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Frequency-Weighted Model Reduction with Applications to Structured Models

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***Abstract*— In this paper, we generalize a recently proposed method for model reduction of linear systems to the frequency-weighted case. The method uses convex optimization and can be used both with sample data and exact models. We also derive simple a priori bounds on the frequency-weighted error. We combine the method with a rank-minimization heuristic, to approximate multi-input–multi-output systems. We also present two applications — environment compensation and simplification of interconnected models — where we argue the proposed methods are useful.**

I. Introduction

The frequency-weighted model reduction problem is arguably a more important problem than the unweighted one; at least in the context of closed-loop control systems. For process models it is usually mainly around the cross-over frequency a good model match is needed, for example. For controller reduction problems, frequency weights are also essential, see [1]. It may seem that weights should not make the problem much more complicated, and that simple extensions of balanced truncation [2] or optimal Hankel-norm approximation [3] should solve the problem. This is only partly true. For example, in [4], an extension of balanced truncation to the frequency-weighted case is proposed. Even though the proposed method often works well, it is hard to prove when it will work and one can even find examples where unstable approximations of stable models result. Many more extensions and improvements have been proposed. The books [1], [5] contain many of these methods.

In [6], [7], a new approach is taken to solve the model reduction problem. Instead of solving high-dimensional Lyapunov equations, as in balanced truncation and Hankel-norm approximation, a relaxation that makes the problem convex is introduced. The problem can be solved using convex optimization with frequency-domain data. In [7], a priori error bounds on the approximation error are obtained. The method is flexible and always delivers stable approximations. In this paper, we show how frequency-weights can be added to the procedure, and how the a priori error bounds are changed. We also combine the method with a rank-minimization heuristic, introduced in [8], to approximate multi-input–multi-output (MIMO) systems.

Furthermore, we present two applications where we argue that frequency-weighted model reduction is useful. The first application deals with finding low-complexity updates to existing feedback controllers. The updates are introduced to compensate for complex environments that disturb the controlled system. The idea of including a model of the environment and to compensate for it has been used in distributed control of vehicle formations, see [9]. Here we deal with a simple linear framework. In [9], a more complicated nonlinear problem was addressed. The other application is the problem of reducing the complexity of an interconnected linear system, whilst taking its structure into account. Model reduction of interconnected systems is a problem that has received some attention recently, see, for example [10]–[12]. This problem is motivated by the many uses of networked control systems [13]. These systems are often very large, and how to systematically reduce their size and complexity is often a difficult task. Further evaluation of the proposed applications is a topic for our future work.

The organization of the paper is as follows: In Section II, the frequency-weighted model reduction technique is described along with an example. In Section III, the application to environment compensation is presented, along with an example. In Section IV, the application to interconnected linear systems is described. In Section V, some conclusions and suggested future work are given.

Notation

H_∞ and H_∞^- denote the sets of stable and anti-stable transfer-function matrices (TFMs), respectively. RH_∞ are the stable rational TFMs, and R_nH_∞ the stable rational TFMs of McMillan degree less or equal to n. Similar definitions hold for H_∞^-. The TFM $G(s)$ belongs to L_∞ if the norm

$$\|G\|_\infty \triangleq \sup_\omega \bar{\sigma}(G(j\omega)),$$

is finite, where $\bar{\sigma}$ is the largest singular value. $\|\cdot\|_H$ denotes the Hankel norm, see [14]. We define $G^\sim(s) \triangleq G(-s)^T$, and $\mathbb{Z}$ ($\mathbb{Z}_+$) are the (non-negative) integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers with j being the imaginary unit and * the complex conjugate.

II. Frequency-Weighted Model Reduction

The problems we end up solving in this paper are frequency-weighted model reduction problems. There are many methods available for solving such problems, see, for example, [1], [4]. Typically these methods use state-space techniques and it is hard to bound their approximation error a priori. Error bounds are important since they can be used to guarantee good approximations. The methods we suggest here do always preserve stability, comes with error bounds,

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and can be used with frequency data samples and with exact models.

