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# Parametric Delay-margin Maximization of Consensus Network Using Local Control Scheme

Linlin Ou, Ye Yuan, and Richard M. Murray

***Abstract*—We consider a network of identical agents with arbitrary linear time-invariant (LTI) dynamics such that the network reaches consensus asymptotically. Uniform time delay is taken into account in the communication channels. The goal of this paper is to maximize the delay (so-called delay margin) that the system can tolerate before becoming unstable by implementing a low-order controller to a single agent. A parametric design method is investigated to guarantee the stability and consensusability. The set of all feasible low-order controllers based on the frequency response data is characterized by combining the argument principle and the generalized Nyquist criterion. Based on this, the algorithm of computing the delay margin is proposed for a given controlled agent. By combining all the possible margins for each controlled agent, we then can obtain the maximal delay margin for the whole network and the corresponding local controller.**

## I. Introduction

COOPERATIVE control of networked multi-agent systems, in which all agents work together to accomplish a global objective, has attracted more and more attention recently. This is mainly motivated by the emergence of inexpensive and reliable wireless communications systems. The potential applications of networked multi-agent system encompass satellite clusters, automated highways, mobile robots, mobile sensors networks and supply chains [1, 2].

The design of efficient consensus algorithms is a current focus of active research in the control community. Many consensus algorithms and criteria have been presented for the homogeneous multi-agent system by relying on algebraic graph theory, matrix theory and control theory [3-6]. A precondition for a multi-agent system to reach consensus is that the system is stable. The network topology plays an important role in the stability and performance of the system.

In practice, the time delays associated with transmission of agent values appear due to finite communication speed. Thus, the time delay becomes another key parameter affecting the system stability. For undirected networks with fixed topology, sufficient and necessary conditions for reaching the consensus were provided in presence of communication delays [3]. Based on the earlier work, different consensus protocols were presented and stability in the delay space was analyzed for the second-order multi-agent system under fixed topology and switching topology by employing linear matrix inequalities (LMI) or Gershgorin's circle theorem [7-9]. Based on the generalized Nyquist criterion, the robustness of consensus schemes for linear multi-agent systems to feedback delays was investigated in [10]. The concept of the responsible eigenvalue was proposed and used to design controllers for the single-delay multi-agent systems [11, 12]. These results present the design approaches of centralized stabilizing controllers for homogeneous multi-agent systems with communication delays.

In order to reduce the complexity in the implementation of multi-agent systems, it is more desirable to design the local and decentralized stabilizing controller, i.e., the controller is implemented on one single agent or part of the agents. There exists great difficulty in the design of the local controller for the multi-agent system with communication delays, especially for large-scale multi-agent system. One main reason lies in the fact that the local controller leads to a heterogeneous multi-agent system. The stability of the heterogeneous multi-agent system with communication delays cannot be equivalently considered as the stability of a set of independent systems depending on the eigenvalues of the graph Laplacian matrix. The other reason is that the communication delays lead to complicated infinite dimensional system although the graph Laplacian has finite number of eigenvalues.

The objective in this paper is to determine the maximal stabilizing range of the uniform communication delay increasing from zero by implementing a low-order controller to a single agent. First of all, the stability relation between the multi-agent system with a local controller and the original multi-agent system is presented analytically. Then based on this, we transform the stability problem of the multi-agent system to a standard stabilization problem of the low-order controller for a SISO (single-input and single-output) system with multiple delays. However, the transfer function of such the SISO system is very complicated since its numerator and denominator are both retarded quasipolynomials with commensurate delays.

There has been a large effort in dealing with the stability problem of systems with multiple delays by using various analytical and numerical tools, see [13-16], and others. Some methods presented the stabilizing delay values exactly by analyzing the asymptotic behavior of the critical zeros on the imaginary axis [17, 18]. However, the analysis of the zero asymptotic behavior is not a simple task since the characteristic zeros of a quasipolynomial may exhibit rather

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Linlin Ou is with the Department of Automation, Zhejiang University of Technology, Hangzhou 310023, China, Email: linlinou@zjut.edu.cn.

