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The Effect of Sensor Health on State Estimation

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***Abstract*— In this paper, we consider the problem of state estimation using the standard Kalman filter recursions which takes account of the available sensor health information. Given a stochastic description of the sensor health, we are able to show that the expected error covariance converges to a unique value for all initial values, while the available previous work only showed the upper bound of the expected error covariance converges. Our approach provides both theoretical value to the analysis as well as the potential to get tighter upper bound. Our results provide a criterion of evaluating the sensor measurement. In the multisensor fusion problem, depending on the system error tolerance levels, it can then be determined whether to fuse a particular sensor measurement or not. Examples and simulations are provided to assist the theory.**

I. INTRODUCTION

State estimation is one of the major areas in the field of control, and one of the most frequently used state estimation tools is the Kalman Filter [1], which deals with linear discrete time systems with both white process and measurement noises. Since [1] was published, the Kalman filter has been the subject of extensive research and applications [2], [3], [4], [5]. For example, the Kalman filter has been widely used in autonomous and assisted navigation.

The Kalman filter in its original form dealt with a single sensor. However as systems become increasingly complex and more fragile to failures [6], redundancy is needed to provide more robustness to the systems, especially when the system is in a severe working environment. For example in DARPA Grand Challenge 2005, the autonomous driving contest, Team Caltech installed a rich set of sensors on Alice (the autonomous car, see [7] for a detailed description on the whole system) which can provide a full estimate of the state of the car. Even if certain sensors fail to work properly, the remaining sensors are still able to provide full estimate.

When there are multiple sensors available, it is natural to fuse the sensor measurements to get an enhanced state estimate. Different algorithms have been proposed in the past few decades. See [8] for a review on the multisensor data fusion algorithms. The Kalman filter can also be used in multisensor state estimation, but it assumes a centralized system structure in the sense that measurements from different sensors are sent to a common center. In [9], the authors proposed a decentralized Kalman filter algorithm to deal with multiple sensors at different locations which can communicate with each other.

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In those sensor fusion schemes, a particular sensor model is usually assumed to be fixed for ease of analyzing and showing certain convergent properties of the error covariance. In many situations, however, the sensor models are changing depending on the environment the sensor is in which is reflected through the sensor health parameter. Take the GPS sensor in Alice [7] for example, where the satellite numbers available to the GPS sensors are changing as Alice crosses different areas. As the measurement noise variance levels decreases when the number of available satellites increase, this makes the measurement noise variance level change from time to time. This is especially true when Alice is traveling in Urban areas where buildings and other obstacles can block the GPS signals.

There has been some recent work on state estimation with sensor models not being fixed. In a slight different aspect, the authors in [10] discussed how packet loss can affect stable state estimation. In their work, they considered two different sensor models, one being that the sensor is working properly, *i.e.*, with a finite noise covariance, and the other being that the sensor fails, *i.e.*, with an infinite noise covariance. They showed there exists a certain threshold of the packet loss rate above which the state estimation diverges in the expected sense. They also provided lower and upper bounds of the threshold value. As we will show later, some of their result is just a special case of ours. In [11], the authors extended the results in [10] to the case when partial observations can be lost. Instead of looking at the expected value of the error covariance, in [12], the authors considered the same problem by looking at the distribution of the error covariance, hence providing a better evaluation of how the packet loss rate would affect the error covariance.

In this paper, we consider the problem of multisensor fusion with each sensor having a health monitor which indicates at any time which model the sensor is using. For example, if the sensor is healthy or working properly, the sensor model is the one with a smaller noise covariance; otherwise, the sensor model is the one with a larger noise covariance. We can for example think this health monitor as the satellite counter in the GPS sensor which indicates what measurement noise covariance should be used. This will be made clear in the next section.

In [13], the authors considered optimal sensor scheduling among a set of sensors. The setting of this paper can be considered as a special case of theirs and it turns out that the upper bound for the error covariance in our paper coincides with their. However, there are a few major differences between our papers. In [13], they only showed the upper bound of the expected error covariance converges, but did not shed any light on how the actual expected error covariance evolves. We in fact are able to prove that the expected error covariance itself converges to a unique value for all initial values. From this fact, we derive the upper bound on the error covariance. Therefore in our paper, we provide both theoretical value to the analysis as well as the potential to get tighter upper bound in the future work. Another difference as we will show in the example session is the different assumptions on the noise covariances. Their results would be trivial if they take our assumption.

The paper is organized as follows. In section 1, we briefly review the motivation and some relevant past work. In section 2, the mathematical model of the problem is given and some lemmas are provided to facilitate the main proof. In section 3, we give the main results of the paper. In section 4, we provide an example to demonstrate the result and simulations are also provided. The paper concludes with a summary of our results and a discussion of the work that lies ahead.

