Submitted, 2015 Conference on Decision and Control (CDC)
http://www.cds.caltech.edu/~murray/papers/pm15-cdc.html

Control System Analysis on Symmetric Cones

Ivan Papusha Richard M. Murray

***Abstract*— Motivated by the desire to analyze high dimensional control systems without explicitly forming computationally expensive linear matrix inequality (LMI) constraints, we seek to exploit special structure in the dynamics matrix. By using Jordan algebraic techniques we show how to analyze continuous time linear dynamical systems whose dynamics are exponentially invariant with respect to a symmetric cone. This allows us to characterize the families of Lyapunov functions that suffice to verify the stability of such systems. We highlight, from a computational viewpoint, a class of systems for which stability verification can be cast as a second order cone program (SOCP), and show how the same framework reduces to linear programming (LP) when the system is internally positive, and to semidefinite programming (SDP) when the system has no special structure.**

I. Introduction

In this paper we wish to verify the stability of a linear dynamical system

$$\dot{x}(t) = Ax(t), \quad A \in \mathbf{R}^{n\times n}, \quad x(t) \in \mathbf{R}^n. \tag{1}$$

It is well known that if A has no special structure, then system (1) is stable if and only if there exists a symmetric matrix $P = P^T \in \mathbf{R}^{n\times n}$ satisfying the linear matrix inequalities

$$P = P^T \succ 0, \quad A^T P + PA \prec 0. \tag{2}$$

Such a matrix P corresponds to a quadratic Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}$,

$$V(x) = \langle x, Px\rangle,$$

which is positive definite and decreasing along trajectories of (1).

Searching for a matrix P satisfying (2) is in general a semidefinite program (SDP), but if A has a special structure, this semidefinite program can sometimes be cast as a linear program (LP)—with advantages in numerical stability, opportunities for parallelism, and better scaling to high dimensions than the general SDP. For example, if A is a Metzler matrix, *i.e.*, its off-diagonal entries are nonnegative,

$$A_{ij} \geq 0, \quad \text{for all } i \neq j,$$

then it suffices to search for a *diagonal* matrix P satisfying (2). Equivalently, it is enough to find an entrywise positive vector $p \in \mathbf{int}\,\mathbf{R}^n_+$ such that $-Ap \in \mathbf{int}\,\mathbf{R}^n_+$ is entrywise positive. Finding such a vector p is, of course, an LP, [1], [2], [3].

The authors are with the Control and Dynamical Systems Department, California Institute of Technology, Pasadena, CA USA.

Autonomous systems for which A is a Metzler matrix are called *internally positive*, because their state space trajectories are guaranteed to remain in the nonnegative orthant $\mathbf{R}^n_+$ if they start in the nonnegative orthant. As we will see, the Metzler structure is (in a certain sense) the only natural matrix structure for which a linear program may generically be composed to verify stability. However, the inclusion

$$\text{LP} \subseteq \text{SOCP} \subseteq \text{SDP}$$

motivates us to determine if stability analysis can be cast, for example, as a second order cone program (SOCP) for some specific subclass of linear dynamics, in the same way that it can be cast as an LP for internally positive systems, or an SDP for general linear systems.

Using Jordan algebraic techniques, which have proved to be effective in unifying interior point methods for conic programming, we will discuss a generalization of the Metzler property known as *cross-positivity* to characterize dynamics that are exponentially invariant with respect to a symmetric cone. In so doing, we demonstrate the strong unifying power of the Jordan product on linear systems and discuss a rich class of systems that admit SOCP-based analysis.

II. Jordan Algebras and Symmetric Cones

The following background is informal and mostly meant to set out notation, adopting conventions familiar to system theory and optimization [4], [5]. Jordan algebras in the context of symmetric cones are rich and well studied field, with an excellent reference [6]. For more recent reviews, see [7], [8], [9].

A. Cones on a vector space.

Let V be a vector space with inner product $\langle\cdot,\cdot\rangle$. A subset $K \subseteq V$ is called a *cone* if it is closed under nonnegative scalar multiplication: for every $x \in K$ and $\theta \geq 0$ we have $\theta x \in K$. A cone K is *pointed* if it contains no line, or equivalently

$$x \in K, -x \in K \quad \Longrightarrow \quad x = 0.$$

A cone is *proper* if it is closed, convex, pointed, and has nonempty interior. Every proper cone induces a partial order $\preceq$ on V given by

$$x, y \in V, \quad x \preceq y \quad \Longleftrightarrow \quad y - x \in K.$$

We have $x \prec y$ if and only if $y - x \in \mathbf{int}\,K$. Similarly we write $x \succeq y$ and $x \succ y$ to mean $y \preceq x$ and $y \prec x$, respectively.