Ye Yuan is with the Department of Engineering, University of Cambridge, UK, CB2 1PZ, Emails: yy311@cam.ac.uk

R. Murray is with the Control & Dynamical Systems department, California Institute of Technology, Pasadena, CA, 91125, Email: murray@cds.caltech.edu.

complex analytical properties. Although the eigenvalue perturbation series proposed most recently in [13, 14] can be applied to the zero asymptotic behavior, the computational load is very heavy when employed to the quasipolynomial derived from the large-scale multi-agent system. This is due to the high order term in the quasipolynomial and the computation complexity in the decomposition of the quasipolynomial into the standard form. In this paper, by making good use of the frequency response data, the stabilizing set of the low-order controller is presented analytically for a given communication delay based on the argument principle and the generalize Nyquist criterion for the retarded quasipolynomials provided in [19]. Subsequently, the maximal stabilizing delay range (so-called delay margin) of the consensus network is obtained by finding the intersection of the stabilizing low-order controllers for the delay values increasing from zero until the intersection becomes empty. Furthermore, the best choice of the controlled agent, which leads to the maximal delay margin, is determined. The proposed algorithm of determining the maximal delay margin and the corresponding low-order controller is applicable to high-order LTI agents and arbitrary directed or undirected networks.

The paper is organized as follows. In Section 2, the design problem of the stabilizing local controller for the parametric delay-margin maximum in the multi-agent system with uniform communication delay is stated. The stability description of the multi-agent system with a local controller is presented and the stability problem of the multi-agent system is reduced to the stabilization problem of the low-order controller for the SISO system with commensurate delays in Section 3. Based on the argument principle and Nyquist criterion, the stabilizing set of the low-order controller for the complex quasipolynomial with distributed delays is derived in Section 4. Then, the algorithm determining the maximal delay margin and the corresponding local controller is presented in Section 5. A numerical example is given in Section 6 and conclusions in Section 7

## II. Problem Statement

Represent the communications between the agents by a directed graph $G=\{V,\varepsilon,A\}$ , where $V=\{V_1,V_2,\cdots,V_n\}$ is the set of agents, and $A=\left[a_{ij}\right]\in\mathbb{R}^{n\times n}$ is the weighted adjacency matrix of $G$ with nonnegtive adjacency elements $a_{ij}$. An edge of $G$ is denoted by $e_{ij}=(v_i,v_j)$. The adjacency elements associated with the edges of the graph are positive, i.e. $e_{ij}\in\varepsilon\Leftrightarrow a_{ij}>0$ . Assume that $a_{ii}=0$ . The neighborhood of agent $i$ is defined as $N_i=\left\{j\in v\middle|a_{ij}=1\right\}$. The cardinal number of $N_i$ is called the degree of $i$ and is denoted by $d_i$. The Laplacian matrix of $G$ is defined as $L=D-A$ , where $D=\mathrm{diag}(d_1,\cdots,d_n)$. The graph is connected if any two nodes $i$ , $j$ of the graph are connected by a path. Recall that a graph is strongly connected if and only if its Laplacian $L$ has a single zero eigenvalue [20].

We consider the consensus network consisting of $n$ identical LTI agents, which is shown in Fig. 1. The dynamics

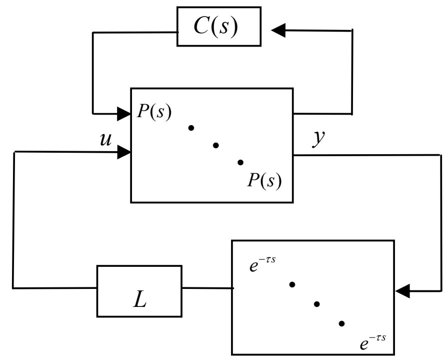



Fig.1. The consensus network structure with a local controller

of each agent is described by the following transfer function

$$P(s)=\frac{N(s)}{D(s)}e^{-\theta s} \tag{1}$$

where $\theta$ is the time delay of the agent, and $N(s)$ and $D(s)$ are coprime polynomials in $s$ , defined as

