Configuration Flatness of Lagrangian Systems Underactuated by One Control

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Abstract

Lagrangian control systems that are differentially flat with flat outputs that only depend on configuration variables are said to be configuration flat. We provide a complete characterisation of configuration flatness for systems with n degrees of freedom and $n-1$ controls whose range of control forces only depends on configuration and whose Lagrangian has the form of kinetic energy minus potential. The method presented allows us to determine if such a system is configuration flat and, if so provides a constructive method for finding all possible configuration flat outputs. Our characterisation relates configuration flatness to Riemannian geometry. We illustrate the method by two examples.

Keywords

Differential Flatness, Nonlinear Control, Lagrangian Mechanics.

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1 Introduction

Roughly speaking an underdetermined system of ordinary differential equations

$$F^k(t, x^1, \ldots, x^N, \dot{x}^1, \ldots, \dot{x}^N) = 0, \qquad k = 1, \ldots, n < N$$

is differentially flat if there is a smooth locally 1-1 correspondence between solutions $x(t)$ of the system and arbitrary functions $y(t)$, of the form

$$\begin{aligned} x(t) &= g(t, y(t), \ldots, y^{(l)}(t)), \\ y(t) &= h(t, x(t), \ldots, x^{(q)}(t)), \end{aligned}$$

where $(y^1, \ldots, y^p) \in \mathbb{R}^p$ and $p = N - n$. Here g, h are smooth maps, $y^{(k)}$ is the k^{th} derivative of y, and l, q are integers. The variables y^j are referred to as flat outputs. The special class of systems given by

$$\dot{x}^i = f^i(t, x^1, \ldots, x^n, u^1, \ldots, u^p), \quad i = 1, \ldots, n$$

are more familiar to control theorists and the flat outputs depend on states, inputs, and derivatives of inputs

$$y^j = h^j(t, x, u, u^{(1)}, \ldots, u^{(q)}), \quad j = 1, \ldots, p.$$

For a detailed discussion of differential flatness see Fliess et al. [3, 4], Martin [9], Pomet [12], van Nieuwstadt et al. [20] and Rathinam and Sluis [13].

The importance of flatness to control applications lies in the fact that it provides a systematic and relatively simple way to generate solution trajectories between two given states. One uses the maps g and h to transform between original system space (states as well as inputs) and the smaller dimensional flat output space. See van Nieuwstadt and Murray [19] and Murray et al. [11] for more details.

For example consider the "kinematic car" shown in Figure 1. Ignoring dynamics we assume the velocity of the mid point between rear wheels and the steering velocity are directly controlled. Then the system is differentially flat with the coordinates of the midpoint between rear wheels providing the two flat outputs (see Tilbury et al. [18]). Given any trajectory for this point one can determine the entire motion of the car: the tangent to the trajectory determines the orientation of the car and the curvature (second derivative) determines the orientation of the front wheels. Hence all feasible paths of the vehicle can be parametrised in terms of the trajectories of the flat output point. A given set of initial and final configurations of the car then determine two end points and first and second order derivatives at these end points for feasible trajectories of the flat output point. One could choose any trajectory for the flat output point that satisfies these end conditions and obtain a feasible trajectory for the car that passes through the given initial and final conditions. In this example flat outputs are rather obvious. This is not the case with many other examples and one needs a theoretical tool to provide a systematic way of finding them if they exist.

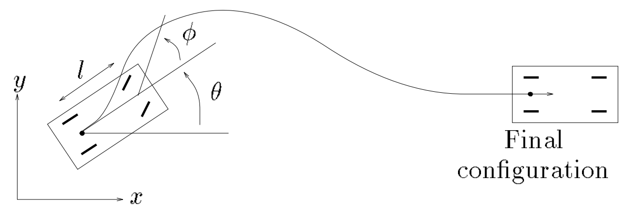

Figure 1: Path Planning for Kinematic Car

In the case of single input systems a complete characterisation of differential flatness is available, see e.g. Shadwick [15]. In that case, flatness is the same as static feedback linearizability. See also [2]. In the framework of exterior differential systems, checking for flatness of a single input system reduces to calculating "derived systems" and checking certain rank and integrability conditions. See van Nieuwstadt et al. [20], Sluis [16] and Sluis and Tilbury [17]. For multi-input systems no complete theory exists.

Many interesting examples of mechanical systems are differentially flat and in most known examples flat outputs have been found that depend only on the configuration variables but not on their derivatives. We refer to such flat outputs as "configuration flat outputs" and systems possessing such outputs as "configuration flat". For instance, the above example of kinematic car is configuration flat. All Lagrangian systems that are fully actuated (number of controls equals number of degrees of freedom) are configuration flat with all the configuration variables as flat outputs. See [11] for a catalogue of other examples. The reasons for studying configuration flatness are as follows. Firstly it is a simpler case than the general case of differential flatness and is possibly the first thing to study if one were to be able to relate the mechanical structure with differential flatness. For instance configuration controllability of mechanical systems has already been studied and related to the mechanical structure (see Lewis and Murray [8]). Secondly the smaller the number of derivatives of configuration variables the flat outputs depend upon the simpler the numerical implementation of the transformations involved in trajectory generation. In this paper we completely characterise configuration flatness for a special class of mechanical systems. The class under consideration involves systems whose dynamics is described by Lagrangian mechanics with a Lagrangian function of the form "kinetic energy minus potential". Also the number of independent controls is assumed to be one less than the number of degrees of freedom (the simplest case next to fully actuated systems) and the possible range of control forces only depends on the configuration and not on the velocity. We describe an algorithm for deciding if such a system is configuration flat and if it is so, we describe a procedure for finding all possible configuration flat outputs. We do not consider systems with nonholonomic constraints. The kinematic car example hence does not fall into the class of systems under our consideration.

The paper is organized as follows. Section 2 introduces some concepts from

Lagrangian control systems theory and also provides a definition of configuration flatness. Section 3 introduces some concepts from Riemannian geometry that are necessary for our theory and also states and proves the main proposition and outlines an algorithm for coordinate calculations to check configuration flatness. Section 4 explores how system symmetries relate to symmetries of the flat outputs. Finally Section 5 gives two examples to illustrate the theory.