II. Problem Set Up

Consider the following discrete-time LTI system

$$x_{k+1} = Ax_k + w_k. \tag{1}$$

A set of sensors $S = \{S_i, \cdots, S_N\}$ produce measurement of the state. The model of sensor $i, i = 1, \cdots, N$, is given by

$$y_k^i = C_i x_k + v_k^{i1}, \tag{2}$$

with probability α_k^i and

$$y_k^i = C_i x_k + v_k^{i2}, \tag{3}$$

with probability $1 - \alpha_k^i$. As usual, $x_k \in \mathbb{R}^n$ is the state vector, $y_k^i \in \mathbb{R}^{m_i}$ is the observation vector of the ith sensor, $w_k \in \mathbb{R}^n$ and $v_k^{ij} \in \mathbb{R}^{m_i}, j = 1, 2$ are all white Gaussian random vectors with zero mean and covariance matrices $Q \geq 0$ and $R_{ij} > 0, j = 1, 2$, respectively. In particular, assume $R_{i2} \geq R_{i1} > 0$ which means that the sensors produce noisier measurement when in mode two. α_k^i indicate how often the sensor is in mode one. Assume (A, Q) is stabilizable and for each i, the pair (A, C_i) is detectable. Finally assume that observation models occur independently at each time step. As we only consider the estimation problem without any control, we are only interested in how the error covariance evolves.

Assume a sensor fusion scheme is given, for example, the decentralized Kalman filter algorithm described in [9], we will then determine whether to include a particular sensor data into to the fusion process or not. It is clear that if R_{i2} is much larger than R_{i1} and $\alpha_k \leq \alpha$ for all k where α is a very small number, intuitively in this case, the measurement from sensor i should be excluded as the measurement contains mostly noise.

As a first step towards this ultimate objective, we restrict ourselves to having only one sensor and hence will drop all the i in the notations from now on. We will generalize the result for the single sensor case to the N sensors case in the future work.

Denote the initial error covariance as $P_0 > 0$ and at time k, the corresponding error covariance as P_k. By using the standard Kalman filter recursions (see [10] for an example how to use those recursions), we obtain the updating equation for the error covariance (to be more precise, the a priori error covariance) as follows.

$$P_{k+1} = \begin{cases} g_1(P_k) & \text{if} \ \ y_k = Cx_k + v_k^1, \\ g_2(P_k) & \text{if} \ \ y_k = Cx_k + v_k^2, \end{cases}$$

where g_1 and g_2 are defined as

$$\begin{aligned} g_1(X) &= AXA' + Q - AXC'[CXC' + R_1]^{-1}CXA', \\ g_2(X) &= AXA' + Q - AXC'[CXC' + R_2]^{-1}CXA'. \end{aligned}$$

Notice that due to the randomness of α_k, P_k is itself a random variable. Therefore we will only consider $E[P_k]$. In the next section, we provide the main results which show the convergence property of $E[P_k]$ and provide an upper bound on its limit. Before we state the main theorem, we introduce a few lemmas to facilitate the proof.

Lemma 1: $X \geq Y > 0$ if and only if $0 < X^{-1} \leq Y^{-1}$.
Proof: See [14], page 471, Corollary 7.7.4.

Lemma 2: For all $X > 0$, $g_2(X) \geq g_1(X)$.
Proof: We only need to show that for all $X > 0$,

$$g_2(X) - g_1(X) \leq 0.$$

As

$$\begin{aligned} & g_2(X) - g_1(X) \\ = \ & AXC'([CXC' + R_1]^{-1} - [CXC' + R_2]^{-1})CXA', \end{aligned}$$

and

$$CXC' + R_2 \geq CXC' + R_1,$$

it follows directly from Lemma 1 that $g_2(X) \geq g_1(X)$. ■

Lemma 3: If $g_i(X) \leq X$, then $g_i^k(X) \leq g_i^{k-1}(X)$ for all k, $i \in \{1, 2\}$. Similarly if $g_i(X) \geq X$, then $g_i^k(X) \geq g_i^{k-1}(X)$ for all k, $i = 1, 2$.

Proof: From Lemma 1-c in [10], if $0 \leq X \leq Y$, then $g_i(X) \leq g_i(Y)$. Therefore, keep applying g_i for k time to both sides of the inequality $g_i(X) \leq X$ or $g_i(X) \geq X$, we get the desired form. ■

Corollary 4: If $g_i(X) \leq X$, then $g_i^k(X) \leq X$ for all k, $i = 1, 2$. Similarly if $g_i(X) \geq X$, then $g_i^k(X) \geq X$ for all k, $i = 1, 2$.