B. Jordan algebras.

A (Euclidean) Jordan algebra is an inner product space $(V, \langle\cdot,\cdot\rangle)$ endowed with a Jordan product $\circ : V \times V \to V$, which satisfies the following properties:

1) *Bilinear*: $x \circ y$ is linear in x for fixed y and vice-versa
2) *Commutative*: $x \circ y = y \circ x$
3) *Jordan identity*: $x^2 \circ (y \circ x) = (x^2 \circ y) \circ x$
4) *Adjoint identity*: $\langle x, y \circ z\rangle = \langle y \circ x, z\rangle$.

In particular, a Jordan product need not be associative. When we interpret $x^2 = x \circ x$, the Jordan identity allows any power x^k, $k \geq 2$ to be inductively defined. An identity element $\mathbf{e}$ satisfies $\mathbf{e} \circ x = x \circ \mathbf{e} = x$ for all $x \in V$, and defines a number $r = \langle \mathbf{e}, \mathbf{e}\rangle$ called the *rank* of V.

The cone of squares K corresponding to the Jordan product $\circ$ is defined as

$$K = \{x \circ x \mid x \in V\}.$$

Every element $x \in V$ has a spectral decomposition

$$x = \sum_{i=1}^{r} \lambda_i e_i,$$

where $\lambda_i \in \mathbf{R}$ are eigenvalues of x and the set of eigenvectors $\{e_1, \ldots, e_r\} \subseteq V$, called a *Jordan frame*, satisfies

$$e_i^2 = e_i, \quad e_i \circ e_j = 0 \text{ for } i \neq j, \quad \sum_{i=1}^{r} e_i = \mathbf{e}.$$

We have $x \in K$ (written $x \succeq_K 0$, or simply $x \succeq 0$ when the context is clear) provided $\lambda_i \geq 0$. Similarly, $x \in \mathbf{int}\, K$ if and only if $\lambda_i > 0$ (written $x \succ_K 0$). The spectral decomposition allows us to define familiar concepts like trace, determinant, and square root of an element x by taking the corresponding function of the eigenvalue. That is,

$$\mathbf{tr}\, x = \sum_{i=1}^{r} \lambda_i, \quad \mathbf{det}\, x = \prod_{i=1}^{r} \lambda_i, \quad x^{1/2} = \sum_{i=1}^{r} \lambda_i^{1/2} e_i,$$

where the last quantity only makes sense if $x \succeq 0$.

Finally, every element $z \in V$ has a quadratic representation, which is a map $P_z : V \to V$ given by

$$P_z(x) = 2(z \circ (z \circ x)) - z^2 \circ x.$$

C. Symmetric cones.

The *closed dual cone* of K is defined as

$$K^* = \{y \in V \mid \langle x, y\rangle \geq 0, \text{ for all } x \in K\},$$

A cone K is *self-dual* if $K^* = K$. The automorphism group $\mathbf{Aut}(K)$ of an open convex cone K is the set of invertible linear transformations $g : V \to V$ that map K to itself,

$$\mathbf{Aut}(K) = \{g \in GL(V) \mid gK = K\}.$$

An open cone K is *homogeneous* if $\mathbf{Aut}(K)$ acts on the cone transitively, in other words, if for every $x, y \in K$ there exists $g \in \mathbf{Aut}(K)$ such that $gx = y$. Finally, K is *symmetric* if it is homogeneous and self-dual. Symmetric cones are an important object of study because they are the cones of squares of a Jordan product, admit a spectral decomposition, and are (in finite dimensions, [6]) isomorphic to a Cartesian product of

- $n \times n$ self-adjoint positive semidefinite matrices with real, complex, or quaternion entries,
- 3×3 self-adjoint positive semidefinite matrices with octonion entries (Albert algebra), and
- Lorentz cone.

In practice, symmetric cones have a differentiable log-det barrier function, allowing numerical optimization via interior point methods [7], [10]. The symmetric cones $\mathbf{R}^n_+ (= \mathbf{S}^1_+ \times \cdots \times \mathbf{S}^1_+)$, $\mathcal{L}^n_+$ and $\mathbf{S}^n_+$ give rise to LP, SOCP, and SDP, respectively. We now define these three cones.