A. Frequency-weighted approximation problem

The problem we would like to solve can be formulated as

$$\min_{\hat{G}} \|W_o(G-\hat{G})W_i\|_\infty \quad \text{subject to} \quad \hat{G} \in R_r H_\infty, \tag{1}$$

where $G \in RH_\infty$ is a given TFM together with frequency-dependent weights $W_i, W_o \in RH_\infty$ and $r \in \mathbb{Z}_+$. To the best knowledge of the authors, no polynomial time algorithm is available to solve (1), and the suboptimal methods mentioned above are frequently used instead. In this paper, we relax the problem (1) and thereby obtain a problem that can be solved with convex optimization.

The first method, presented in Section II-B, only deals with single-input–single-output (SISO) models. The second method, presented in Section II-C, can be applied to multi-input–multi-output (MIMO) models as well. How the methods can be combined is also discussed in Section II-C.

B. SISO frequency-weighted approximation

It is not known how to solve the desired approximation problem (1) using convex optimization. In [6], a relaxation technique that makes the unweighted discrete-time problem convex is introduced. We here use an analogous relaxation for the weighted continuous-time problem. Instead of the desired problem (1), we suggest to solve the problem

$$\begin{aligned} &\min_{a,b,c} \gamma \quad \text{subject to} \\ &\left\| \frac{w_1}{w_2}\left(G - \frac{b}{a} - \frac{c}{a^\sim}\right)\right\|_\infty < \gamma, \end{aligned} \tag{2}$$

where a, w_1, w_2 are Hurwitz polynomials, $G \in RH_\infty$ and SISO, and

$$\begin{aligned} a(s) &= s^r + a_{r-1}s^{r-1} + \ldots + a_1 s + a_0 \\ b(s) &= b_r s^r + b_{r-1}s^{r-1} + \ldots + b_1 s + b_0 \\ c(s) &= c_{r-1}s^{r-1} + c_{r-2}s^{r-2} + \ldots + c_1 s + c_0 \\ \frac{w_1}{w_2}, \frac{w_2}{w_1} &\in R_d H_\infty, \quad \frac{b}{a} \in R_r H_\infty, \quad \frac{c}{a^\sim} \in R_r H_\infty^-. \end{aligned}$$

Just as in [6], we can re-parameterize the problem. Define

$$\frac{B(s)}{A(s)} \triangleq \frac{a^\sim(s)b(s) + a(s)c(s)}{a^\sim(s)a(s)}$$

where

$$\begin{aligned} A(s) &= (-1)^r s^{2r} + A_{2r-2}s^{2r-2} + \ldots + A_2 s^2 + A_0 \\ B(s) &= B_{2r}s^{2r} + B_{2r-1}s^{2r-1} + \ldots + B_1 s + B_0. \end{aligned}$$

There is a one-to-one correspondence between the set of polynomials $\{A(s), B(s)\}$ and $\{a(s), b(s), c(s)\}$. The direction $\{a(s), b(s), c(s)\} \to \{A(s), B(s)\}$ is obvious. The other direction follows if we enforce the condition

$$A(j\omega) > 0 \quad \text{for all} \quad \omega.$$

Since $A(j\omega) > 0$, and $A(s) = A(-s)$ by construction, we can compute a spectral factor $a(s)$ of $A(s)$. It follows that we can choose $a(s)$ as a Hurwitz polynomial of degree r. Once $a(s)$ is determined, we can solve for $b(s), c(s)$ as the unique solution to the polynomial equation

$$a^\sim(s)b(s) + a(s)c(s) = B(s),$$

for instance by constructing a Sylvester matrix from $a(s)$ and $a^\sim(s)$ (which are coprime). Hence, instead of solving (2), we can equivalently solve the the *quasi-convex* optimization problem

$$\begin{aligned} &\min_{A,B} \gamma \quad \text{subject to} \\ &\left| \frac{w_1(j\omega)}{w_2(j\omega)}\left(G(j\omega)A(j\omega) - B(j\omega)\right)\right| < \gamma A(j\omega), \qquad (3) \\ &A(j\omega) > 0 \quad \text{for all} \quad \omega. \qquad (4) \end{aligned}$$

The above problem is quasi-convex because for each fixed γ, the constraints (3)–(4) are convex in the unknown polynomial coefficients $\{A_k\}, \{B_k\}$. The approximation accuracy γ can be minimized using a bisection algorithm.