2 Lagrangian control systems and configuration flatness

Consider a Lagrangian system with configuration manifold Q of dimension n and a Lagrangian $L : TQ \to \mathbb{R}$. When no external (generalised) forces are applied, the motion of this system satisfies the Euler-Lagrange equations, written in coordinates $(q^1, \dots, q^n)$ as

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n. \tag{1}$$

In a control situation external control forces are applied and it is natural to think of forces as covectors on the manifold Q. In other words, for a configuration $q \in Q$ the total external force acting on the system can be represented by an element of T_q^*Q. This is because forces naturally pair with velocities, which can be thought of as elements of T_qQ, to give instantaneous power. The possible range of control forces lies in a subspace of T_q^*Q which may depend on position q as well as velocity v_q. In other words the control forces can be described by a horizontal valued codistribution $\bar{P} \subset T^*(TQ)$, and $p = \dim \bar{P}$ is the number of independent controls. For an interesting and wide class of systems this subspace only depends on configuration q and hence can be described by a codistribution $P \subset T^*Q$ of dimension p. For the rest of the discussion we shall only consider this case. All feasible paths (solutions) of such a system are characterised by the following underdetermined system of ODEs in coordinates $(q^1, \dots, q^n)$:

$$a_k^i(\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i}) = 0, \quad k = 1, \dots, n-p \tag{2}$$

where $a_k^i \frac{\partial}{\partial q^i}$ for $k = 1, \dots, n-p$ span the annihilator of P, denoted ann P.

It is useful to think in terms of the associated submanifold $\mathscr{E} \subset J^2(\mathbb{R}, Q)$ of the second order jet space (see [14]), which geometrically describes such a second order system of equations. $\mathscr{E}$ has codimension $n-p$ and in local coordinates $(t, q, \dot{q}, \ddot{q})$ is cut out by the common zeroes of the functions

$$a_k^i(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial L}{\partial q^i}), \quad k = 1, \dots, n-p.$$

Let $q \in Q$ be a point and let $y : U \subset Q \to \mathbb{R}^p$ be a submersion locally defined around q. Let $y = (y^1, \dots, y^p)$. We say $y^1, \dots, y^p$ are *differentially independent* around q if $y^1, \dots, y^p$ do not have to satisfy an ODE along solutions local to q. More precisely, when restricted to $\mathscr{E}$, $dy^1, \dots, dy^p, d\dot{y}^1, \dots, d\dot{y}^p, d\ddot{y}^1, \dots, d\ddot{y}^p$ are linearly independent for generic points on $\pi_2^{-1}(V) \cap \mathscr{E}$ where $V \subset U$ is an open neighbourhood of q and $\pi_2 : J^2(\mathbb{R}, Q) \to Q$ is the standard projection. If $dy^1, \dots, dy^p, d\dot{y}^1, \dots, d\dot{y}^p, d\ddot{y}^1, \dots, d\ddot{y}^p$ are linearly dependent when restricted to $\mathscr{E}$, for points on $\pi_2^{-1}(V) \cap \mathscr{E}$ where $V \subset U$ is an open neighbourhood of q then $y^1, \dots, y^p$ are *differentially dependent* around q.

Suppose $y^1, \dots, y^p$ are differentially independent around q. If there are functions f^i and a neighbourhood W of q such that along a generic solution $c : \mathbb{R} \to W \subset Q$,

$$(z^i \circ c)(t) = f^i((y \circ c)(t), \dots, \frac{d^r}{dt^r}(y \circ c)(t)), \quad i = 1, \dots, n-p \tag{3}$$

where $z^1, \dots, z^{n-p}$ are any complementary coordinates to $y^1, \dots, y^p$, then $y^1, \dots, y^p$ are said to be *configuration flat outputs* around q and the system is *configuration flat* around q. In other words, given $y^1(t), \dots, y^p(t)$ we can determine a (locally) unique trajectory for the Lagrangian system (2).

We present the following lemma which will be of use later.

Lemma 1 Let $q \in Q$, U an open neighbourhood of q, and $y : U \to \mathbb{R}^p$ be a configuration flat output. Then generically the set of solutions $c : \mathbb{R} \to U$ that project down to the same curve $y \circ c$ are all isolated.

Proof By definition of flatness along generic solutions, given $y(t)$ the complementary coordinates $z(t)$ are locally uniquely determined by equations (3). ∎

3 Mechanical systems with n degrees of freedom and n − 1 controls

Consider the mechanical system whose Lagrangian is given by

$$L(v) = \frac{1}{2} g(v, v) - V \circ \tau_Q(v), \tag{4}$$

where g is the Riemannian metric (assumed to be non degenerate) corresponding to kinetic energy and V is the potential energy function on Q and $\tau_Q : TQ \to Q$ is the tangent bundle projection. Suppose the number of controls $p = n - 1$, in other words $\dim P = n - 1$. In this section we shall present a method for determining if this system is configuration flat. If the system is configuration flat our approach provides us with a constructive method for finding all possible (configuration) flat outputs.

Before proceeding further we present some concepts from Riemannian geometry. Given a metric g we have a notion of differentiation of objects on the manifold such as functions, vector fields, differential forms and tensors along a given vector field Z. This is the covariant derivative ∇ given by the Levi-Civita connection (see [1]). ∇_Z denotes covariant derivative along a vectorfield Z and is related to parallel (with respect to metric) transport of objects along the integral curves of Z. The covariant derivative of a function f along Z denoted $\nabla_Z f$ is just the familiar directional derivative $Z(f)$ or the Lie derivative. But covariant derivative of a vectorfield X along Z denoted $\nabla_Z X$ is not the same as the Lie derivative $[Z, X]$. Some properties of ∇ are

$$\begin{aligned}
\nabla_Z(X_1 + X_2) &= \nabla_Z X_1 + \nabla_Z X_2 && (5)\\
\nabla_Z(fX) &= \nabla_Z X + Z(f)X && (6)\\
\nabla_{fZ} X &= f\nabla_Z X && (7)\\
\nabla_Z X - \nabla_X Z &= [Z, X] && (8)
\end{aligned}$$

where X, X_1, X_2, Z are arbitrary vector fields and f is an arbitrary function on the manifold. In a coordinate system $(q^1, \ldots, q^n)$ on manifold Q the covariant derivatives are calculated with the aid of Christoffel symbols Γ^i_{jk} where $i, j, k = 1, \ldots, n$ and Christoffel symbols are defined by

$$\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k} = \Gamma^i_{jk} \frac{\partial}{\partial q^i}. \tag{9}$$