Let $l_j \in \{1, 2\}$ which stands for the sensor model at time $j, j = 1, \cdots, k$. Further define

$$I_1 = I_1(l_k \cdots l_1) = \sum_{j=1}^{k} (l_j \mod 2),$$

and

$$I_2 = I_2(l_k \cdots l_1) = k - I_1(l_k \cdots l_1).$$

Define

$$f_k \triangleq \sum_{l_k \cdots l_1} \alpha^{I_1} (1 - \alpha)^{I_2} g_{l_k} \cdots g_{l_1},$$

where the sum is running over all the possible 2^k realizations of the sensor equations. It is easy to show that f_k is related to f_{k-1} by

$$f_k = \alpha f_{k-1} \circ g_1 + (1-\alpha) f_{k-1} \circ g_2,$$

where $f_0 = I$ is the identity map.

Lemma 5: For all time k, f_k is concave, *i.e.*, for any $\alpha \in [0,1]$ and any $X \geq 0$, $Y \geq 0$,

$$f_k(\alpha X + (1-\alpha)Y) \geq \alpha f_k(X) + (1-\alpha) f_k(Y).$$

Furthermore, if $0 \leq X \leq Y$, then $f_k(X) \leq f_k(Y)$ for all k.

Proof: From Lemma 1-e in [10], both g_1 and g_2 are concave functions, *i.e.*,

$$\begin{aligned} g_1(\alpha X + (1-\alpha)Y) &\geq \alpha g_1(X) + (1-\alpha) g_1(Y), \\ g_2(\alpha X + (1-\alpha)Y) &\geq \alpha g_2(X) + (1-\alpha) g_2(Y). \end{aligned}$$

Hence for $i = 1$ or 2 and $j = 1$ or 2,

$$\begin{aligned} g_i g_j(\alpha X + (1-\alpha)Y) &\geq g_i(\alpha g_j(X) + (1-\alpha) g_j(Y)), \\ &\geq \alpha g_i g_j(X) + (1-\alpha) g_i g_j(Y). \end{aligned}$$

It is easily verified via induction that $g_{l_k} \cdots g_{l_1}$ is concave too for any value of l_j, $j = 1, \cdots, k$. Hence it follows that f_k is concave for all k. When $0 \leq X \leq Y$, it directly follows from Lemma 1-c in [10] that $f_k(X) \leq f_k(Y)$ as each individual g_i is an increasing function. ■

Lemma 6: If $\alpha_k \geq \alpha$ for all k, then $E[P_k] \leq f_k(P_0)$ for all k.

Proof: Let h_k be defined by

$$h_k = \alpha_k h_{k-1} \circ g_1 + (1-\alpha_k) h_{k-1} \circ g_2,$$

with $h_0 = I$ being the identity map. Then $E[P_k] = h_k(P_0)$. Hence we only need to show

$$h_k \leq f_k$$

for all k. From the definition of h_k, it is easy to show that if $0 \leq X \leq Y$, then $h_k(X) \leq h_k(Y)$ for all k by using the increasing property of g_1 and g_2. Clearly this holds for the case when $k = 0$. Assume for $k \geq 0$,

$$h_{k-1} \leq f_{k-1}.$$

Write $\alpha_k = \alpha + \Delta_k$, where $0 \leq \Delta_k \leq 1$ for all k. Then

$$f_k - h_k = S_1 + S_2 + S_3,$$

where

$$\begin{aligned} S_1 &= \alpha(f_{k-1} \circ g_1 - h_{k-1} \circ g_1) \geq 0, \\ S_2 &= (1-\alpha)(f_{k-1} \circ g_1 - h_{k-1} \circ g_2) \geq 0, \\ S_3 &= \Delta_k(h_{k-1} \circ g_2 - h_{k-1} \circ g_1) \geq 0. \end{aligned}$$

The first two hold as $f_{k-1} \geq h_{k-1}$ from the assumption and the last one holds as $g_2 \geq g_1$ from Lemma 2 and h_{k-1} is an increasing function. As a result $f_k \geq h_k$ for all k. ■

In light of Lemma 6, we can assume from now on that $\alpha_k = \alpha$ for all k, *i.e.*, we consider the worse case scenario of the error covariance. The tradeoff is that the analysis becomes much simpler as seen in the main theorem in the next section.

III. Main Results

In this section, we provide the main results of this paper. We show that $E[P_k]$ converges to a unique value asymptotically regardless of where $P_0 > 0$ starts. We further show that this unique value is upper bounded by a known matrix. This result will help us to determine which sensor data to include in fusion in the multisensor fusion problem given the error tolerance level of the system.