The constraint (4) can be enforced for all ω with a linear matrix inequality (LMI) using the Kalman-Yakubovich-Popov (KYP) lemma [15], or sum-of-squares techniques, see, for example, [16], [17].

Whereas (4) should always be enforced for all ω to guarantee existence of a stable spectral factor $a(s)$, (3) can be enforced on a grid $\{\omega_k\}$. This is of interest if $G(j\omega)$ only is known on this grid. Enforcing (3) for a high-order $G(s)$ for all ω using the KYP lemma leads to an LMI of high dimension, and may not be possible to solve. It can then be effective to sample $G(j\omega)$ on a grid. This of course requires that $G(j\omega)$ does not vary much between the samples.

In the problem (2), an unstable term $c/a^\sim$ is introduced to make the problem convex. This may seem like an odd thing to do, but a similar idea is also used in optimal Hankel-norm approximation, see [3]. The following theorem shows that the unstable term $c/a^\sim$ can be bounded, and how stable approximations $\hat{G}$ can be chosen. A discrete-time unweighted counterpart is given in [7].

Theorem 1: Assume that

$$\left\| \frac{w_1}{w_2}\left(G - \frac{b}{a} - \frac{c}{a^\sim}\right)\right\|_\infty < \gamma \tag{5}$$

where $G \in RH_\infty$ and w_1, w_2, a, b, c satisfy the assumptions in (2).

(i) Define $\hat{G}_1 = \frac{b}{a} \in R_r H_\infty$. Then

$$\left\| \frac{w_1}{w_2}\left(G - \hat{G}_1\right)\right\|_\infty < \gamma\left(1 + 2r\left\|\frac{w_1}{w_2}\right\|_\infty \left\|\frac{w_2}{w_1}\right\|_\infty\right).$$

(ii) Define $\hat{G}_2 = \frac{b}{a} + \frac{c_2}{w_1} \in R_{r+d}H_\infty$ where

$$\frac{w_1(s)c(s)}{w_2(s)a^\sim(s)} = \frac{c_1(s)}{a^\sim(s)} + \frac{c_2(s)}{w_2(s)}$$

is a stable/anti-stable decomposition. Then

$$\left\| \frac{w_1}{w_2}\left(G - \hat{G}_2\right)\right\|_\infty < \gamma(1 + 2r).$$

(iii) Assume that $\gamma_{\min}$ is the smallest γ such that (5) holds. Then

$$\min_{\hat{G}\in R_r H_\infty} \left\| \frac{w_1}{w_2}\left(G-\hat{G}\right)\right\|_\infty \geq \gamma_{\min}.$$

Proof:

(i) We first give a bound on $\|c/a^\sim\|_\infty$. This can be done by using an argument from [7] which is based on Nehari's theorem [14]. We have that

$$\left\|G-\frac{b}{a}-\frac{c}{a^\sim}\right\|_\infty < \gamma \left\|\frac{w_2}{w_1}\right\|_\infty.$$

Since $(G-b/a)\in RH_\infty$ and $c/a^\sim \in R_r H_\infty^-$, we have that

$$\left\|\frac{c}{a^\sim}\right\|_H < \gamma \left\|\frac{w_2}{w_1}\right\|_\infty,$$

and using a standard bound from [14] we have

$$\left\|\frac{c}{a^\sim}\right\|_\infty < 2r\gamma \left\|\frac{w_2}{w_1}\right\|_\infty.$$

Using this and the triangle inequality in (5) gives the stated bound.

(ii) Using the stable/anti-stable decomposition, we have that

$$\left\|\frac{w_1}{w_2}\left(G-\frac{b}{a}-\frac{c_2}{w_1}\right)-\frac{c_1}{a^\sim}\right\|_\infty < \gamma.$$

Now, use a similar Nehari theorem argument as in (i) to obtain $\|c_1/a^\sim\|_\infty < 2r\gamma$ (notice that the weight does not appear now). The bound in the statement again follows from the triangle inequality.