From the properties (8) of ∇ it follows that $\Gamma^i_{jk} = \Gamma^i_{kj}$. Γ^i_{jk} can be computed from metric g by the formula

$$\Gamma^m_{jk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^i}\right) g^{im}, \quad j, m = 1, \ldots, n, \tag{10}$$

where $g^{ik} g_{kj} = \delta^i_j$ (g^{ik} are components of the inverse of matrix g_{ik}). Then the covariant derivative of vectorfield $X = X^k \frac{\partial}{\partial q^k}$ along $Z = Z^j \frac{\partial}{\partial q^j}$ is given by

$$\nabla_Z X = Z^j X^k \Gamma^i_{jk} \frac{\partial}{\partial q^i} + Z^j \frac{\partial X^k}{\partial q^j} \frac{\partial}{\partial q^k}. \tag{11}$$

For the mechanical system under consideration let us define an associated distribution D by

$$D = \operatorname{span}\{\xi, \nabla_Z \xi \ : \ Z \in \mathfrak{X}(Q)\}, \tag{12}$$

where ξ is any vector field such that $\operatorname{ann} P = \operatorname{span}\{\xi\}$ and $\mathfrak{X}(Q)$ is the set of all smooth vector fields on Q.

It is easy to check that D doesn't depend on the choice of $\xi \in \operatorname{ann} P$. By the linearity of covariant derivative it follows that

$$D = \operatorname{span}\{\xi, \nabla_{\frac{\partial}{\partial q^i}} \xi \ : \ i = 1, \ldots, n\} \tag{13}$$

where $(q^1, \ldots, q^n)$ are any set of coordinates. Hence D is easily calculated using equations (10), (11) and (13). The following proposition characterises configuration flat outputs $y^1, \ldots, y^p$ by conditions on $\ker Ty$, which in coordinates is the null space of the Jacobian of the map y.

Proposition 2 Let q be a point on Q, U an open neighbourhood of q and suppose $y : U \subset Q \to \mathbb{R}^p$ is a submersion. If $y^1, \ldots, y^p$ are configuration flat outputs, then

$$g(\ker Ty, D) = 0. \tag{14}$$

Conversely if $g(\ker Ty, D) = 0$ and if certain regularity condition holds at q then $y^1, \ldots, y^p$ are configuration flat outputs around q.

The regularity condition is that the ratios of functions in the following set should not all be the same at q:

$$\{\nabla_\eta(g(\xi, Z)) : g(\xi, Z), \nabla_\eta(g(\nabla_{Z_1} Z_2, \xi)) : g(\nabla_{Z_1} Z_2, \xi), \nabla_\eta(\xi(V)) : \xi(V)\}, \tag{15}$$

where Z, Z_1, Z_2 are arbitrary vector fields around q that are y-related to some vectorfield on $\mathbb{R}^p$ and ξ, η are fixed nonvanishing vector fields such that $\operatorname{ann} P = \operatorname{span}\{\xi\}$ and $\ker Ty = \operatorname{span}\{\eta\}$.

Remark 3 Proposition 2 states the conditions for configuration flatness in intrinsic geometric terms. In coordinates the algorithm for deciding if the system is configuration flat is as follows. Calculate D using equation (13). If $D = TQ$ then system is not configuration flat, since for any y, one can find some vector field $Z \in D = TQ$, such that $g(\ker Ty, Z) \neq 0$. Suppose $\dim D \leq n - 1$. Then choose a one dimensional distribution, say spanned by a vectorfield η, that is orthogonal to D. Since a one dimensional distribution is integrable locally, one can find independent functions $y^1, \ldots, y^p$ $(p = n - 1)$ around q that "cut out" the leaves of the corresponding foliation. These will be flat outputs provided the regularity conditions are met.

The regularity conditions can be checked in coordinates as follows. Choose a function z that completes $y^1, \ldots, y^p$ to a coordinate system. Then $y^1, \ldots, y^p$ will be flat outputs if the following ratios of functions are not all identically equal in a local neighbourhood:

$$\begin{aligned}
\frac{\partial}{\partial z}\Big(g\big(\xi, \frac{\partial}{\partial y^j}\big)\Big) : g\big(\xi, \frac{\partial}{\partial y^j}\big), & \qquad j = 1, \ldots, p \\
\frac{\partial}{\partial z}\Big(g\big(\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j}, \xi\big)\Big) : g\big(\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j}, \xi\big)\big), & \qquad j, k = 1, \ldots, p \\
\frac{\partial}{\partial z}(\xi(V)) : \xi(V). &
\end{aligned} \tag{16}$$

If these are all identically equal that means $y^1, \ldots, y^p$ are differentially dependent and another one dimensional distribution must be tried.

Remark 4 It is readily seen that configuration flatness is determined primarily by the kinetic energy metric g since the role of potential function V only enters via the regularity conditions. This explains why in many known examples (see [11]) the presence or absence of gravity does not alter the configuration flat outputs but only the solution curves where singularities occur. However, we present an example in next section where the potential function plays a crucial role via the regularity conditions.