$$N(s)=\upsilon_b s^b+\upsilon_{b-1}s^{b-1}+\cdots+\upsilon_1 s+\upsilon_0$$

and

$$D(s)=s^a+\mu_{a-1}s^{a-1}+\cdots+\mu_1 s+\mu_0 .$$

Here, $\upsilon_0,\upsilon_1,\cdots\upsilon_b$ and $\mu_0,\mu_1,\cdots\mu_{a-1}$ are real numbers, and $a>b$. In Fig. 1, $\tau$ is the uniform communication delay, $L$ is the graph Laplacian and $C(s)$ is a local controller imposed on the $l$ th agent, where $l\in\{1,2,\cdots,n\}$. The subsystem from the input $u_l(s)$ to the output $y_l(s)$ is a closed-loop unity feedback one composed of the controller $C(s)$ and the agent $P(s)$. We have

$$\frac{y_l}{u_l}=\frac{P(s)C(s)}{1+P(s)C(s)} \tag{2}$$

For the other agents, the relation between the input $u_i(s)$ and output $y_i(s)$ (here, $i\neq l$ ) is still determined by $P(s)$. It is seen that the multi-agent system with identical agents becomes a heterogeneous one when a local controller is introduced. This causes great difficulty in the stability analysis and control synthesis of the multi-agent system.

If a controlled agent is known, the delay margin $\tau^*$ is defined as the maximal amount of the delay that the consensus network can tolerate from $\tau=0$ before the system becomes unstable. The objective is to design a local controller $C(s)$ so that the whole interconnected system is stable and the delay margin $\tau^*$ is explicitly characterized. Furthermore, we investigate which agent should be locally controlled in order to produce the maximal delay margin.

## III. Stability of The Multi-agent System with a Local Controller

The agents in the original system without the local controller are identical, and thus the system can be decomposed into a set of smaller, independent modal systems with time delay. This leads to the simple and intuitive stability analysis for this kind of system based on the classical stability theory. Since the system becomes a heterogeneous one after the local controller is introduced, the decomposition is not applicable. In this section, the stability description of the multi-agent system with the local controller is analytically presented with respect to the characteristic function of the original system and the stability of the multi-agent system with the local controller is then formulated to the stabilization of the local controller for the SISO system with commensurate delays.

**Lemma 1:** Denote the characteristic function of the original system as $\rho(s)$. Then, the system in Fig. 1 is stable if and only if all the zeros of the following equation lie in the left-half complex plane.

$$\delta(s)=\rho(s)\left\{1+\bar{C}(s)\cdot e_l^T\left[I+P(s)Le^{-\tau s}\right]^{-1}e_l\right\} \tag{3}$$

where $e_l^T=[0,\cdots,0,1_{l^{th}},0,\cdots,0]\in\mathbb{R}^{1\times n}$,

$$\bar{C}(s)=\frac{1+P(s)C(s)}{C(s)}-1 \tag{4}$$

and

$$\rho(s)=\det\left(I+P(s)\bar{L}e^{-\tau s}\right).$$

Proof: Assume that the local controller is added to the first agent without loss of generality. The characteristic function of the whole system is

$$\delta(s)=\det\left(I+\begin{pmatrix}\frac{P(s)C(s)}{1+P(s)C(s)} & & & \\ & P(s) & & \\ & & \ddots & \\ & & & P(s)\end{pmatrix}Le^{-\tau s}\right)$$

$$=\frac{C(s)}{1+P(s)C(s)}\det\left(\begin{pmatrix}\frac{1+P(s)C(s)}{C(s)} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}+P(s)Le^{-\tau s}\right) \tag{5}$$

By using the block decomposition of the matrix, we have

$$\det\left(\begin{pmatrix}\frac{1+P(s)C(s)}{C(s)} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}+P(s)Le^{-\tau s}\right)-\det\left(I+P(s)Le^{-\tau s}\right)$$

$$=\left[\frac{1+P(s)C(s)}{C(s)}-1\right]\cdot\det\left(\begin{bmatrix}0 & I_{n-1}\end{bmatrix}\left(I+P(s)Le^{-\tau s}\right)\begin{bmatrix}0\\ I_{n-1}\end{bmatrix}\right)$$

$$=\left[\frac{1+P(s)C(s)}{C(s)}-1\right]\cdot\det\left(I+P(s)\bar{L}e^{-\tau s}\right) \tag{6}$$

where

$$\bar{L}=\begin{bmatrix}0 & I_{n-1}\end{bmatrix}L\begin{bmatrix}0\\ I_{n-1}\end{bmatrix} \tag{7}$$