From now on, let $\bar{P}_i$ be such that $g_i(\bar{P}_i) = \bar{P}_i$, $i = 1, 2$, *i.e.*, $\bar{P}_i$ is the unique solution to the corresponding algebraic riccati equation. We summarize the main result of this paper in the following theorem.

Theorem 7: Let

$$D(\alpha, P_0) = \lim_{k \to \infty} E[P_k].$$

Then $D(\alpha)$ has the following properties.

1) $D(0, P_0) = \bar{P}_2$ and $D(1, P_0) = \bar{P}_1$.
2) $D(\alpha, P_0)$ exists for all $\alpha \in [0,1]$ and all $P_0 > 0$ and it does not depend on the value of P_0. Furthermore, it satisfies
$$\bar{P}_1 \leq D(\alpha, P_0) \leq \bar{P}_2.$$
3) $D(\alpha, P_0) \leq \bar{D}(\alpha)$, $\bar{P}_1 \leq \bar{D}(\alpha) \leq \bar{P}_2$ where $\bar{D}(\alpha)$ is the unique positive definite solution to
$$\alpha g_1(\bar{D}) + (1-\alpha) g_2(\bar{D}) = \bar{D}. \tag{4}$$
4) If $\alpha_1 \leq \alpha_2$, then $\bar{D}(\alpha_1) \geq \bar{D}(\alpha_2)$.

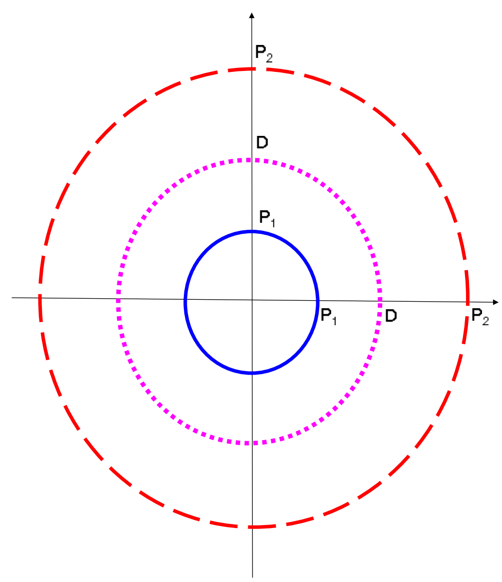

Fig. 1. system error tolerance levels.

Before proving the theorem, we give an intuitive explanation of what the theorem means for scalar system. In Figure 1, the region inside the (blue) solid line is where the actual error eventually converges to when the sensor is healthy (mode 1). The region inside the (red) dashed line corresponds to when the sensor is unhealthy (mode 2). Suppose the system error tolerance level is within the (pink) dotted line, then we can study by how often the sensor is in mode 2 that the error could jump outside the (pink) dotted line. In our theorem, $\bar{D}$ provides a sufficient condition to evaluate how healthy a sensor is.

Proof: Using the previous notations, with initial error covariance P_0, we have $E[P_k] = f_k(P_0)$. To simplify the notation, let us write $D(\alpha, P_0) = D(\alpha)$.

1) Notice that when $\alpha = 0$, $f_k = g_2^k$, *i.e.*, f_k equals the composition of the function g_2 for k times. Hence

$$D(0) = \lim_{k\to\infty} g_2^k(P_0) = \bar{P}_2.$$

Similarly if $\alpha = 1$, then $f_k = g_1^k$, hence

$$D(1) = \lim_{k\to\infty} g_1^k(P_0) = \bar{P}_1.$$

2) We first show that if $P_0 = \bar{P}_2$, $D(\alpha)$ exists. In this case,

$$\begin{aligned} E[P_{k+1}] &= f_{k+1}(\bar{P}_2) \\ &= \alpha f_k \circ g_1(\bar{P}_2) + (1-\alpha) f_k(\bar{P}_2) \\ &\le f_k(\alpha g_1(\bar{P}_2) + (1-\alpha)\bar{P}_2) \text{ (Lemma 5)} \\ &\le f_k(\alpha \bar{P}_2 + (1-\alpha)\bar{P}_2) \text{ (Lemma 2)} \\ &= f_k(\bar{P}_2) \\ &= E[P_k]. \end{aligned}$$