Proof (of Proposition 2) : Given a submersion $y : Q \to \mathbb{R}^p$, one can choose a local coordinate chart on Q such that y is the canonical submersion of $\mathbb{R}^n$ onto $\mathbb{R}^p$. Let the corresponding coordinates on Q be $(q^1, \ldots, q^n)$. Then, $y^j(q) = q^j$ for $j = 1, \ldots, p = n-1$. Let $\xi = a^i \frac{\partial}{\partial q^i}$ span ann P. Then all solutions of the system satisfy the single ODE

$$a^i\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right) = 0. \tag{17}$$

Suppose in these coordinates g is given by g_{ij}. Then we can rewrite equation (17) as

$$a^i\left(g_{ij}\ddot{q}^j + \frac{\partial g_{ik}}{\partial q^j}\dot{q}^j\dot{q}^k - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i}\right) = 0. \tag{18}$$

Using the formula (10) for the Christoffel symbols and using $q^j = y^j$ for $j = 1, \ldots, p$ to separate the terms involving $\dot{q}^n$ and $\ddot{q}^n$, we rewrite equations (18) as,

$$a^i\left(g_{ij}\ddot{y}^j + \Gamma^m_{jk} g_{mi}\dot{y}^j\dot{y}^k + \frac{\partial V}{\partial q^i} + g_{in}\ddot{q}^n + \Gamma^m_{nn} g_{mi}(\dot{q}^n)^2 + \Gamma^m_{jn} g_{mi}\dot{y}^j\dot{q}^n\right) = 0 \tag{19}$$

where range of summation of various indices is clear.

Necessity: Suppose y are flat outputs. Then it follows that the coefficient of $\ddot{q}^n$ in the above ODE must to be zero. Otherwise we can rewrite the equation as

$$\frac{d\dot{q}^n}{dt} = f(y, \dot{y}, \ddot{y}, q^n, \dot{q}^n)$$

for some smooth function f, and by existence theorem of solutions to ODEs, given any curve $y(t)$ we get a 2-parameter family of solutions $q(t)$ (parametrised by initial conditions $q^n(t_0), \dot{q}^n(t_0)$) that project to $y(t)$ and they are not isolated from each other and hence by Lemma 1 y cannot be flat, contradicting our assumption. So $a^i g_{in} = 0$ and this leaves us with an ODE of the form

$$A(y)(\dot{q}^n)^2 + B(y, \dot{y})\dot{q}^n + C(y, \dot{y}, \ddot{y}, q^n) = 0.$$

A similar reasoning tells us that the term $\dot{q}^n$ should be absent, in other words $A(y) = 0$ and $B(y, \dot{y}) = 0$. Here A and B are given by,

$$A = a^i \Gamma^m_{nn} g_{mi} \quad B = a^i \Gamma^m_{jn} g_{mi} \dot{y}^j.$$

Observe that B is linear in terms $\dot{y}$ with coefficients that are functions only of (y, q^n). Hence the condition $B = 0$ can be written as $n-1$ equations that set the coefficients of $\dot{y}^j$ to be zero. The equation $A = 0$ has the same form as these, and we get the following n equations:

$$a^i \Gamma^m_{jn} g_{im} = 0, \quad j = 1, \ldots, n.$$

So all together flatness of y implies the following equations,

$$\begin{aligned} a^i g_{in} &= 0 \\ a^i \Gamma^m_{jn} g_{im} &= 0, \quad j = 1, \dots, n. \end{aligned} \tag{20}$$

If $\ker Ty = \operatorname{span}\{\eta\}$, then in our choice of coordinates $\eta = \lambda \frac{\partial}{\partial q^n}$ where λ is some nonvanishing function on Q. Hence, $g(\xi, \eta) = a^i g_{in} = 0$ by the first condition, where $\xi = a^i \frac{\partial}{\partial q^i}$ spans $\operatorname{ann} P$. Also since

$$\nabla_{\frac{\partial}{\partial q^j}} \eta = \lambda \Gamma^m_{jn} \frac{\partial}{\partial q^m} + \frac{\partial \lambda}{\partial q^j} \frac{\partial}{\partial q^n},$$

it follows that

$$g(\nabla_{\frac{\partial}{\partial q^j}} \eta, \xi) = \lambda a^i \Gamma^m_{jn} g_{im} + \frac{\partial \lambda}{\partial q^j} a^i g_{in} = 0.$$

But, by derivation property,

$$\nabla_Z(g(\xi, \eta)) = (\nabla_Z g)(\xi, \eta) + g(\nabla_Z \xi, \eta) + g(\xi, \nabla_Z \eta)$$

and since $\nabla_Z g = 0$ for any $Z \in \mathfrak{X}(Q)$ (by the property of Levi-Civita connection) and since $g(\eta, \xi) = 0$ it follows that

$$g(\nabla_{\frac{\partial}{\partial q^j}} \xi, \eta) = 0, \quad j = 1, \dots, n.$$

By linearity of ∇ it follows that

$$g(\nabla_Z \xi, \eta) = 0, \quad \forall Z \in \mathfrak{X}(Q).$$

Hence, $\ker Ty$ is orthogonal to D.

Sufficiency: Conversely, if $\ker Ty$ is orthogonal to D, previous reasoning shows that, in the same coordinate system the equations (20) hold. As seen before these imply that the solution curves of the system are given by the ODE

$$E(q^n, y, \dot{y}, \ddot{y}) = 0,$$

where

$$E = a^i g_{ij} \ddot{y}^j + a^i g_{im} \Gamma^m_{jk} \dot{y}^j \dot{y}^k + a^i \frac{\partial V}{\partial q^i}.$$

This is not sufficient for flatness of $y^1, \dots, y^p$ since it is possible that $y^1, \dots, y^p$ are differentially dependent and this happens when E does not depend on q^n. More precisely $y^1, \dots, y^p$ are differentially dependent around q when there exists a neighbourhood V of q such that $\frac{\partial E}{\partial q^n}$ is identically zero on $(\pi_2^{-1}(V) \cap \{E = 0\}) \subset J^2(\mathbb{R}, Q)$ where $\pi_2 : J^2(\mathbb{R}, Q) \to Q$ is the standard projection. The functions E and $\frac{\partial E}{\partial q^n}$ are both affine in $\ddot{y}$ and quadratic in $\dot{y}$ with the coefficients functions only of (y, q^n) and E depends on $\ddot{y}$ non trivially since metric g is non degenerate. Hence $\frac{\partial E}{\partial q^n}$ is either identically zero on $\pi_2^{-1}(q) \cap \{E = 0\}$ or it is non zero for generic points on $\pi_2^{-1}(q) \cap \{E = 0\}$. Further more $\frac{\partial E}{\partial q^n}$ is identically zero on $\pi_2^{-1}(q) \cap \{E = 0\}$ if and only if it is a multiple of E as a polynomial in $\dot{y}$ and $\ddot{y}$ for points on $\pi_2^{-1}(q)$. Hence the regularity condition we impose is that $\frac{\partial E}{\partial q^n}$ is a not a multiple of E as a polynomial in $\ddot{y}$ and $\dot{y}$ for points on $\pi_2^{-1}(q)$. Then it would follow from continuity and implicit function theorem that for generic points on $\pi_2^{-1}(V) \cap \{E = 0\}$ where V is some neighbourhood of q, q^n can be locally solved for in terms of $y, \dot{y}, \ddot{y}$, implying flatness around q.