D. Three important symmetric cones.

1) Nonnegative orthant $\mathbf{R}^n_+$: the cone of squares associated with the standard Euclidean space $\mathbf{R}^n$ and Jordan product

$$(x \circ y)_i = x_i y_i, \quad i = 1, \ldots, n,$$

i.e., the entrywise (or Hadamard) product. The identity element is $\mathbf{e} = \mathbf{1} = (1, \ldots, 1)$, and the Jordan frame comprises the standard basis vectors. The quadratic representation of an element $z \in \mathbf{R}^n$ is given by the diagonal matrix $P_z = \mathrm{diag}(z)^2$

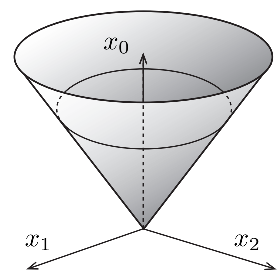

Fig. 1. Second-order (Lorentz) cone $\mathcal{L}^3_+$.

2) Lorentz cone $\mathcal{L}^n_+$: partition every element of $\mathbf{R}^n$ as $x = (x_0, x_1) \in \mathbf{R} \times \mathbf{R}^{n-1}$ and define the Lorentz cone (also known as *second-order* or *norm* cone) as

$$\mathcal{L}^n_+ = \{(x_0, x_1) \in \mathbf{R} \times \mathbf{R}^{n-1} \mid \|x_1\|_2 \leq x_0\} \subseteq \mathbf{R}^n.$$

This cone is illustrated in Figure 1 for the case $n = 3$. It can be shown that $\mathcal{L}^n_+$ is the cone of squares corresponding to the Jordan product

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \circ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} \langle x, y\rangle \\ x_0 y_1 + y_0 x_1 \end{bmatrix}.$$

The rank of this algebra is 2, giving a particularly simple spectral decomposition

$$x = \lambda_1 e_1 + \lambda_2 e_2,$$

where the eigenvalues and Jordan frame are

$$\lambda_1 = x_0 + \|x_1\|_2, \qquad \lambda_2 = x_0 - \|x_1\|_2,$$
$$e_1 = \frac{1}{2}\begin{bmatrix} 1 \\ x_1/\|x_1\|_2 \end{bmatrix}, \qquad e_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -x_1/\|x_1\|_2 \end{bmatrix}.$$

The identity element is $\mathbf{e} = (1, 0)$ and the quadratic representation of an element $z \in \mathbf{R}^n$ is the matrix

$$P_z = zz^T - \frac{z^T J_n z}{2} J_n,$$

where $J_n = \text{diag}(1, -1, \ldots, -1) \in \mathbf{R}^{n\times n}$ is a $n \times n$ inertia matrix with signature $(1, n-1)$. Note that P_z is a positive semidefinite matrix if and only if $z \succeq 0$.

3) Positive semidefinite cone $\mathbf{S}^n_+$*:* consider the vector space $\mathbf{S}^n$ of real, symmetric $n \times n$ matrices with the trace inner product. If we define the Jordan product as the symmetrized matrix product,

$$X \circ Y = \frac{1}{2}(XY + YX),$$

then the cone of squares is $\mathbf{S}^n_+$, the positive semidefinite matrices with real entries. The spectral decomposition of an element $X \in \mathbf{S}^n$ follows from the eigenvalue decomposition

$$X = \sum_{i=1}^{n} \lambda_i v_i v_i^T, \quad \Longrightarrow \quad e_i = v_i v_i^T, \quad i = 1, \ldots, n.$$

The quadratic representation of an element $Z \in \mathbf{S}^n$ is given by the map $P_Z : \mathbf{S}^n \to \mathbf{S}^n$,

$$P_Z(X) = 2(Z \circ (Z \circ X)) - (Z \circ Z) \circ X = ZXZ.$$

III. Computational approach to Dynamics on a Symmetric Cone

Instead of discussing the original dynamics (1) on $\mathbf{R}^n$, we consider instead general linear dynamics on a Jordan algebra V. This will allow us to present a unified treatment of the cones $\mathbf{R}^n_+$, $\mathcal{L}^n_+$, as well as restate familiar results from classical state space theory by specializing to $\mathbf{S}^n_+$. We list most notable results by Gowda et al. [8], [9], [11].

A. Cross-positive linear operators.

A linear operator $L : V \to V$ is *cross-positive* with respect to a proper cone K if

$$x \in K, y \in K^*, \text{ and } \langle x, y\rangle = 0 \quad \Longrightarrow \quad \langle L(x), y\rangle \geq 0.$$

Exponentials of cross-positive operators leave the cone invariant: if $L : V \to V$ is cross-positive with respect to K, then $e^{tL}(K) \subseteq K$ for all $t \geq 0$. In fact, the converse is also true [12, Theorem 3].