(iii) The approximation $b/a + c/a^\sim$ belongs to a set that is larger than (and contains) $R_r H_\infty$. Hence,

$$\gamma_{\min} \triangleq \min_{a,b,c}\left\|\frac{w_1}{w_2}\left(G-\frac{b}{a}-\frac{c}{a^\sim}\right)\right\|_\infty \leq \min_{\hat{G}\in R_r H_\infty}\left\|\frac{w_1}{w_2}\left(G-\hat{G}\right)\right\|_\infty.$$

■

It should be remembered that the bounds derived in Theorem 1 mainly should be seen as a theoretical justification for the method and to help guide in the choice of approximations and weights. We expect the bound in (i) to be conservative in general. This is because we used the submultiplicative property of the L_∞-norm to derive it. If the weight attains large and small values, the bound is always large. However, numerical experiments show that $\hat{G}_1$ often are good approximations. It may be possible to construct examples where the worst-case bound really is attained.

The bound in (ii) is more attractive since it only depends on γ and the approximation order r. The price is that $\hat{G}_2$ has d more states than $\hat{G}_1$. It is then important to choose low-complexity weights. A typical low-complexity weight can be in the form

$$\frac{w_1(s)}{w_2(s)} = \frac{k\omega_0^2(s/\alpha+1)^2}{s^2+2\zeta\omega_0 s+\omega_0^2}.$$

Here ω_0 is typically chosen to be close to the cross-over frequency of G (where good approximation is desired), with a damping parameter $0<\zeta<1$. The constant $k>0$ determines how important it is to have good model matching at low frequencies. The factor $(s/\alpha+1)^2$ in the numerator is introduced to make the weight biproper. A biproper weight is a common assumption in weighted model reduction, see [1], and is also assumed in Theorem 1. Often it is reasonable to choose $\alpha \gg \omega_0$. Then $\hat{G}_2$ has two poles in $-\alpha$ whose break points are far away from the cross-over frequency. In this case, it may be reasonable to discard these poles and simply use

$$\hat{G}_2 = \frac{b}{a} + \frac{c_2(0)}{w_1(0)} \in R_r H_\infty,$$

as approximation (if the break points in $c_2(s)$ also are large).

The bound (iii) is interesting since it shows that the problem we solve actually gives a lower bound on what can be achieved at all with *any* stable model of McMillan degree r. The upper and lower bounds together give us an a priori estimate on how far away our approximations $\hat{G}_1$ and $\hat{G}_2$ at worst are from an optimal solution.

C. *MIMO frequency-weighted approximation*

The MIMO method we suggest here requires that the stable poles $\{p_i\}_{i=1}^r$ of $\hat{G}(s)$ are fixed from the start. The poles could be determined by first running the SISO approximation technique in Section II-B on each entry $G_{ij}(s)$ of the $(p\times m)$-dimensional TFM $G(s)$, for example.

Once the poles are fixed, the problem is to find the zeros of $\hat{G}$ such that the McMillan degree of $\hat{G}$ is as small as possible, while the weighted error $\|W_o(G-\hat{G})W_i\|_\infty$ is also small. To solve this problem, we use a heuristic similar to the one suggested in [8]. Assuming that p_i are distinct[1], use the parametrization

$$\hat{G}(s) = \hat{G}_0 + \sum_{i=1}^r \frac{1}{s-p_i}\hat{G}_i, \tag{6}$$

and we shall fix $\hat{G}_i \in \mathbb{C}^{p\times m}$, where $\hat{G}_i^* = \hat{G}_j$ when $p_i^* = p_j$. The McMillan degree of (6) is given by

$$\sum_{i=1}^r \operatorname{rank}\hat{G}_i.$$

Minimization of the rank of a matrix subject to LMIs is known as a difficult and nonconvex problem. However, there exist simple and effective heuristics, such as the one in [8]. There the trace-class (or nuclear) norm of $\hat{G}_i$,

$$\|\hat{G}_i\|_1 = \sum_{k=1}^{\min\{p,m\}} \sigma_k(\hat{G}_i),$$

where σ_i are the singular values, is minimized instead of the rank. The minimization problem we solve is the following: Fix a desired approximation accuracy $\gamma>0$. Then solve

$$\min_{\hat{G}_i} \sum_{i=1}^r \|\hat{G}_i\|_1 \quad \text{subject to}$$
$$\|W_o(G-\hat{G})W_i\|_\infty < \gamma, \tag{7}$$