Rest of the proof is concerned with showing that this condition translates to the regularity condition stated in the proposition. It is sufficient to show that $\frac{\partial E}{\partial q^n}$ is a multiple of E as polynomials in $\dot{y}, \ddot{y}$ with the ratio being a smooth function on Q is equivalent to the set of ratios of functions (15) all being identically equal in a neighbourhood of q.

Let η span $\ker Ty$. Then $\eta = \lambda \frac{\partial}{\partial q^n}$ for some nonvanishing function λ. Also let $\xi = a^i \frac{\partial}{\partial q^i}$ span $\operatorname{ann} P$. Suppose $\frac{\partial E}{\partial q^n} = fE$ for some function f defined in a neighbourhood of q on Q. Considering coefficients of $\dddot{y}^j$ terms we get

$$\frac{\partial}{\partial q^n}(a^i g_{ij}) = f a^i g_{ij} \quad j = 1, \ldots, p. \tag{21}$$

Also observe that any vectorfield Z on Q is y-related if and only if it has the form $Z^j(y)\frac{\partial}{\partial y^j} + Z^n(y, q^n)\frac{\partial}{\partial q^n}$. Hence

$$\nabla_\eta(g(\xi, Z)) = \lambda \frac{\partial}{\partial q^n}(Z^j a^i g_{ij})$$
$$= \lambda Z^j \frac{\partial}{\partial q^n}(a^i g_{ij}) = \lambda f Z^j a^i g_{ij},$$

where we have used $a^i g_{in} = 0$ and equation (21). Hence equation (21) is equivalent to

$$\nabla_\eta(g(\xi, Z)) = \lambda f g(\xi, Z), \tag{22}$$

where Z is any arbitrary y-related vectorfield.

Considering coefficients of $\dot{y}^j \dot{y}^k$ we get

$$\frac{\partial}{\partial q^n}(a^i g_{im} \Gamma^m_{jk}) = f a^i g_{im} \Gamma^m_{jk}, \quad j, k = 1, \ldots, p. \tag{23}$$

Assuming equation (21), this is equivalent to

$$\nabla_\eta(g(\nabla_{Z_1} Z_2, \xi)) = f\lambda g(\nabla_{Z_1} Z_2, \xi), \tag{24}$$

where Z_1, Z_2 are arbitrary y-related vectorfields. This is because substituting $Z_l = Z_l^j(y)\frac{\partial}{\partial y^j} + Z_l^n(y, q^n)\frac{\partial}{\partial q^n}$ for $l = 1, 2$ we get

$$g(\nabla_{Z_1} Z_2, \xi) = Z_1^j Z_2^k g(\Gamma^m_{jk} \frac{\partial}{\partial y^m}, \xi) + Z_1^j \frac{\partial Z_2^k}{\partial y^j} g(\frac{\partial}{\partial y^k}, \xi),$$

where we have used $a^i g_{in} = 0$, $a^i \Gamma^m kn g_{im} = 0$ (since $\ker Ty$ is orthogonal to D) and $\frac{\partial Z_2^k}{\partial q^n} = 0$ for $k = 1, \dots, p$. Hence

$$\begin{aligned}
&\nabla_\eta (g(\nabla_{Z_1} Z_2, \xi)) \\
&= \lambda Z_1^j Z_2^k \frac{\partial}{\partial q^n}(a^i g_{im} \Gamma^m_{jk}) + \lambda Z_1^j \frac{\partial Z_2^k}{\partial y^j} \frac{\partial}{\partial q^n}(a^i g_{ik}) \\
&= \lambda f Z_1^j Z_2^k a^i g_{im} \Gamma^m_{jk} + \lambda f Z_1^j \frac{\partial Z_2^k}{\partial y^j} a^i g_{ik}
\end{aligned}$$

where we have used equations (21) and (23). This simplifies to

$$\nabla_\eta (g(\nabla_{Z_1} Z_2, \xi)) = \lambda f g(\nabla_{Z_1} Z_2, \xi). \tag{25}$$

Finally considering the coefficients of the terms independent of $\dot{y}$ and $\ddot{y}$ we get

$$\frac{\partial}{\partial q^n}(a^i \frac{\partial V}{\partial q^i}) = f a^i \frac{\partial V}{\partial q^i}.$$

Clearly this is equivalent to

$$\nabla_\eta (\xi(V)) = \lambda f \xi(V), \tag{26}$$

completing the proof. ∎

4 Systems with n degrees of freedom, n − 1 controls and symmetry

In this section we shall consider systems of the type considered in last section that also exhibit symmetries. We shall suppose that a Lie group G acts on our configuration space Q with action Φ_h corresponding to $h \in G$ and that

$$\Phi_h^* g = g, \qquad \Phi_h^* P = P \quad \forall h \in G. \tag{27}$$

In other words the kinetic energy of the system as well as the range of control forces both are invariant under the group action. However we do not assume that V is invariant under the group action. Many mechanical systems fall under this category. Rigid body systems moving in Euclidean space actuated by body fixed forces are typical examples where the group is $G = SE(3)$, even though the equations of motion often do not have $SE(3)$ as a symmetry group since potential forces due to gravity break the symmetry. But since V plays a very limited role in configuration flatness we may expect that when the system is configuration flat that it would be possible to find flat outputs that reflect this symmetry. We believe this to be true and shall prove it for the case $\dim D = n - 1$. The general case $\dim D < n - 1$ has not yet been resolved completely (see Remark 8).

Lemma 5 Consider a system satisfying (27). Let D be defined as in (12). Then $\Phi_{h_*} D = D$.