From (5) and (6), we have

$$\delta(s)=\rho(s)\left[I+\bar{C}(s)\cdot\frac{\det\left(I+P(s)\bar{L}e^{-\tau s}\right)}{\det\left(I+P(s)Le^{-\tau s}\right)}\right]$$

$$=\rho(s)\left[I+\bar{C}(s)\cdot e_1^T\left[I+P(s)Le^{-\tau s}\right]^{-1}e_1\right] \tag{8}$$

where

$$\bar{C}(s)=\frac{1+P(s)C(s)}{C(s)}-1$$

If the local controller is added to other single agent, the similar result as in (8) can also be obtained. This concludes the proof.

***Remark 1***: The term $e_l^T\left[I+P(s)Le^{-\tau s}\right]^{-1}e_l$ in (3) is the transfer function from the input of Agent $l$ to its output in the original system and it can be obtained by using Mason's rule in the signal flow graph [21].

Given a topology of the system and the dynamics of the agent, it is seen from (8) that the stability of the whole system is determined by $C(s)$. Let $\bar{L}_l$ denote the matrix that is obtained by omitting the $l$ th row and $l$ th column of $L$. Rewriting (8) yields

$$\delta(s)=\prod_{i=1}^{n}\left[1+\lambda_i P(s)e^{-\tau s}\right]+\bar{C}(s)\cdot\prod_{i=1}^{n-1}\left[1+\gamma_i P(s)e^{-\tau s}\right] \tag{9}$$

where $\lambda_i$ and $\gamma_i$ denote the eigenvalues of $L$ and $\bar{L}_l$, respectively. Take

$$Q(s)=\prod_{i=1}^{n'}\left[1+\lambda_i P(s)e^{-\tau s}\right],\ R(s)=\prod_{i=1}^{m'}\left[1+\gamma_i P(s)e^{-\tau s}\right] \tag{10}$$

Here, $n'$ and $m'$ are the number of the non-zero eigenvalues of $L$ and $\bar{L}_i$, respectively. Equation (9) can be converted into

$$\delta(s)=\frac{1}{C(s)}\left\{R(s)+C(s)\left[Q(s)+P(s)R(s)-R(s)\right]\right\} \tag{11}$$

Although several methods to find the delay margin of $\tau$ have been presented recently, it is difficult to be implemented for (9) due to the computation complexity, especially for the case that the number of agents is large or the dynamics of the agent is complex. However, if the stabilizing set of the control parameters in $C(s)$ can be found for each $\tau$ increasing from 0 until the intersection of these stabilizing sets becomes empty, then the maximum of $\tau$ can be treated as the delay margin. Thus, the key task is to determine the set of $C(s)$ that can guarantee the stability of (11) for a given $\tau$. From (11), it is seen that the stability problem of (11) is equivalent to the stabilization problem of the controller $C(s)$ for the following SISO plant with multiple delays:

$$G(s)=\frac{Q(s)+P(s)R(s)-R(s)}{R(s)}$$

Both the numerator and denominator of $G(s)$ are the retarded quasipolynomials with commensurate delays.

## IV. Stabilizing Set of a Low-order Controller for the Quasipolynomial with Distributed Delays

Consider the retarded quasipolynomial of the form

$$\zeta(s)=U(s)+KV(s) \tag{12}$$

where

$$U(s)=s^u+\sum_{i=0}^{u-1}\sum_{j=1}^{h_i}\alpha_{ij}s^i e^{-s\tau_{ij}} \text{ and } V(s)=\beta_v s^v+\sum_{i=0}^{v-1}\sum_{j=1}^{f_i}\beta_{ij}s^i e^{-s\theta_{ij}}$$

Here, $u$, $v$, $h_i$ and $f_i$ are the integers not less than zero, and $u>v$. Let

$$F(s)=\frac{V(s)}{U(s)} \tag{13}$$

The following definitions are given before presenting the main theorem.