Hence $E[P_k]$ forms a monotonically decreasing sequence for $P_0 = \bar{P}_2$. Also notice that $D(\alpha)$ is bounded below by $\bar{P}_1$ as

$$\begin{aligned} E[P_k] &= f_k(\bar{P}_2) \\ &= \sum_{l_k\cdots l_1} \alpha^{I_1}(1-\alpha)^{I_2} g_{l_k}\cdots g_{l_1}(\bar{P}_2) \\ &\ge \sum_{l_1\cdots l_k} \alpha^{I_1}(1-\alpha)^{I_2} g_1\cdots g_1(\bar{P}_2) \\ &= g_1^k(\bar{P}_2) \\ &\ge \bar{P}_1. \end{aligned}$$

The last inequality follows from the fact that

$$\bar{P}_2 = g_2(\bar{P}_2) \ge g_1(\bar{P}_2) \ge g_1^2(\bar{P}_2) \ge \cdots \ge g_1^k(\bar{P}_2)$$

which converges to $\bar{P}_1$.
Hence we have a monotonic nonincreasing sequence of matrices bounded below. It is a simple matter to show that the sequence converges. At this point, we do not know where the sequence $E[P_k]$ initiated at $P_0 = \bar{P}_2$ (*i.e.*, the sequence $f_k(\bar{P}_2)$) converges to. Since $f_k(\bar{P}_2)$ forms a convergent sequence, let us call its limit as $f_\infty(\bar{P}_2)$. From definition, we can write $f_k(\bar{P}_2)$ as follows.

$$\begin{aligned} f_k(\bar{P}_2) &= \alpha f_{k-1}(g_1(\bar{P}_2)) + (1-\alpha) f_{k-1}(g_2(\bar{P}_2)) \\ &= \alpha f_{k-1}(g_1(\bar{P}_2)) + (1-\alpha) f_{k-1}(\bar{P}_2) \end{aligned}$$

or we can write

$$f_{k-1}(g_1(\bar{P}_2)) = \frac{1}{\alpha}(f_k(\bar{P}_2) - (1-\alpha) f_{k-1}(\bar{P}_2)).$$

As both the sequences $f_k(\bar{P}_2)$ and $f_{k-1}(\bar{P}_2)$ converges, $f_{k-1}(g_1(\bar{P}_2))$ must also converge. Let us call its limit as $f_\infty(g_1(\bar{P}_2))$. Notice that $f_k(\bar{P}_2)$ converges to $f_\infty(\bar{P}_2)$ and $(1-\alpha)f_{k-1}(\bar{P}_2)$ converges to $(1-\alpha)f_\infty(\bar{P}_2)$, hence we obtain

$$f_\infty(\bar{P}_2) = f_\infty(g_1(\bar{P}_2)).$$

As $f_k(X) \le f_k(Y)$ for $X \le Y$ and $g_1(\bar{P}_2) \le \bar{P}_2$, then it is true that $f_\infty(P_0)$ equals $f_\infty(\bar{P}_2)$ for at least all P_0 such that $g_1(\bar{P}_2) \le P_0 \le \bar{P}_2$.
Next we further expand $f_k(\bar{P}_2)$ as follows.

$$\begin{aligned} f_k(\bar{P}_2) = {} & \alpha^2 f_{k-2}(g_1^2(\bar{P}_2)) + \\ & (1-\alpha)\alpha f_{k-2}(g_2(g_1(\bar{P}_2))) + \\ & \alpha(1-\alpha) f_{k-2}(g_1(g_2(\bar{P}_2))) + \\ & (1-\alpha)^2 f_{k-2}(g_2^2(\bar{P}_2)). \end{aligned} \quad (5)$$

Notice that

$$g_2(g_1(\bar{P}_2)) \le g_2(\bar{P}_2) = \bar{P}_2,$$

and

$$g_1(g_2(\bar{P}_2)) = g_1(\bar{P}_2), g_2^2(\bar{P}_2) = \bar{P}_2.$$

We also have

$$f_{k-2}(g_1^2(\bar{P}_2)) \le f_{k-2}(g_2(g_1(\bar{P}_2))).$$

As a result,

$$\text{LHS} \le f_{k-2}(g_2(g_1(\bar{P}_2)) \le \text{RHS}$$

where

$$\begin{aligned} \text{LHS} = {} & \frac{1}{\alpha}\{f_k(\bar{P}_2) - \\ & \alpha(1-\alpha) f_{k-2}(g_1(\bar{P}_2)) - (1-\alpha)^2 f_{k-2}(\bar{P}_2)\} \end{aligned}$$