[1]Generically p_i are distinct. If not, we have to modify the parametrization in (6) slightly.

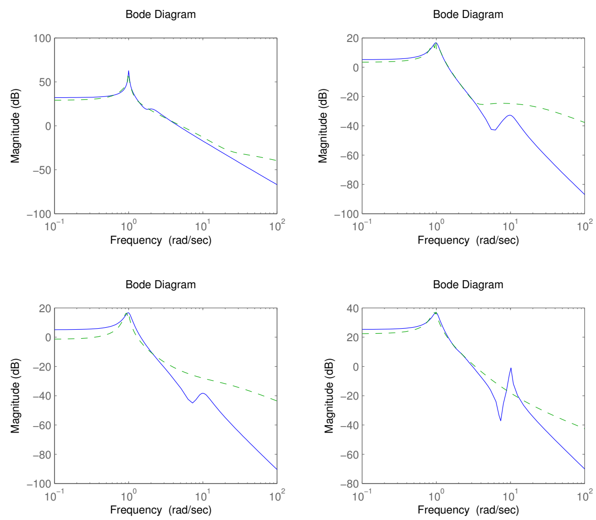

Fig. 1. The 208th-order model G (solid) and the 13th-order approximation $\hat{G}$ (dashed) from Example 1. The relative approximation criteria (8) has been used.

where $\hat{G}$ is given by (6). This is a convex optimization problem. How to solve this problem by means of LMIs is shown in [8] for the case when (7) is enforced on a frequency grid $\{\omega_k\}$ and without weights. To add weights only require minor changes. To enforce (7) for all ω, one can again use the KYP lemma. A problem is that the resulting LMI is often of high dimension. How to enforce similar conditions more effectively for all ω is shown in [18].

As γ is decreased to obtain a better approximation, the McMillan degree of $\hat{G}$ increases until there no longer is a feasible solution to (7). Hence, there is a trade off between approximation accuracy and complexity.

An upper bound on the McMillan degree of $\hat{G}$ is $r \cdot \min\{p, m\}$. If G is SISO, the McMillan degree of $\hat{G}$ is always equal to r. Then we do not need to minimize the sum of trace norms. Instead we can simply minimize γ as in Section II-B.

D. Implementation and an example

We implements and solve the above methods using the LMI solver SeDuMi [19] with YALMIP [20].

We use the method in Section II-B to fix the poles for the method in Section II-C, which delivers the final approximation.

Example 1: Here we seek to minimize the relative error

$$\|(G - \hat{G})G^{-1}\|_\infty. \tag{8}$$

This is a very common criteria in model reduction, see [1]. We have a model $G(s)$ of McMillan degree 208, shown in Fig. 1, and $W_o = I$ and $W_i = G^{-1}$. We assume knowledge of $G(j\omega)$ on a 75-point frequency grid in the interval $[0.1, 3]$ rad/s.

In the first step of the approximation procedure, described in Section II-B, we approximate each entry of $G(s)$ separately,

$$\min_{\hat{G}_{ij}} \quad \|(G_{ij} - \hat{G}_{ij})/G_{ij}\|_\infty, \quad i, j = 1, 2,$$

using $r = 2$ or $r = 4$ depending on the entry. This gives us a set of 10 stable poles $\{p_i\}_{i=1}^{10}$ to be used in the second step.

In the method in Section II-C, we have a trade off between the accuracy γ, and the degree of $\hat{G}$. In Table I, this trade off is shown. We choose $\gamma = 0.34$, which gives a 13th-order approximation. $\hat{G}$ is plotted together with G in Fig. 1, and there is seen to be a good fit over the interval $[0.1, 3]$ rad/s.