**Definition 1** Assume that $\omega^*$ is sufficiently large. Let $0=\omega_0<\omega_1<\omega_2<\cdots<\omega_{c-1}$ be the real and distinct zeros of $\mathrm{Im}\left[F(j\omega)\right]=0$ for $u+v$ odd (or be the real and distinct zeros of $\mathrm{Im}\left[F(j\omega)\right]=0$ in $[0,\omega^*)$ for $u+v$ even). Then, define $i_0,i_1,i_2,\cdots,i_{c-1}$ as follows:

(i) If $V(-j\omega_t)=0$ for some $t=1,2,\cdots,c-1$, then $i_t=0$;

(ii) If $V(-s)$ has a zero at the origin, then

$$i_0=\mathrm{sgn}\left(\frac{d}{d\omega}\left[V_r(\omega)U_r(\omega)+V_i(\omega)U_i(\omega)\right]\Big|_{\omega=0}\right), \tag{14}$$

where $V_r(\omega)$, $V_i(\omega)$, $U_r(\omega)$, and $U_i(\omega)$ are the real and imaginary parts of $V(j\omega)$ and $U(j\omega)$, respectively.

(iii) For all other $t=0,1,2,\cdots c-1$, let $i_t=1$ or $-1$.

It is worth noting that in (iii), that $i_t$ equals 1 or -1 is determined by the stability conditions.

**Definition 2** Let $I=\{i_0,i_1,\cdots\}$ and denote $\gamma(I)$ by

$$\gamma(I)=\begin{cases}\left[i_0-2i_1+2i_2+\cdots+(-1)^{c-2}2i_{c-2}+(-1)^{c-1}i_{c-1}\right] & \\ \cdot(-1)^{c-1}\,\mathrm{sgn}\left\{\mathrm{Im}\left[F(j\omega_{c-1}^+)\right]\right\} & \text{for } u+v \text{ even}\\ \left[i_0-2i_1+2i_2+\cdots+(-1)^{c-1}2i_{c-1}\right] & \\ \cdot(-1)^{c-1}\,\mathrm{sgn}\left\{\mathrm{Im}\left[F(j\omega_{c-1}^+)\right]\right\} & \text{for } u+v \text{ odd}\end{cases} \tag{15}$$

Based on the argument principle and the generalized Nyquist criterion for the retarded quasipolynomial [19], the following theorem is presented to determine the range of $K$ that can guarantee the stability of $\zeta(s)$ in terms of the frequency domain data of $F(s)$.

**Theorem 1** $\zeta(s)$ is stable if and only if

$$\frac{u}{2}\pi-\underset{s=j\omega,\omega\in(0,+\infty)}{\Delta\arg}\left[V(s)\right]=\frac{\pi}{2}\gamma(I) \tag{16}$$

where $\gamma(I)$ is given in Definition 2. If there exists the stabilizing $K$, the stability set of the gain $K$ is equal to

$$\bigcup_{e=1}^{h}\left(\max_{i_t\in I_e,i_t>0}\left[-\frac{1}{F(j\omega_t)}\right],\min_{i_t\in I_e,i_t<0}\left[-\frac{1}{F(j\omega_t)}\right]\right) \tag{17}$$

There does not exist the stabilizing $K$ if one of the two conditions holds:

(i) There does not exist $I_e$ satisfying (16);

(ii) $\min_{i_t\in I,i_t<0}\left[-1/F(j\omega_t)\right]\le\max_{i_t\in I,i_t>0}\left[-1/F(j\omega_t)\right]$ for all feasible $I_e$ satisfying (16).