For example, a matrix $A \in \mathbf{R}^{n\times n}$, thought of as a linear transformation $A : \mathbf{R}^n \to \mathbf{R}^n$, is cross-positive with respect to the cone $\mathbf{R}^n_+$ if and only if A is a Metzler matrix, *i.e.*,

$$A_{ij} \geq 0, \quad \text{ for all } i \neq j,$$

or equivalently that trajectories $x(t)$ of the system

$$\dot{x}(t) = Ax(t)$$

remain in $\mathbf{R}^n_+$ whenever they enter $\mathbf{R}^n_+$. Therefore, cross-positivity can be thought of as a generalization of the Metzler property.

Examples of operators L that satisfy $e^L(K) \subseteq K$ are

- *Nonnegative orthant:* $L(x) = Ax$, where $A \in \mathbf{R}^{n\times n}$ is Metzler.
- *Lorentz cone:* $L(x) = Ax$, where $A^T J_n + J_n A \succeq \xi J_n$ for some $\xi \in \mathbf{R}$.
- *Positive semidefinite matrices:* $L(X) = AX + XA^T$ for any matrix A.

In fact, due to a result in [13], the examples above precisely characterize cross-positive operators on $\mathbf{R}^n_+$ and $\mathcal{L}^n_+$ (but not $\mathbf{S}^n_+$).

B. A class of Lyapunov functions

Let $L : V \to V$ be a linear operator, and consider the linear dynamical system

$$\dot{x}(t) = L(x), \quad x(0) = x_0, \tag{3}$$

where $x_0 \in V$ is an initial condition. We make the assumption that $e^L(K) \subseteq K$, or equivalently, that L is cross-positive. Systems that obey this this assumption are, of course, very special, because any trajectory that starts in the cone of squares K will remain within K for all time $t \geq 0$,

$$x_0 \in K \quad \Longrightarrow \quad x(t) \in K \text{ for all } t \geq 0,$$

in other words, such systems are *exponentially invariant* with respect to the cone K. We consider a generalization of the class of quadratic Lyapunov functions on V given by

$$V_z : V \to \mathbf{R}, \quad V_z(x) = \langle x, P_z(x)\rangle,$$

where $z \in V$ is a parameter, and P_z is the quadratic representation of z in the Jordan algebra V. The following theorem gives a necessary and sufficient condition that V_z is a Lyapunov function

Theorem 1 (Gowda et al. 2009)**.** *Let $L : V \to V$ be a linear operator on a Jordan algebra V with corresponding symmetric cone of squares K, and assume that $e^L(K) \subseteq K$. The following statements are equivalent:*

(a) There exists $p \succ_K 0$ such that $-L(p) \succ_K 0$
(b) There exists $z \succ_K 0$ such that $LP_z + P_z L^T$ is negative definite on V.
(c) The system $\dot{x}(t) = L(x)$ with initial condition $x_0 \in K$ is asymptotically stable.

Proof. See [11, Theorem 11]. □

This result allows the existence of a distinguished element $p \succ_K 0$ in the interior of K, where the vector field points in the direction of $-K$, or equivalently $-L(p) \succ_K 0$, to be a certificate for the existence of a quadratic Lyapunov function of the form

$$V_z : V \to \mathbf{R}, \quad V_z(x) = \langle x, P_z(x)\rangle.$$

In fact, the derivative of V_z along trajectories of (3) is precisely

$$\begin{aligned}\dot{V}_z(x) &= \langle \dot{x}, P_z(x)\rangle + \langle x, P_z(\dot{x})\rangle \\ &= \langle L(x), P_z(x)\rangle + \langle x, P_z(L(x))\rangle \\ &= \langle x, L^T(P_z(x))\rangle + \langle x, P_z(L(x))\rangle \\ &= \langle x, (P_z L + L^T P_z)(x)\rangle\end{aligned}$$

For the algebra $\mathbf{R}^n$ and the corresponding cone $\mathbf{R}^n_+$, Theorem 1 translates as follows. The quadratic representation of an element $z \succ_{\mathbf{R}^n_+} 0$ is a diagonal matrix D with strictly positive diagonal. Let A be a cross-positive (*i.e.*, Metzler) matrix. The system $\dot{x}(t) = Ax(t)$ is stable (*i.e.*, Hurwiz), if and only if there exists an entrywise positive vector p such that

$$p \succ_{\mathbf{R}^n_+} 0, \quad -Ap \succ_{\mathbf{R}^n_+} 0,$$

which happens if and only if there exists a diagonal Lyapunov function $V(x) = x^T Dx$,

$$D \succ_{\mathbf{S}^n_+} 0, \quad AD + DA^T \prec_{\mathbf{S}^n_+} 0.$$

Finding such a p is a LP for a fixed A. In addition, if we know (or impose, through a feedback interconnection) that A has the Metzler structure, then a diagonal Lyapunov function candidate suffices to ensure stability. This trick has been used in, *e.g.*, [3] to greatly simplify (and in certain cases parallelize) stability analysis and controller synthesis.