and

$$\text{RHS} = f_{k-2}(\bar{P}_2).$$

Since both RHS and LHS converge to $f_\infty(\bar{P}_2)$, $f_k(g_2(g_1(\bar{P}_2)))$ must also converge to $f_\infty(\bar{P}_2)$, *i.e.*, $f_\infty(g_2(g_1(\bar{P}_2))) = f_\infty(\bar{P}_2)$. Next substitute this back to equation (5), it is easy to obtain that $f_k(g_1^2(\bar{P}_2)))$ converges to $f_\infty(\bar{P}_2)$. Hence $f_\infty(P_0)$ equals $f_\infty(\bar{P}_2)$ for at least all P_0 such that $g_1^2(\bar{P}_2) \le P_0 \le \bar{P}_2$.
By induction, we can easily show that

$$f_\infty(\bar{P}_2) = f_\infty(g_1^k(\bar{P}_2)),$$

for all k. Since it is true that

$$\lim_{k\to\infty} g_1^k(\bar{P}_2) = \bar{P}_1,$$

it follows immediately $f_\infty(P_0)$ is a constant for all P_0 such that $\bar{P}_1 \le P_0 \le \bar{P}_2$. This in fact includes the case where $P_0 \not\ge \bar{P}_j$ and $P_0 \not\le \bar{P}_j$, $j = 1$ or 2, *i.e.*, the eigenvalues of P_0 are not all bigger than or all smaller than the corresponding eigenvalue of $\bar{P}_j$, $j = 1$ or 2. This is because for all $P_0 > 0$,

$$g_1^k(P_0) \le f_k(P_0) \le g_2^k(P_0),$$

and

$$\lim_{k\to\infty} g_1^k(P_0) = \bar{P}_1, \lim_{k\to\infty} g_2^k(P_0) = \bar{P}_2.$$

Hence eventually, $f_k(P_0)$ will satisfy $\bar{P}_1 \le f_k(P_0) \le \bar{P}_2$. This completes the proof for part 2.

3) Let $\bar{D}$ be such that

$$\alpha g_1(\bar{D}) + (1-\alpha)g_2(\bar{D}) = \bar{D}.$$

Notice that

$$\bar{D} \le \alpha g_2(\bar{D}) + (1-\alpha)g_2(\bar{D}) = g_2(\bar{D})$$

and

$$\bar{D} \ge \alpha g_1(\bar{D}) + (1-\alpha)g_1(\bar{D}) = g_1(\bar{D}).$$

Therefore by Corollary 4,

$$g_1^k(\bar{D}) \le \bar{D} \le g_2^k(\bar{D})$$

for all k. Therefore it follows that $\bar{P}_1 \le \bar{D} \le \bar{P}_2$. By part 2 of the theorem, $f_\infty(\bar{D}) = D(\alpha)$. Now take $P_0 = \bar{D}$, then

$$\begin{aligned} E[P_{k+1}] &= f_{k+1}(\bar{D}) \\ &= \alpha f_k(g_1(\bar{D})) + (1-\alpha) f_k(g_2(\bar{D})) \\ &\le f_k(\alpha g_1(\bar{D}) + (1-\alpha)g_2(\bar{D})) \\ &= f_k(\bar{D}) \\ &= E[P_k]. \end{aligned}$$

Hence $E[P_k]$ also forms a monotonically decreasing sequence. Therefore $f_\infty(\bar{D}) \le \bar{D}$, *i.e.*, $D(\alpha) \le \bar{D}$.

4) Assume $0 \le \alpha_1 \le \alpha_2 \le 1$ and let $\alpha_2 = \alpha_1 + \Delta$, where $0 \le \Delta \le 1$. Write

$$\bar{D}(\alpha_1) = \bar{D}_1, \bar{D}(\alpha_2) = \bar{D}_2$$

and

$$T(X) = \alpha_1 g_1(X) + (1-\alpha_1)g_2(X).$$

Then $\bar{D}_1 = T(\bar{D}_1)$ and $\bar{D}_2 = T(\bar{D}_2) + M$, where

$$M = \Delta(g_1(\bar{D}_2) - g_2(\bar{D}_2)) \le 0.$$

From the expression of $\bar{D}_i, i = 1, 2$, we can in fact write

$$\bar{D}_i = \lim_{k\to\infty} \bar{D}_i(k), i = 1, 2$$

where

$$\bar{D}_1(k+1) = T(\bar{D}_1(k))$$

and

$$\bar{D}_2(k+1) = T(\bar{D}_2(k)) + M(k)$$

with $\bar{D}_i(0) = I$ and

$$M(k) = \Delta(g_1(\bar{D}_2(k)) - g_2(\bar{D}_2(k))) \le 0.$$

Clearly, $\bar{D}_1(0) \ge \bar{D}_2(0)$ as $M(0) \le 0$. Now assume $\bar{D}_1(k) \ge \bar{D}_2(k)$, then

$$\begin{aligned} \bar{D}_1(k+1) &= T(\bar{D}_1(k)) \\ &\ge T(\bar{D}_2(k)) \\ &\ge T(\bar{D}_2(k)) + M(k), \\ &= \bar{D}_2(k+1). \end{aligned}$$

Taking the limit as $k \to \infty$, we see that $\bar{D}(\alpha_1) \ge \bar{D}(\alpha_2)$. ∎

Notice that in the third part of the theorem, if $\alpha = 1$, then $\bar{D} = \bar{P}_1$ and similarly if $\alpha = 0$, $\bar{D} = \bar{P}_2$ which agree with our intuition.