Proof Let ξ span $\operatorname{ann} P$. Clearly $\Phi_{h_*}(\operatorname{ann} P) = \operatorname{ann} P$. Hence $\Phi_{h_*}\xi = \lambda_h \xi \in D$ where λ_h is some smooth function. Since Φ_h is an isometry by (27), it follows that $\Phi_{h_*}(\nabla_Z \xi) = \nabla_{\Phi_{h_*} Z}(\Phi_{h_*}\xi)$ by properties of ∇ (see, for example, [5] page 161). Hence

$$\begin{aligned} \Phi_{h_*} \nabla_Z \xi &= \nabla_{\Phi_{h_*} Z}(\lambda_h \xi) \\ &= \lambda_h \nabla_{\Phi_{h_*} Z} \xi + (\nabla_{\Phi_{h_*} Z} \lambda_h) \xi \in D. \end{aligned} \tag{28}$$

So we have $\Phi_{h_*} D \subset D$. Since Φ_h is a diffeomorphism, the result follows by dimension count. ∎

Let $y : Q \to \mathbb{R}^p$ be a map defined locally around $q \in Q$. We shall say $y^1, \ldots, y^p$ are *G-equivariant* if

$$\Phi_{h_*} \ker Ty = \ker Ty.$$

This means level sets of y are mapped to level sets by the group action.

Proposition 6 Consider a system satisfying (27). Suppose $\dim D = n - 1$ and that the system is configuration flat. Then the flat outputs are G-equivariant.

Proof Follows from the fact that $\ker Ty$ is the orthogonal complement to D and Lemma 5. ∎

Remark 7 The case $\dim D = n - 1$ is not as restrictive as it may seem. Typically $\dim D = n$, implying that the system is not configuration flat. When the system is configuration flat ($\dim D \leq n - 1$), most likely $\dim D = n - 1$. In fact many examples of systems that are configuration flat fall into this category including the first example in next section as well as the "ducted fan with stand" in [19] and the "planar coupled rigid bodies" example in [13].

Remark 8 In the case when $\dim D < n - 1$, given the system is flat with flat outputs $y : Q \to \mathbb{R}^p$ around $q \in Q$, it is possible to construct outputs $\tilde{y} : Q \to \mathbb{R}^p$ around q that are G-equivariant and satisfy $g(\ker T\tilde{y}, D) = 0$. But it hasn't been resolved whether it is possible to construct $\tilde{y}$ in such a way that it also satisfies the regularity conditions (15). The authors are currently trying to resolve this technical issue but suspect that at least in typical cases this construction should work. The second example in next section falls into the case $\dim D = n - 2$ and we see that it possesses G-equivariant flat outputs.

5 Examples

In this section we shall consider some examples to illustrate the theory developed in the previous section.

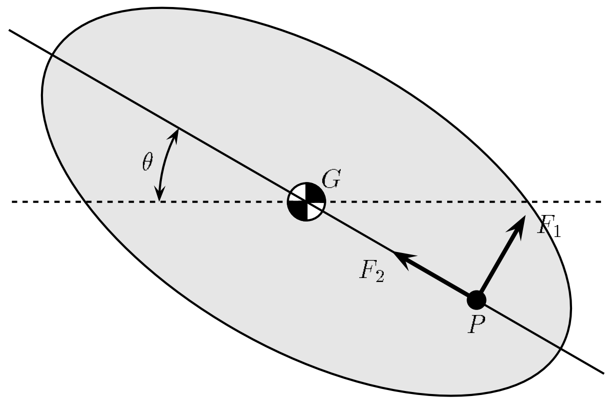

Figure 2: Underwater vehicle in $\mathbb{R}^2$

5.1 Underwater Vehicle

We shall study a simple model of an underwater vehicle that is controlled by a force applied through a fixed point P on the body whose magnitude and direction can be independently controlled.

Only the motion in the vertical plane is considered and hence our configuration space is $SE(2) = \mathbb{R}^2 \times S^1$. This is reasonable when the vehicle has symmetries about 3 orthogonal planes. In addition if we assume that the centre of buoyancy is coincident with centre of mass, the kinetic energy is given by

$$\frac{1}{2}(m+\delta m)(\dot{x}_1\cos\theta - \dot{x}_2\sin\theta)^2 + \frac{1}{2}(m-\delta m)(\dot{x}_1\sin\theta + \dot{x}_2\cos\theta)^2 + \frac{1}{2}I(\dot{\theta})^2, \tag{29}$$

where (x_1, x_2) are horizontal and vertical coordinates of the centre of mass G, θ is the orientation (measured clockwise) of line PG with respect to horizontal axis, $m = M + (m_1 + m_2)/2$ and $\delta m = (m_1 - m_2)/2$ where M is the mass of the vehicle and m_1 and m_2 are added mass terms that take into account inertia of the fluid, and I is the effective moment of inertia taking into account the fluid. This model assumes an incompressible, irrotational flow and neglects viscosity effects. It is assumed that the motion of the fluid is entirely due to that of the solid. The body and the fluid together are considered to form a dynamical system and the kinetic energy is the combined energy of body and fluid. See [7] and [6] for details. The analysis in [7] assumes a neutrally buoyant model, but we need not make this assumption since this only alters the form of the potential function but does not affect the kinetic energy. In fact for the first part of the analysis we shall not assume any specific form for potential V. If the vehicle is in air (strictly speaking vacuum) $m_1 = m_2 = 0$, so $m = M$ and $\delta m = 0$ and the kinetic energy takes the familiar form

$$\frac{1}{2}(m(\dot{x}_1)^2 + m(\dot{x}_2)^2 + I(\dot{\theta})^2)$$

where I is the usual moment of inertia and the model is the same as that of VTOL (see [10]).