TABLE I
ACCURACY γ AND CORRESPONDING MCMILLAN DEGREE OF $\hat{G}$.

γ	Degree
0.24	17
0.29	17
0.34	13
0.39	10

III. APPLICATION 1: ENVIRONMENT COMPENSATOR

In this section, we study a TFM $P(s)$ ("the plant") that is interacting with an environment modeled by $G(s)$:

$$\begin{aligned} y &= P(u + w) \\ w &= -Gy + l, \end{aligned} \tag{9}$$

where the output $y(t) \in \mathbb{R}^m$ is available for feedback control of P and also influences the environment. The control signal is $u(t) \in \mathbb{R}^o$. The signal $w(t) \in \mathbb{R}^p$ represents the influence from the environment on the plant, and $l(t) \in \mathbb{R}^p$ is an additional external disturbance. The environment is assumed to be stable and is modeled by the $(p \times m)$-dimensional TFM $G \in RH_\infty$. It influences P through the feedback (9). We assume throughout that the feedback connection of P and G is internally stable, and hence $(I + PG)^{-1} \in RH_\infty$. The environment G may be a TFM of high McMillan degree.

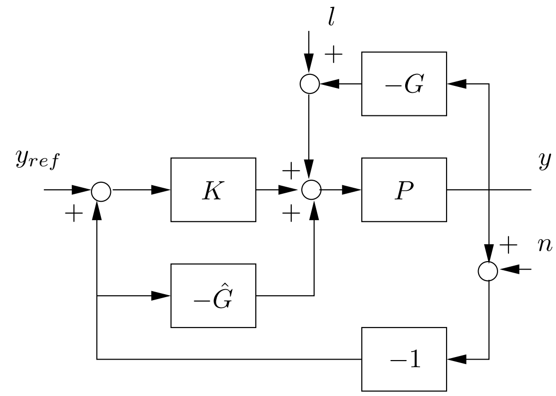

Fig. 2. The environment compensator using the feedback (10).

The problem we consider here is to find a low-complexity feedback controller for $P(s)$ that compensates for the disturbances that are generated by the environment $G(s)$. The controller will consist of two parts: One part depends on $P(s)$, and is assumed to be fixed. The other part depends on the environment $G(s)$.

This problem should be of interest when a plant is working in a possibly changing and complex environment. Applications we have in mind for this set up include

- a vehicle P driving in a formation of other vehicles G, compare with [9];
- a generator or subnetwork P acting in a larger power system G; and
- a cell P producing proteins in an hostile environment G.

We will not discuss these applications further here. Instead, we focus on how the problem can be formulated and solved using weighted model reduction. In particular, we show how the methods proposed in Section II are useful.

We use the feedback

$$u = K(y_{ref} - y) + \hat{G}y, \tag{10}$$

where the TFM K is a well-tuned controller for P, designed without taking the environment G into account. y_{ref} is a reference signal, and $\hat{G}$ an approximate model of the environment. The controlled system is shown in Fig. 2. The closed-loop transfer function is

$$y = (I + P(K + \Delta))^{-1}P(Ky_{ref} + l + (\hat{G} - K)n),$$

where $\Delta = G - \hat{G}$ is the environment model error. If $\Delta = 0$, the response to references, y_{ref}, and to disturbances, l, is the same as when P is not connected to the environment G. Notice, however, that the response to measurement noise, n, depends on $\hat{G}$.

One rationale for choosing $\hat{G}$ when elimination of load disturbances l is of interest, is to match the closed-loop transfer functions from l to y. If K has been chosen to fulfill requirements from the unconnected system, we then choose $\hat{G}$ so that the error

$$(I + PK)^{-1}P - (I + PK + P\Delta)^{-1}P \approx (I + PK)^{-1}P\Delta(I + PK)^{-1}P,$$

is small. Here we have used a first-order Taylor expansion, which is valid for small $P\Delta$. This leads to the weights $W_o = W_i = (I + PK)^{-1}P$ in the approximation problem (1).