We can interpret the above result in many different ways. For example, consider the scenario at [10] where they considered the problem of Kalman filter with packet losses. It turns out that their results on the upper bound of the expected error covariance is just a special case of our results by letting $R_2 \to \infty$. Then $\bar{D} = \bar{V}$ where $\bar{V}$ is their notation for the upper bound on the expected error covariance. Furthermore we have shown that $E[P_k]$ converges to the same value for all initial conditions.

IV. Example and Simulations

We consider an example which is taken from that in [13] with slight modification. In their original example, the authors assumed $R_1 \le R_2$ is *not true* to make interesting result, or otherwise, α_1 would be just 1 for all the time, hence there is no point to optimize the probability distribution. In fact the main difference between our paper and theirs is that we try to evaluate a given α for a sensor and eventually a given set of α^i and a set of sensors $\{S_i\}, i = 1, \cdots, N$, while they try to design the optimal α for a set of sensors. Let the system parameters (equations(1, 2, 3)) be the following

$$\begin{aligned} A &= \begin{bmatrix} 1.0 & 0.0 & 0.2 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.2 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \\ C &= \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix} \\ Q &= \begin{bmatrix} 0.002 & 0.005 & 0.02 & 0.005 \\ 0.005 & 0.002 & 0.005 & 0.02 \\ 0.02 & 0.005 & 0.2 & 0.05 \\ 0.005 & 0.02 & 0.05 & 0.2 \end{bmatrix} \\ R_1 &= \begin{bmatrix} 0.4 & 0.0 \\ 0.0 & 0.4 \end{bmatrix} \\ R_2 &= \begin{bmatrix} 20.4 & 0.0 \\ 0.0 & 20.4 \end{bmatrix} \end{aligned}$$

It is easily verified that all the assumptions are satisfied. This also assumed in sensor mode two, both components produce noisier data than when it is in mode one. For this system, it is calculated that

$$\begin{aligned} \bar{P}_1 &= \begin{bmatrix} 0.2780 & 0.0226 & 0.3660 & 0.0525 \\ 0.0226 & 0.2780 & 0.0525 & 0.3660 \\ 0.3660 & 0.0525 & 0.8543 & 0.1686 \\ 0.0525 & 0.3660 & 0.1686 & 0.8543 \end{bmatrix} \\ \bar{P}_2 &= \begin{bmatrix} 4.4622 & 0.3132 & 0.3660 & 0.0525 \\ 0.3132 & 4.4622 & 0.0525 & 0.3660 \\ 0.3660 & 0.0525 & 0.8543 & 0.1686 \\ 0.0525 & 0.3660 & 0.1686 & 0.8543 \end{bmatrix} \end{aligned}$$

And their corresponding traces are $\text{Trace}(\bar{P}_1) = 2.2646$ and $\text{Trace}(\bar{P}_2) = 13.1254$. For a given $\alpha \in [0,1]$, we also calculate the corresponding value of $\bar{D}$ according to equation (4). The (red) dotted curve in Figure 2 shows the relationship between the trace of $\bar{D}$ and α.

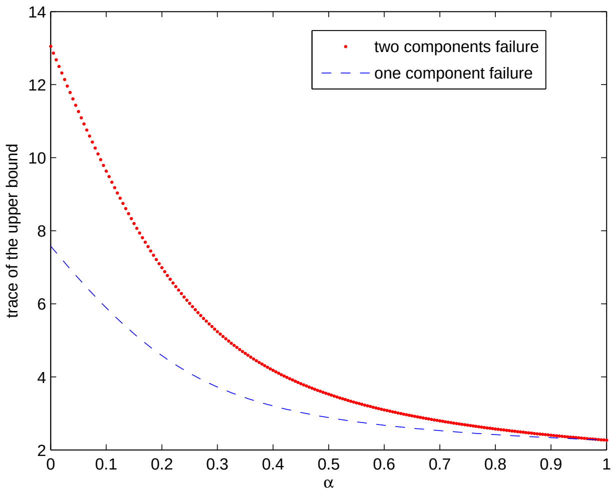

Fig. 2. α versus the trace of the upper bound $\bar{D}$.