The metric g in coordinates x_1, x_2, θ is given by the matrix

$$\begin{bmatrix} m+\delta m\cos 2\theta & -\delta m\sin 2\theta & 0 \\ -\delta m\sin 2\theta & m-\delta m\cos 2\theta & 0 \\ 0 & 0 & I \end{bmatrix}.$$

The control forces lie in the codistribution

$$\begin{aligned} P &= \operatorname{span}\{d(x_1+R\cos\theta), d(x_2-R\sin\theta)\} \\ &= \operatorname{span}\{dx_1 - R\sin\theta d\theta, dx_2 - R\cos\theta\ d\theta\} \end{aligned}$$

and $\xi = \frac{\partial}{\partial\theta} + R\sin\theta\frac{\partial}{\partial x_1} + R\cos\theta\frac{\partial}{\partial x_2}$ spans $\operatorname{ann} P$ where R is the length of PG.

The Christoffel symbols Γ^i_{jk} can be computed from g using equation (10). Then using formula (11) we see that

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_1}}\xi &= -\frac{m\delta m}{m^2-(\delta m)^2}\sin 2\theta\frac{\partial}{\partial x_1} - \frac{\delta m}{m^2-(\delta m)^2}(\delta m + m\cos 2\theta)\frac{\partial}{\partial x_2} \\ &\quad + \frac{R\delta m\cos\theta}{I}\frac{\partial}{\partial\theta} \\ \nabla_{\frac{\partial}{\partial x_2}}\xi &= -\frac{\delta m}{m^2-(\delta m)^2}(-\delta m + m\cos 2\theta)\frac{\partial}{\partial x_1} + \frac{m\delta m}{m^2-(\delta m)^2}\sin 2\theta\frac{\partial}{\partial x_2} \\ &\quad - \frac{R\delta m\sin\theta}{I}\frac{\partial}{\partial\theta} \\ \nabla_{\frac{\partial}{\partial\theta}}\xi &= \frac{mR\cos\theta}{m+\delta m}\frac{\partial}{\partial x_1} - \frac{mR\sin\theta}{m+\delta m}\frac{\partial}{\partial x_2} \end{aligned} \tag{30}$$

It can be seen by computation that the above vector fields together with ξ span the full tangent space for generic points and generic parameter values $m, \delta m, I, R$. Since by equation (13)

$$D = \operatorname{span}\{\nabla_{\frac{\partial}{\partial x_1}}\xi, \nabla_{\frac{\partial}{\partial x_2}}\xi, \nabla_{\frac{\partial}{\partial\theta}}\xi, \xi\},$$

it follows that $D = TQ$ for generic points on Q and for generic parameter values and hence the system is not configuration flat for generic parameter values regardless of the potential energy function.

However for the case $\delta m = 0$, we see that

$$D = \operatorname{span}\{R\cos\theta\frac{\partial}{\partial x_1} - R\sin\theta\frac{\partial}{\partial x_2}, R\sin\theta\frac{\partial}{\partial x_1} + R\cos\theta\frac{\partial}{\partial x_2} + \frac{\partial}{\partial\theta}\}.$$

Hence $\dim D = 2$ and $\eta = \frac{\partial}{\partial\theta} - \frac{I}{mR}\sin\theta\frac{\partial}{\partial x_1} - \frac{I}{mR}\cos\theta\frac{\partial}{\partial x_2}$ spans the orthogonal complement to D. Since D has codimension 1, up to a diffeomorphism there is at most 1 set of flat outputs. One set of functions that "cut out" the foliation due to η is

$$y_1 = x_1 - \frac{I}{mR}\cos\theta, \quad y_2 = x_2 + \frac{I}{mR}\sin\theta.$$

To ensure that y_1, y_2 are indeed flat outputs we must check the regularity conditions (16). Let us choose $z = \theta$ as a complementary coordinate to y_1, y_2. Then,

$$\begin{aligned} \frac{\partial}{\partial y_1} &= \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial y_2} = \frac{\partial}{\partial x_2}, \\ \frac{\partial}{\partial z} &= -\frac{I}{mR}\sin\theta\frac{\partial}{\partial x_1} - \frac{I}{mR}\sin\theta\frac{\partial}{\partial x_1} + \frac{\partial}{\partial \theta}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial z}\left(g(\xi, \frac{\partial}{\partial y_1})\right) : g(\xi, \frac{\partial}{\partial y_1}) &= -\sin z : \cos z \\ \frac{\partial}{\partial z}\left(g(\xi, \frac{\partial}{\partial y_2})\right) : g(\xi, \frac{\partial}{\partial y_2}) &= \cos z : \sin z. \end{aligned} \tag{31}$$

So at any point $q = (y_1, y_2, z)$ these two ratios are unequal. This ensures that y_1, y_2 are indeed flat outputs everywhere.

When the vehicle is in air (strictly speaking vacuum) $\delta m = 0$, and in this case it is already known to be flat (see [10, 11]). We have just shown that up to a diffeomorphism these are the only configuration flat outputs. Also we have covered the case of underwater vehicle of spherical shape (since then $m_1 = m_2$) and this result is independent of any assumptions we make on the potential function V.

Now let us suppose the system is moving under gravity in air and the potential energy is given by $V = mgx_2$ where $g \approx 9.8\ \mathrm{m/s^2}$ is the acceleration due to gravity. Then the solutions of the system in coordinates y_1, y_2, z satisfy the ODE

$$\ddot{y}_1 \sin z + \ddot{y}_2 \cos z + g\cos z = 0.$$

So along generic solution curves we get,

$$z(t) = \tan^{-1}\frac{\ddot{y}_2 + g}{\ddot{y}_1}$$

or

$$z(t) = \tan^{-1}\frac{\ddot{y}_2 + g}{\ddot{y}_1} + \pi.$$

The exception being the singularity at $\ddot{y}_1 = 0, \ddot{y}_2 + g = 0$. Note that this singularity is not a point on Q but corresponds to a submanifold in the jet space $J^2(\mathbb{R}, Q)$, the space with coordinates $(t, q, \dot{q}, \ddot{q})$ and such singularities are very common in practical examples. We still want to regard such systems as flat and this is the reason why our definition of flatness refers to generic curves as opposed to all curves. Also note that though potential V does not affect the flat outputs of the system it influences where the singularities occur.