Of course, the above technique can be generalized to other cases, such as when reference following is the main concern.

Example 2: In this example, we assume that the plant is $P(s) = 1/(s + 1)^4$ and we choose K as a PID-controller

$$K(s) = 2\left(1 + \frac{1}{2.5s} + \frac{s}{1 + 0.05s}\right).$$

A Bode diagram of the environment G is shown in Fig. 3. It is seen to be a highly resonant system with many poles and zeros close to the imaginary axis. Such systems are generally hard to approximate with low-order systems. We use $W_i = W_o = (1 + PK)^{-1}P$, which is also shown in Fig. 3. Furthermore, we assume knowledge of frequency samples on an uniform grid $\{\omega_k\} = \{0.2, 0.21, \ldots, 1.2\}$.

To obtain $\hat{G}$, we first use the method in Section II-B with $r = 4$ and $r = 6$ to fix the poles of $\hat{G}$. When $r = 4$ we obtain $\gamma = 0.24$ and when $r = 6$ we obtain $\gamma = 0.04$.

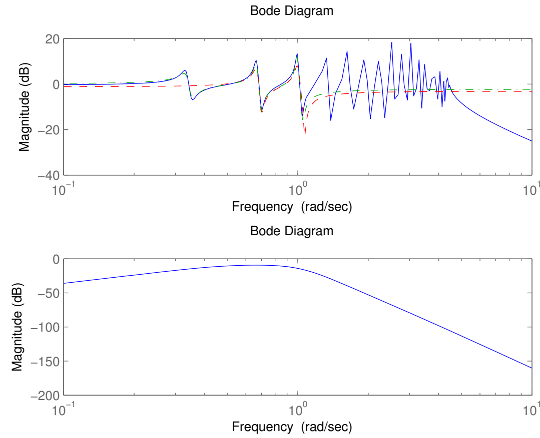

Fig. 3. The upper plot shows magnitude data from the environment G (solid), a fourth-order approximation $\hat{G}$ (dashed), and a sixth-order approximation $\hat{G}$ (dash-dotted) from Example 2. The lower plot shows the weight $P^2/(1+PK)^2$ used in the approximation. There is good agreement between the models for the relevant frequencies.

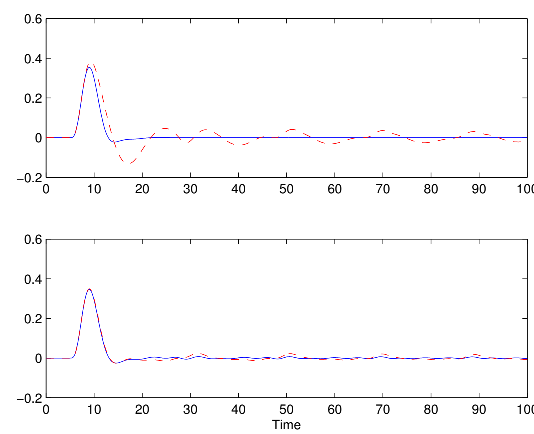

Fig. 4. The load step response test in Example 2. In the upper plot, P is just controlled by K. In the nominal case (solid) it is not connected to the environment G, and in the other case (dashed) it is connected to G. The resonant environment G introduces oscillations in the system. In the lower plot, fourth- and sixth-order models $\hat{G}$ (dashed and solid, respectively) are added to the feedback (10), and are seen to almost bring the performance back to nominal.

Instead of using $\hat{G}_1$ or $\hat{G}_2$ as approximations, we extract the poles $\{p_i\}$, and use them to get improved approximations with the method in Section II-C.

Using the method in Section II-C, we obtain the minimum $\gamma \approx 0.04$ with both $r = 4$ and $r = 6$. The resulting $\hat{G}$ are shown in Fig. 3. Notice, that since this example is SISO, the McMillan degree of $\hat{G}$ is always equal to r. A load step response test is shown in Fig. 4, with and without the environment model $\hat{G}$ in (10). As can be seen, adding just a low-order model $\hat{G}$ almost brings the behavior back to nominal, even though the environment G is very complex.