In some applications, maybe only certain components become unhealthy. We model that as a new R_2, where

$$R_2 = \begin{bmatrix} 0.4 & 0.0 \\ 0.0 & 20.4 \end{bmatrix}$$

This basically says that the second component of the sensor is not working properly while the first one remains healthy. The newly calculated values are:

$$\bar{P}_2 = \begin{bmatrix} 0.2796 & 0.0492 & 0.3686 & 0.0831 \\ 0.0492 & 4.4219 & 0.0831 & 2.1766 \\ 0.3686 & 0.0831 & 0.8585 & 0.2086 \\ 0.0831 & 2.1766 & 0.2086 & 2.0622 \end{bmatrix}$$

and $\text{Trace}(\bar{P}_2) = 7.6223$. We again plot the relationship between the trace of $\bar{D}$ and α in the (blue) dashed curve in Figure 2.

Much can be said from Figure 2. For example, for the sensor with two components unhealthy, $\alpha \geq 0.425$ suffices to provide that

$$\text{Trace}(\lim_{k\to\infty} E[P_k]) \leq 4,$$

which could represent the system error tolerance level. α can be further reduced to just 0.265 to guarantee the same performance when there is just one unhealthy component.

V. Conclusions and Future Work

As system becomes much more complex, redundancy in terms of multiple sensors/acutators is necessary to guarantee performance in severe working environments. Hence multi-sensor fusion especially with changing sensor models, is of great theoretical as well as practical importance for lots of applications. In this paper, we considered a first step towards multisensor fusion where each individual sensor has a health monitor which indicates the different modes the sensor is in. As is demonstrated in the example, we have provided sufficient conditions to evaluate sensor measurement given a system error tolerance level.

We will next apply the result obtained in the one sensor case to a set of sensors. Each individual sensor is assumed to have their own representations of of the model Equations(2, 3). We will then determine whether to include a sensor measurement into the final fusion or not given a system error tolerance level. Tradeoffs in terms of the noise covariances and α values will be given for certain guaranteed system performance.

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References

[1] R.E.Kalman, "A new approach to linear filtering and prediction problems," *Trasactions of ASME, Journal of Basic Engineering on Automatic Control*, vol. 82(D), pp. 35–45, 1960.

[2] M. S. Grewal and A. P. Andrews, *Kalman Filtering Theory and Practice*. Prentice Hall, 1993.

[3] P. S. Maybeck, *Stochastic Models, Estimation, and Control*. Academic Press, Inc, 1979, vol. 1.

[4] T.Kailath, A.Sayed, and B.Hassibi, *Linear Estimation*. Prentice Hall, 2000.

[5] B.Anderson and J.Moore, *Optimal Filtering*. Prentice Hall, 1990.

[6] X. Liu, M. Fazel, and J. Doyle, "Complexity and robustness in lattice percolation models and linear programs," *Working notes, Control and Dynamical Systems, Caltech, Pasadena, USA*, 2005.

[7] L. B. Cremean, T. B. Foote, J. H. Gillula, G. H. Hines, D. Kogan, K. L. Kriechbaum, J. C. Lamb, J. Leibs, L. Lindzey, A. D. Stewart, J. W. Burdick, and R. M. Murray, "An information-rich autonomous vehicle for high-speed desert navigation," *Journal of Field Robotics, Submitted*, Dec 2005.

[8] D. L.Hall and J. Llinas, "An introduction to multisensor data fusion," *Proceedings of the IEEE*, vol. Vol 85, No. 1, pp. 6 – 23, Jan 1997.

[9] B.S.Rao and H.F.Durrant-Whyte, "Fully decentralized algorithm for multisensor kalman filtering," in *IEE Proceedings-Control Theory and Applications*, vol. 138(5), Sep 1991, pp. 413–420.

[10] B.Sinopoli, L.Schenato, M.Franceschetti, K.Poolla, M.Jordan, and S.Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, 2004.

[11] X. Liu and A. Goldsmith, "Kalman filtering with partial observation losses." IEEE Control and Decision, 2004.

[12] L. Shi, M. Epstein, A. Tiwari, and Richard.M.Murray, "Estimation with information loss: Asymptotic analysis and error bounds." CDC-ECC05, Seville, Spain, Dec 2005.

[13] V. Gupta, T. Chung, B. Hassibi, and R. M. Murray, "On a stochastic sensor selection algorithm with applications in sensor scheduling and dynamic sensor coverage," *Automatica*, vol. 42, no. 2, pp. 251–260, 2006.

[14] R. A.Horn and C. R.Johnson, *Matrix Analysis*. Cambridge University Press, 1985.