Proof: It can be seen from (12) that both the real and imaginary parts of $\zeta(s)$ depend on the gain $K$ and this causes difficulty in investigating the stability of $\zeta(s)$. To overcome this problem, we construct a new quasipolynomial in which the imaginary part is independent of $K$. Multiplying two sides of (12) by $V(-s)$ and taking $s=j\omega$, we have

$$\zeta'(j\omega)=V(-j\omega)\delta(j\omega)=p(K,\omega)+jq(\omega) \tag{18}$$

where

$$p(K,\omega)=V_r(\omega)U_r(\omega)+V_i(\omega)U_i(\omega)+K\left[V_r^2(\omega)+V_i^2(\omega)\right] \tag{19}$$

$$q(\omega)=V_r(\omega)U_i(\omega)-V_i(\omega)U_r(\omega) \tag{20}$$

In terms of the result in [19], it is derived that $\zeta(s)$ is stable if and only if

$$\underset{s=j\omega,\omega\in(0,+\infty)}{\Delta\arg}\left[\zeta(s)\right]=\frac{u+1}{2}\pi \tag{21}$$

This means that the argument of $\zeta'(j\omega)$ has to satisfy

$$\underset{\omega\in(0,+\infty)}{\Delta\arg}\left[\zeta'(j\omega)\right]=\frac{u}{2}\pi-\underset{\omega\in(0,+\infty)}{\Delta\arg}\left[V(j\omega)\right] \tag{22}$$

Then, consider the relation between $\Delta\arg\left[\zeta'(j\omega)\right]$ and $\zeta'(j\omega)$. In terms of the result in [22], we have

(i) If $\omega_t$, $\omega_{t+1}$ are both zeros of $q(\omega)$, then

$$\underset{\omega\in(\omega_t,\omega_{t+1})}{\Delta\arg}\left[\zeta'(j\omega)\right]=\frac{\pi}{2}\left\{\mathrm{sgn}[p(\omega_t,K)]-\mathrm{sgn}[p(\omega_{t+1},K)]\right\}\cdot\mathrm{sgn}[q(\omega_t^+)] \tag{23}$$

(ii) If $\omega_t$ is a zero of $q(\omega)$ and $\omega_{t+1}$ is not a zero of $q(\omega)$ (for a $\omega^*$ value tending to infinity, such the situation occurs only when $\omega_{t+1}$ is a zero of $p(\omega,K)$), then

$$\underset{\omega\in(\omega_t,\omega_{t+1})}{\Delta\arg}\left[\zeta'(j\omega)\right]=\frac{\pi}{2}\mathrm{sgn}[p(\omega_t,K)]\cdot\mathrm{sgn}[q(\omega_t^+)] \tag{24}$$

(iii) For $t=0,1,2,\cdots c-2$,

$$\mathrm{sgn}[q(\omega_{t+1}^+)]=-\mathrm{sgn}[q(\omega_t^+)] \tag{25}$$

when $\omega\to\infty$, we have

$$\zeta'(j\omega)\approx(j\omega)^{u+v}$$

This means that if $u+v$ is odd, the curve of $\zeta'(j\omega)$ tends to the imaginary axis as $\omega\to\infty$, i.e. $p(K,\omega)$ has infinite number of zeros and $q(\omega)$ has finite number of zeros. Similarly, if $u+v$ is even, the curve of $\zeta'(j\omega)$ tends to the real axis as $\omega\to\infty$, i.e. $p(K,\omega)$ has finite number of zeros and $q(\omega)$ has infinite number of zeros. Thus, $\Delta\arg\left[\zeta'(j\omega)\right]$ for $\omega$ from 0 to $\omega^*$ will not change with the increase of $\omega^*$ when the values of $\omega^*$ is sufficiently large. Due to this, we consider all the finite zeros of $q(\omega)$ for $u+v$ odd and its zeros in $[0,\omega^*]$ for $u+v$ even, which was also mentioned in Definition 1. Thus, for $u+v$ odd, $\underset{\omega\in(0,+\infty)}{\Delta\arg}\left[\zeta'(j\omega)\right]$ can be described as