We also see that the general system (no assumptions on δm) possesses an $SE(2)$ symmetry when the potential function is ignored. If we consider translating and rotating our spatial frame of reference the expression for kinetic energy as well as the the expression for P are invariant. We may state this more precisely as follows. Consider the following action of $SE(2)$ on $Q = SE(2)$. Given $h = (\alpha_1, \alpha_2, \phi) \in SE(2)$ the action Φ_h corresponds to first rotating the spatial frame counter clockwise by ϕ about its origin and then with respect to this frame translate the frame without rotation by $(-\alpha_1, -\alpha_2)$. Hence if $q = (x_1, x_2, \theta) \in Q$ then

$$\Phi_h(q) = (x_1 \cos\phi + x_2 \sin\phi + \alpha_1, -x_1 \sin\phi + x_2 \sin\phi + \alpha_2, \theta + \phi).$$

The corresponding tangent map $T\Phi_h$ is given by

$$\begin{aligned} \frac{\partial}{\partial x_1} &\to \cos\phi \frac{\partial}{\partial x_1} + \sin\phi \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} &\to -\sin\phi \frac{\partial}{\partial x_1} + \cos\phi \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial \theta} &\to \frac{\partial}{\partial \theta}. \end{aligned} \tag{32}$$

It is easy to verify this preserves g. Recalling that $\xi = \frac{\partial}{\partial\theta} + R\sin\theta \frac{\partial}{\partial x_1} + R\cos\theta \frac{\partial}{\partial x_2}$ spans $\operatorname{ann} P$, we see that $\Phi_{h_*}\xi = \xi$, implying $\Phi_h^* P = P$. In particular these statements are true for the $\delta m = 0$ case as well. Hence by Proposition 6 the flat outputs are G-equivariant. This is indeed true since $\eta = \frac{\partial}{\partial\theta} - \frac{I}{mR}\sin\theta \frac{\partial}{\partial x_1} - \frac{I}{mR}\cos\theta \frac{\partial}{\partial x_2}$ spans $\ker Ty$ and $\Phi_{h_*}\eta = \eta$.

5.2 Particle in force field

This example does not necessarily correspond to an engineering example, but illustrates the regularity conditions. We consider a particle of unit mass moving in 3 dimensional Euclidean space in the presence of a potential field $V = x_2 x_3$. Hence the kinetic energy metric is given by the 3×3 identity matrix in orthogonal coordinates x_1, x_2, x_3. Suppose we control independently the forces along x_1 and x_3 directions. Hence $P = \operatorname{span}\{dx_1, dx_3\}$ and $\xi = \frac{\partial}{\partial x_2}$ spans $\operatorname{ann} P$. We see that Christoffel symbols are all zero by (10) (which is a feature of Euclidean space) and using (11) and (13) we obtain $D = \operatorname{span}\{\frac{\partial}{\partial x_2}\}$ and hence the orthogonal complement to D is $\operatorname{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}\}$ which is two dimensional. Hence we have infinitely many "candidates" for flat outputs that are not equivalent via a diffeomorphism. But these "candidates" may not satisfy the regularity conditions (16). Following the method outlined in Remark refremalg we pick some η, say $\eta = \frac{\partial}{\partial x_3}$ which is orthogonal to D. Then $y_1 = x_1, y_2 = x_2$ are a possible choice of corresponding "candidates" for flat outputs (since they cut out the one dimensional foliation by η). We may choose $z = x_3$ to complete the coordinate system and then we see that the ratio of functions $\frac{\partial}{\partial z}(\xi(V)) : \xi(V)$ in the set (16) is $1 : x_3$ where as the ratio of $\frac{\partial}{\partial z}(g(\xi, \frac{\partial}{\partial y^2}))$ is $0 : 1$. Hence x_1, x_2

are configuration flat outputs (globally). But alternatively another choice could have been $\eta = \frac{\partial}{\partial x_1}$ with corresponding candidates $y_1 = x_2, y_2 = x_3$. Choosing $z = x_1$ we see that all the ratios in (16) are zero and hence equal. Hence x_2, x_3 are not flat outputs as they are differentially dependent. This example is simple enough that the above conclusions can be reached by inspecting the equations of motion for the system

$$\ddot{x}_1 - \frac{\partial V}{\partial x_1} = F_1 \tag{33}$$

$$\ddot{x}_2 - \frac{\partial V}{\partial x_2} = 0 \tag{34}$$

$$\ddot{x}_3 - \frac{\partial V}{\partial x_3} = F_3 \tag{35}$$

where F_1, F_3 are the forces along x_1, x_3 directions. The equation (34) alone characterises all solution trajectories of system and substituting $V = x_2 x_3$ we obtain,

$$\ddot{x}_2 - x_3 = 0. \tag{36}$$

It is clear from the equation that x_2, x_3 are differentially dependent and hence are not flat outputs. However it is also clear from the equations that x_1, x_2 are flat outputs since along solution curves,

$$x_3(t) = \frac{d^2 x_2(t)}{dt^2}$$

and x_1, x_2 do not satisfy an ODE.

Also note that the system is globally controllable since it is globally flat. However if $V = 0$ then the system is not configuration flat and not even locally accessible.

It is easy to see that translations by the group $\mathbb{R}^3$ leave g and P invariant. But Proposition 6 does not apply since $\dim D = n-2$. However as mentioned in Remark 8 we see that G-equivariant flat outputs exist. In fact $y = (x_1, x_2)$ are G-equivariant, although not all (configuration) flat outputs are G-equivariant, since $\tilde{y} = (f(x_1, x_3), x_2)$ where f is an arbitrary smooth function with $\frac{\partial f}{\partial x_1} \neq 0$, are not G-equivariant for a typical f, but are configuration flat outputs.

6 Conclusions and future work

We have presented a method for determining configuration flatness of Lagrangian control systems with n degrees of freedom and $n-1$ controls. Our method is constructive and provides a way for finding configuration flat outputs if they exist. We assumed a Lagrangian of the form "kinetic energy minus potential". We also assumed that the range of control forces only depends on configuration. These assumptions are not unreasonable since a wide class of systems fall into this category. However $n-1$ controls is a special case and is the simplest case next to fully actuated (n controls) systems which are allways flat. In that sense we regard this as a first step towards a general theory of configuration flatness of Lagrangian systems. The authors are currently working on generalising this result to arbitrary number of controls.

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