Now consider the algebra $\mathbf{S}^n$ and the corresponding cone $\mathbf{S}^n_+$. One can define many cross-positive operators, but one comes to mind: the Lyapunov transformation $L : \mathbf{S}^n \to \mathbf{S}^n$,

$$L(X) = AX + XA^T,$$

where $A \in \mathbf{R}^n$ is a given matrix. (By construction, L is cross-positive). Following the theorem, we now restate some widely known facts about linear systems. The matrix differential equation

$$\dot{X}(t) = AX + XA^T, \quad X(0) = X_0 \succ_{\mathbf{S}^n_+} 0$$

is asymptotically stable if and only if there exists a matrix $P \succ 0$ such that $L(P) = AP + PA^T \prec 0$, if and only if there exists a matrix $Z \succ 0$ such that the function

$$V_Z(X) = \langle X, P_Z(X)\rangle = \mathbf{Tr}(XZXZ) = \|XZ\|_F^2$$

is a Lyapunov function. This happens if and only if A is a (Hurwiz) stable matrix.

IV. Case study: dynamics on a Lorentz cone

The cones $\mathbf{R}^n_+$ and $\mathbf{S}^n_+$ have been well studied in the literature. The "intermediate case"—the cone $\mathcal{L}^n_+$—is, however, quite strange. This cone has received relatively little attention in the control community. Recent efforts have been in the context of model matching [14]. Our work here can be seen as a complement.

A. Enforcing $\mathcal{L}^n_+$-invariance

To apply the main theorem, we require that $e^L(K) \subseteq K$. In the case $K = \mathcal{L}^n_+$, the dynamics matrix A should *a priori* satisfy the LMI

$$A^T J_n + J_n A - \xi J_n \succeq 0, \quad \xi \in \mathbf{R}. \tag{4}$$

If A were to affinely depend on optimization variables (as it would if it were a closed loop matrix in a linear feedback synthesis problem), then enforcing $\mathcal{L}^n_+$-invariance would also be an LMI—we might as well dispense with any special structure, and resort to algebraic Riccati, bounded-real, or general LMI-based analysis, *e.g.*, [15].

However, if A has some extra structure, for example, it has an embedded positive block transverse to the Lorentz cone axis, then the LMI (4) can be simplified, as we discuss below after introducing the concept of diagonal dominance.

B. Diagonal dominance.

A square matrix $A \in \mathbf{R}^{n\times n}$ is (weakly) diagonally dominant if its entries satisfy

$$|A_{ii}| \geq \sum_{j\neq i} |A_{ij}|, \quad \text{for all } i = 1, \dots, n.$$

As a simple consequence of Gershgorin's circle theorem, diagonally dominant matrices with nonnegative diagonal entries are positive semidefinite.

If there exists a positive diagonal matrix $D \in \mathbf{R}^{n\times n}$ such that AD is diagonally dominant, then the matrix A is *generalized* or *scaled* diagonally dominant. Equivalently, there exists a positive vector $y \succ_{\mathbf{R}^n_+} 0$ such that

$$|A_{ii}|y_i \geq \sum_{j\neq i} |A_{ij}|y_j, \quad \text{for all } i = 1, \dots, n.$$

Note that generalized diagonally dominant matrices are also positive semidefinite, and include diagonally dominant matrices as a special case. Symmetric generalized diagonally dominant matrices with nonnegative diagonal entries are also known as H^+-matrices, and have a very nice characterization as matrices with factor width of at most two, see [16, Theorem 9].

One consequence of this characterization is that H^+ matrices can be written as a sum of positive semidefinite matrices that are nonzero only on a 2×2 principal submatrix, see [17]. For example, a 3×3 H^+-matrix $A = A^T$ is the sum of terms of the form

$$A = \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} y_1 & 0 & y_2 \\ 0 & 0 & 0 \\ y_2 & 0 & y_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & z_1 & z_2 \\ 0 & z_2 & z_3 \end{bmatrix},$$

where the sub-matrices are all positive semidefinite,

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \succeq 0.$$

Recently, this fact has been exploited in [18], [19] to extend the reach of sum-of-squares techniques to high dimensional dynamical systems without imposing full LMI constraints on the Gram matrix, and in [17] to preprocess SDPs for numerical stability.