IV. Application 2: Approximation of Interconnected Linear Systems

In this section, we will formulate two weighted model reduction problems that are relevant for simplification of interconnected linear systems. We use the model setup from [12]. Consider a collection of n TFMs $G_i(s)$ that models the subsystems in the interconnected structure. The interconnected linear system is given by

$$\begin{aligned} b(s) &= \begin{pmatrix} b_1(s) \\ \vdots \\ b_n(s) \end{pmatrix} = \begin{pmatrix} G_1(s) & & 0 \\ & \ddots & \\ 0 & & G_n(s) \end{pmatrix} \begin{pmatrix} a_1(s) \\ \vdots \\ a_n(s) \end{pmatrix} \\ &\triangleq G(s)a(s) \\ a(s) &= Kb(s) + Hu(s) \\ y(s) &= Fb(s) \end{aligned}$$

where u is the external input, y the output, and a, b are interconnection signals. K, H, and F are real constant matrices of appropriate dimensions. K is the connectivity matrix and contains the interconnection structure of $G_i(s)$. We assume that the interconnected system is internally stable so that $(I-KG)^{-1} \in RH_\infty$. The TFM of the interconnected system is given by

$$y(s) = F(I - G(s)K)^{-1}G(s)Hu(s).$$

A. Simplification of subsystem dynamics $G_i(s)$

We are looking for an approximation $\hat{G}(s)$ with the same block-diagonal structure as $G(s)$. Using the method in Section II-C, we can seek approximations in the form

$$\hat{G}(s) = \hat{G}_0 + \sum_{i=1}^{r} \frac{1}{s - p_i}\hat{G}_i,$$

where $\hat{G}_i$ has the same block-diagonal structure as $G(s)$. Such a structure is easily enforced in LMI solvers.

To simplify the subsystem dynamics, one can for example solve

$$\begin{aligned} &\min_{\hat{G}_i} \sum_{i=1}^{r} \|\hat{G}_i\|_1 \quad \text{subject to} \\ &\|(I - KG)^{-1}K(\hat{G} - G)\|_\infty < \gamma, \\ &\text{block structure}(G(s)) = \text{block structure}(\hat{G}_i), \end{aligned}$$

where the weight comes from the small-gain theorem. If $\gamma < 1$, it guarantees that the interconnected system is stable using $\hat{G}$ instead of G. Other weights result if we try to match closed-loop transfer functions, as was done in Section III.

B. Simplification of interconnection structure K

Another interesting problem is to simplify the interconnection structure. One complexity measure of the interconnection structure is the rank of K. If the rank of K is equal to l, then there are only l independent signals that connects the subsystems. If we simplify the system with the respect to the rank of K, we gain insight about what signals are most important in the structure.

We can use the rank minimization heuristic in Section II-C to simplify the interconnection structure whilst maintaining the stability of the interconnected system. For example, using the small-gain theorem, we obtain

$$\begin{aligned} &\min_{\hat{K}} \|\hat{K}\|_1 \quad \text{subject to} \\ &\|(I - GK)^{-1}G(\hat{K} - K)\|_\infty < \gamma. \end{aligned}$$

If $\gamma < 1$, we have guaranteed stability. As the approximation error tolerance γ is increased, we obtain an approximation $\hat{K}$ of lower rank.

V. Conclusion and Future Work

In this paper, we have shown how a recently proposed model-reduction technique [6] can be used together with frequency weights. Incorporation of weights is often very important when closed-loop systems are dealt with. Yet it is not a straightforward thing to do using the traditional methods, see [1]. We also derived upper and lower error bounds on the weighted error, and they turned out to be simple. The technique can also be combined with a rank-minimization heuristic to deal with MIMO systems.

We presented two applications where the suggested techniques are useful. First, there was an application where the problem was to update an existing feedback controller with a simple compensator for complex environments. We called this environment compensation, and gave a simple example. Second, we showed how the methods can be used to simplify linear models while maintaining their interconnection structure.

Future work will include more work on the applications in Sections III and IV.

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