	$\boxed{2.07 \cdot 10^{-1}}$	$\boxed{3.28 \cdot 10^{-1}}$	∞	$4.02 \cdot 10^{-1}$
2	$3.99 \cdot 10^{-1}$	$3.98 \cdot 10^{-1}$	$\boxed{4.20 \cdot 10^{-1}}$	$\boxed{\infty}$	$4.02 \cdot 10^{-1}$
0	$4.84 \cdot 10^{-1}$	$4.82 \cdot 10^{-1}$	$4.40 \cdot 10^{-1}$	$\boxed{\infty}$	$\boxed{4.02 \cdot 10^{-1}}$

Table II. The approximation error $\|\mathcal{F}_l(N,G) - \hat{G}\|_\infty$, where $\hat{G}$ is an unstructured approximation computed with regular balanced truncation with order r, when $k = 10$. The boxed values from Table I are also shown for comparison, together with the ratio of the actual errors.

r	Reg. error	Method 1 error	Ratio
18	0.00	0.00	-
16	$6.75 \cdot 10^{-7}$	$1.23 \cdot 10^{-5}$	$1.81 \cdot 10^{1}$
14	$1.12 \cdot 10^{-3}$	$7.63 \cdot 10^{-3}$	6.80
12	$3.58 \cdot 10^{-2}$	$3.58 \cdot 10^{-2}$	1.00
10	$7.00 \cdot 10^{-2}$	$2.07 \cdot 10^{-1}$	2.95
8	$2.42 \cdot 10^{-1}$	$3.28 \cdot 10^{-1}$	1.36
6	$2.57 \cdot 10^{-1}$	$4.20 \cdot 10^{-1}$	1.63
4	$2.65 \cdot 10^{-1}$	∞	∞
2	$3.48 \cdot 10^{-1}$	∞	∞
0	$4.02 \cdot 10^{-1}$	$4.02 \cdot 10^{-1}$	1.00

Table III. The approximation error $\|\mathcal{F}_l(N,G)-\mathcal{F}_l(N,\hat{G})\|_\infty$, where $\hat{G}$ is computed with Method 1, and $k = 0.1$. The orders of $\hat{G}_1$ and $\hat{G}_2$ are r_1 and r_2, respectively. The boxed values represent the choices of r_1 and r_2 using the policy in Section 3.2.1. The norm of the system is $\|\mathcal{F}_l(N,G)\|_\infty = 0.5663$.

$r_2 \setminus r_1$	8	6	4	2	0
10	$\boxed{0.00}$	$3.18 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$4.56 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
8	$\boxed{2.71 \cdot 10^{-8}}$	$3.18 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$4.56 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
6	$\boxed{1.02 \cdot 10^{-5}}$	$3.18 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$4.56 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
4	$\boxed{3.50 \cdot 10^{-4}}$	$3.18 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$4.56 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
2	$\boxed{3.41 \cdot 10^{-3}}$	$\boxed{3.18 \cdot 10^{-2}}$	$1.94 \cdot 10^{-1}$	$4.56 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
0	$2.59 \cdot 10^{-2}$	$\boxed{3.18 \cdot 10^{-2}}$	$\boxed{1.94 \cdot 10^{-1}}$	$\boxed{4.56 \cdot 10^{-1}}$	$\boxed{5.66 \cdot 10^{-1}}$

Table IV. The approximation error $\|\mathcal{F}_l(N,G)-\mathcal{F}_l(N,\hat{G})\|_\infty$, where $\hat{G}$ is computed with Method 2, and $k = 0.1$. The orders of $\hat{G}_1$ and $\hat{G}_2$ are r_1 and r_2, respectively. The boxed values represent the choices of r_1 and r_2 using the policy in Section 3.3.1. The norm of the system is $\|\mathcal{F}_l(N,G)\|_\infty = 0.5663$.

$r_2 \setminus r_1$	8	6	4	2	0
10	$\boxed{0.00}$	$3.18 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$4.57 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
8	$\boxed{6.19 \cdot 10^{-8}}$	$3.18 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$4.57 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
6	$\boxed{1.00 \cdot 10^{-5}}$	$3.18 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$4.57 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
4	$\boxed{3.50 \cdot 10^{-4}}$	$\boxed{3.18 \cdot 10^{-2}}$	$1.94 \cdot 10^{-1}$	$4.57 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
2	$3.41 \cdot 10^{-3}$	$\boxed{3.18 \cdot 10^{-2}}$	$\boxed{1.94 \cdot 10^{-1}}$	$4.57 \cdot 10^{-1}$	$5.66 \cdot 10^{-1}$
0	$2.59 \cdot 10^{-2}$	$3.18 \cdot 10^{-2}$	$\boxed{1.94 \cdot 10^{-1}}$	$\boxed{4.57 \cdot 10^{-1}}$	$\boxed{5.66 \cdot 10^{-1}}$

Table V. The approximation errors for regular balanced truncation and Method 2, when $k = 0.1$. In parenthesis, the error bound in [2, 3] and in Theorem 3 are given. The conservatism is indicated by the ratio of the bound and the actual error.

r	Reg. error (Bound)	Reg. cons.	Method 2 error (Bound)	Method 2 cons.
18	0.00 (0.00)	-	0.00 (0.00)	-
16	$5.30 \cdot 10^{-9}$ $(1.06 \cdot 10^{-8})$	2.00	$6.19 \cdot 10^{-8}$ $(3.08 \cdot 10^{-5})$	$4.97 \cdot 10^{2}$
14	$6.89 \cdot 10^{-6}$ $(1.38 \cdot 10^{-5})$	2.00	$1.00 \cdot 10^{-5}$ $(3.43 \cdot 10^{-3})$	$3.42 \cdot 10^{2}$
12	$3.48 \cdot 10^{-4}$ $(7.10 \cdot 10^{-4})$	2.04	$3.50 \cdot 10^{-4}$ $(4.75 \cdot 10^{-2})$	$1.36 \cdot 10^{2}$
10	$3.37 \cdot 10^{-3}$ $(7.45 \cdot 10^{-3})$	2.21	$3.18 \cdot 10^{-2}$ $(1.15 \cdot 10^{-1})$	3.61
8	$2.44 \cdot 10^{-2}$ $(5.59 \cdot 10^{-2})$	2.29	$3.18 \cdot 10^{-2}$ $(3.03 \cdot 10^{-1})$	9.51
6	$3.18 \cdot 10^{-2}$ $(1.20 \cdot 10^{-1})$	3.76	$1.94 \cdot 10^{-1}$ $(7.15 \cdot 10^{-1})$	3.69
4	$1.94 \cdot 10^{-1}$ $(5.07 \cdot 10^{-1})$	2.62	$1.94 \cdot 10^{-1}$ (1.34)	6.94
2	$4.56 \cdot 10^{-1}$ (1.42)	3.12	$4.57 \cdot 10^{-1}$ (2.35)	5.13
0	$5.66 \cdot 10^{-1}$ (2.56)	4.52	$5.66 \cdot 10^{-1}$ (3.72)	6.57