C. Rotated quadratic constraints.

Real, symmetric, positive semidefinite matrices of size 2×2, as they occur in the characterization of H^+-matrices above, are special, because they satisfy a restricted hyperbolic constraint; hence their definiteness can be enforced with a SOCP rather than SDP [7]. Specifically, we have,

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \succeq 0 \iff x_1 \geq 0, \quad x_3 \geq 0, \quad x_1 x_3 - x_2^2 \geq 0$$
$$\iff \left\| \begin{bmatrix} 2x_2 \\ x_1 - x_3 \end{bmatrix} \right\|_2 \leq x_1 + x_3$$
$$\iff (x_1 + x_3, 2x_2, x_1 - x_3) \in \mathcal{L}^3_+,$$

for scalars x_1, x_2, and x_3.

D. Embedded positive block.

We are now equipped to consider the simple, augmented $(n+1)$-dimensional dynamics

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} a_0 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \quad a_0 \in \mathbf{R}, \quad A \in \mathbf{R}^{n \times n},$$

with trajectory $x(t) = (x_0(t), x_1(t)) \in \mathbf{R} \times \mathbf{R}^n$, on the cone $\mathcal{L}^{n+1}_+$. After writing out the LMI (4) in block form, we determine that the augmented system is $\mathcal{L}^{n+1}_+$-invariant if and only if

$$2a_0 I - (A + A^T) \succeq 0. \tag{5}$$

It is stable if and only if A is Hurwiz and $a_0 < 0$. Roughly speaking it is $\mathcal{L}^{n+1}_+$-invariant and stable if the stability degree of A (*i.e.*, minus the maximum real part of the eigenvalues of A) is at least $|a_0|$.

In general, the stability degree constraint (5) is an LMI in the variables (A, a_0), however, if A is Metzler then the LMI can be replaced with the constraint

$$2a_0 I - (A + A^T) \text{ is a } H^+\text{-matrix}, \tag{6}$$

without any loss. From previous sections, the H^+-matrix constraint (6) is a SOCP.

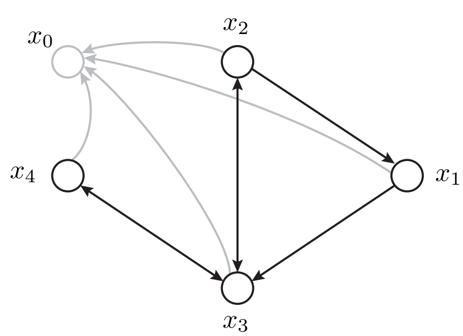

Fig. 2. Transportation network $x_1, \ldots, x_4$ from [3, Figure 2], augmented with a catch-all buffer x_0.

E. Example in 5D.

We consider the linear transportation network shown in Figure 2, which is inspired by the one studied in [3]. The network might represent a base system of four buffers (solid nodes $x_1, \ldots, x_4$) exchanging and consuming material in a way that preserves the network structure. The base system is augmented with an extra catch-all buffer (grayed out node x_0), leading to the augmented dynamics

$$\begin{bmatrix} \dot{x}_0 \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} a_0 & h^T \\ 0 & A \end{bmatrix} \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}, \quad x_0(t) \in \mathbf{R}, \quad \bar{x}(t) \in \mathbf{R}^4, \tag{7}$$

where the base dynamics $\dot{\bar{x}} = A\bar{x}$ are given by the internally positive system

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}}_{\dot{\bar{x}}(t)} = \underbrace{\begin{bmatrix} -1 - \ell_{31} & \ell_{12} & 0 & 0 \\ 0 & -\ell_{12} - \ell_{32} & \ell_{23} & 0 \\ \ell_{31} & \ell_{32} & -\ell_{23} - \ell_{43} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4 - \ell_{34} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\bar{x}(t)}.$$

Here, the state x_i represents the amount of material at node i, $\ell_{ij} \geq 0$ the transfer rate between nodes i and j in the base system, $a_0 \in \mathbf{R}$ the self-degradation rate of the catch-all buffer, and h_i the rate of transfer between node i and the catch-all node. Note that A is Metzler by construction.

Several problems are now readily solved: by checking (5) (or (6)), with the ℓ_{ij} treated as variables and $h = 0$ fixed, the augmented system is $\mathcal{L}^5_+$-invariant only if $a_0 \geq -1.25$, which is an upper bound on the Metzler eigenvalue of A. Thus the catch-all node must consume material no faster than with rate constant 1.25. In addition, if ℓ_{ij} and h_i are known, and the augmented system (7) is $\mathcal{L}^5_+$-invariant, then it is stable provided $a_0 < 0$. Of course, these types of problems can be cast as LP or SOCP.

F. Other examples on $\mathcal{L}^n_+$.

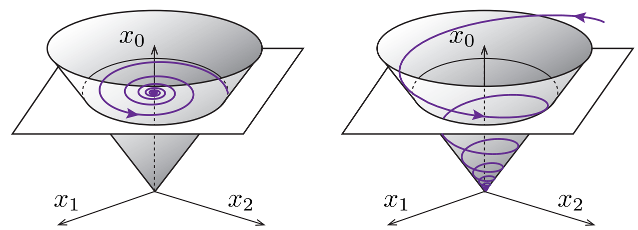

Fig. 3. Embedded focus along x_0-axis

1) Embedded block: For the dynamics

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} a_0 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix},$$

This system is $\mathcal{L}^n_+$-invariant if and only if $A + A^T \preceq 2a_0 I$. It is stable if and only if A is Hurwiz and $a_0 < 0$. Figure 3 illustrates embedded block dynamics without a shear term h.

2) Twist system: Suppose $A + A^T = 0$ (skew-symmetric). The system

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} a_0 & h^T \\ 0 & A \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix},$$

is $\mathcal{L}^n_+$-invariant if and only if $\|h\|_2 \leq a_0$, *i.e.*, the point $(a_0, h) \in \mathcal{L}^n_+$. Furthermore, if $a_0 = 0$, this is a twist in homogeneous coordinates.

3) Proper orthochronous Lorentz transformations: The restricted Lorentz group $SO^+(1, n-1)$, which is generated by spatial rotations and Lorentz boosts, consists of linear transformations that keep the quadratic form $x^T J_n x$ invariant. In particular, elements of $SO^+(1, n-1)$ correspond to $\mathcal{L}^n_+$-invariant dynamics matrices.

Algebra:	Real	Lorentz	Symmetric
V	$\mathbf{R}^n$	$\mathbf{R}^n$	$\mathbf{S}^n$
K	$\mathbf{R}^n_+$	$\mathcal{L}^n_+$	$\mathbf{S}^n_+$
$\langle x, y\rangle$	$x^T y$	$x^T y$	$\mathbf{Tr}(XY^T)$
$x \circ y$	$x_i y_i$	$(x^T y, x_0 y_1 + y_0 x_1)$	$\frac{1}{2}(XY + YX)$
P_z, $z \in \mathbf{int}\, K$	$\mathrm{diag}(z)^2$	$zz^T - \frac{z^T J_n z}{2} J_n$	$X \mapsto ZXZ$
$V(x) = \langle x, P_z(x)\rangle$	$x^T \mathrm{diag}(z)^2 x$	$x^T \left(zz^T - \frac{z^T J_n z}{2} J_n\right) x$	$\Vert XZ\Vert_F^2$
Free variables in $V(x)$	n	n	$n(n+1)/2$
dynamics L	$x \mapsto Ax$	$x \mapsto Ax$	$X \mapsto AX + XA^T$
L is cross-positive	A is Metzler	A satisfies (4)	by construction
$-L(p) \succ_K 0$	$(Ap)_i < 0$	$\Vert(Ap)_1\Vert_2 \leq (-Ap)_0$	$AP + PA^T \prec 0$
Stability verification	LP	SOCP	SDP

TABLE I

SUMMARY OF DYNAMICS PRESERVING A CONE

V. CONCLUSION

In this work we used Jordan algebraic techniques to analyze linear systems with special structure. In particular, we put analysis of internally positive systems, which have recently been in vogue, in the same theoretical framework as systems that are exponentially invariant with respect to the Lorentz cone, as well as general linear systems. It is evident that Lyapunov functions for such systems can be significantly simpler, computationally and representationally, than if the special cone-invariant dynamic structure were not exploited. In this framework, summarized in Table I, the relevant structure is cross-positivity of the dynamics matrix on a symmetric cone.

Unfortunately, while cross-positivity has a fairly simple LP characterization in terms of the corresponding dynamics matrix (via the Metzler structure) for internally positive systems, the same condition is generically much more complicated for $\mathcal{L}^n_+$-invariant dynamics: the condition is a LMI in the dynamics matrix. In fact, the condition corresponding to entrywise positivity of the dynamics matrix is also a LMI for discrete time systems [20], [21]. As a result, $\mathcal{L}^n_+$-invariance, by itself, is not computationally attractive for high dimensional systems without taking advantage of *even more* special structure. We gave a simple (almost trivial) example of how this might be done by augmenting an internally positive system with a catch-all block.

Future work will aim to apply these techniques to control synthesis and as well as to further study special structure. An interesting question is whether it is possible to verify stability of certain subclasses of, *e.g.*, difference of positive or ellipsoidal cone invariant systems via LP or SOCP, rather than via SDP.

ACKNOWLEDGMENTS

We thank S. You, N. Matni, and A. Swaminathan for helpful discussions. This work was supported in part by a Department of Defense NDSEG Fellowship, Terraswarm, one of six centers of STARnet, a Semiconductor Research Corporation program sponsored by MARCO and DARPA, and by the Boeing company.

REFERENCES

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, ser. Classics in Applied Mathematics. SIAM, 1994.

[2] P. D. Leenheer and D. Aeyels, "Stabilization of positive linear systems," *Systems and Control Letters*, vol. 44, no. 4, pp. 259–271, 2001.

[3] A. Rantzer, "Distributed control of positive systems," arXiv:1203.0047 [math.OC], Feb. 2012.

[4] R. T. Rockafellar, *Convex Analysis*. Princeton University Press, 1970.

[5] S. P. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[6] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, ser. Oxford Mathematical Monographs. Oxford University Press, 1994.

[7] F. Alizadeh and D. Goldfarb, "Second-order cone programming," *Mathematical Programming*, vol. 95, no. 1, pp. 3–51, 2003.

[8] M. S. Gowda, R. Sznajder, and J. Tao, "Some P-properties for linear transformations on Euclidean Jordan algebras," *Linear Algebra and its Applications*, vol. 393, no. 0, pp. 203–232, 2004.

[9] M. S. Gowda and R. Sznajder, "Some global uniqueness and solvability results for linear complementarity problems over symmetric cones," *SIAM Journal on Optimization*, vol. 18, no. 2, pp. 461–481, 2007.

[10] L. Vandenberghe, "Symmetric cones," EE236C Lecture Notes, 2013–2014.

[11] M. S. Gowda and J. Tao, "Z-transformations on proper and symmetric cones," *Mathematical Programming*, vol. 117, no. 1-2, pp. 195–221, 2009.

[12] H. Schneider and M. Vidyasagar, "Cross-positive matrices," *SIAM Journal on Numerical Analysis*, vol. 7, no. 4, pp. 508–519, 1970.

[13] R. J. Stern and H. Wolkowicz, "Exponential nonnegativity on the ice cream cone," *SIAM Journal on Matrix Analysis and Applications*, vol. 12, no. 1, pp. 160–165, 1991.

[14] C. Grussler and A. Rantzer, "Modified balanced truncation preserving ellipsoidal cone-invariance," in *IEEE Conference on Decision and Control (CDC)*, Dec. 2014, pp. 2365–2370.

[15] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied and Numerical Mathematics. SIAM, 1994, vol. 15.

[16] E. G. Boman, D. Chen, O. Parekh, and S. Toledo, "On factor width and symmetric H-matrices," *Linear Algebra and its Applications*, vol. 405, no. 0, pp. 239–248, 2005.

[17] F. Permenter and P. Parrilo, "Partial facial reduction: simplified, equivalent SDPs via approximations of the PSD cone," arXiv:1408.4685 [math.OC], Nov. 2014.

[18] A. A. Ahmadi and A. Majumdar, "DSOS and SDSOS optimization: LP and SOCP-based alternatives to sum of squares optimization," in *Conference on Information Sciences and Systems (CISS)*, Mar. 2014, pp. 1–5.

[19] A. Majumdar, A. A. Ahmadi, and R. Tedrake, "Control and verification of high-dimensional systems with DSOS and SDSOS programming," in *IEEE Conference on Decision and Control (CDC)*, Dec. 2014, pp. 394–401.

[20] R. Hildebrand, "An LMI description for the cone of Lorentz-positive maps," *Linear and Multilinear Algebra*, vol. 55, no. 6, pp. 551–573, 2007.

[21] ——, "An LMI description for the cone of Lorentz-positive maps II," *Linear and Multilinear Algebra*, vol. 59, no. 7, pp. 719–731, 2011.