$$\begin{aligned}\underset{\omega\in(0,+\infty)}{\Delta\arg}\left[\zeta'(j\omega)\right]&=\sum_{t=0}^{c-2}\underset{\omega\in(\omega_t,\omega_{t+1})}{\Delta\arg}\left[\zeta'(j\omega)\right]+\underset{\omega\in(\omega_{c-2},\omega_{c-1})}{\Delta\arg}\left[\zeta'(j\omega)\right]\\&=\frac{\pi}{2}\left\{\mathrm{sgn}[p(\omega_0,K)]-2\,\mathrm{sgn}[p(\omega_1,K)]+2\,\mathrm{sgn}[p(\omega_2,K)]+\cdots\right.\\&\quad\left.+(-1)^{c-1}2\,\mathrm{sgn}[p(\omega_{c-1},K)]\right\}\cdot(-1)^{c-1}\,\mathrm{sgn}\left[q(\omega_{c-1}^+)\right]\end{aligned} \tag{26}$$

and for $u+v$ even,

$$\begin{aligned}\underset{\omega\in(0,+\infty)}{\Delta\arg}\left[\zeta'(j\omega)\right]&=\sum_{t=0}^{c-2}\underset{\omega\in(\omega_t,\omega_{t+1})}{\Delta\arg}\left[\zeta'(j\omega)\right]+\underset{\omega\in(\omega_{c-2},\omega_{c-1})}{\Delta\arg}\left[\zeta'(j\omega)\right]\\&=\frac{\pi}{2}\left\{\mathrm{sgn}[p(\omega_0,K)]-2\,\mathrm{sgn}[p(\omega_1,K)]+2\,\mathrm{sgn}[p(\omega_2,K)]+\cdots\right.\\&\quad\left.+(-1)^{c-1}\,\mathrm{sgn}[p(\omega_{c-1},K)]\right\}\cdot(-1)^{c-1}\,\mathrm{sgn}\left[q(\omega_{c-1}^+)\right]\end{aligned} \tag{27}$$

It is seen that the zeros of $q(\omega)$ are the same as the roots of $\mathrm{Im}\left[F(j\omega)\right]=0$. By combining (22), (26) and (27) and using Definitions 1 and 2, we can conclude that $\delta(s)$ is stable if and only if Equation (16) holds.

Finally, the necessary and sufficient condition for the existence of the stabilizing $K$ values is determined with respect to the string set of $I$ satisfying (16). Let us consider two cases: $V(-s)$ has zeros on the imaginary axis or not.

If $V(-s)$ has no zeros on the imaginary axis, $\delta(s)V(-s)$ also has no zeros on the imaginary axis for a stabilizing $K$, which leads to $i_t\in\{-1,1\}$ for $t=0,1,2,\cdots,c-1$ by Definition 1. If $i_t=-1$, from (19) the stability requirement is

$$K<-\frac{V_r(\omega_t)U_r(\omega_t)+V_i(\omega_t)U_i(\omega_t)}{V_r^2(\omega_t)+V_i^2(\omega_t)} \tag{28}$$

Since

$$\frac{1}{F(j\omega_t)}=\frac{V_r(\omega_t)U_r(\omega_t)+V_i(\omega_t)U_i(\omega_t)}{V_r^2(\omega_t)+V_i^2(\omega_t)}$$

The inequality (28) is equivalent to $K<-1/F(j\omega_t)$. Similarly, for $i_t=1$, the stability requirement is $K>-1/F(j\omega_t)$.

If $F(-s)$ has zeros on the imaginary axis including a zero at the origin, $\zeta(s)V(-s)$ also has the same set of zeros. Assume that $F(-s)$ has a zero at $j\omega'$. It is clear that $\omega'$ is a subset of $\{\omega_0,\omega_1,\cdots,\omega_{c-1}\}$. From the results in [22], we have
(i) In the case $\omega_t\neq 0$, $i_t=0$ and it is independent of $K$;
(ii) In the case $\omega_t=0$, that is, $\omega_0=0$, Equation (14) has to hold although no constraints on $K$ appear.

***Remark 2:*** $\Delta\arg\left[V(j\omega)\right]$ for $\omega$ from 0 to $+\infty$ can be derived analytically in terms of the zeros of $\mathrm{Re}\left[V(j\omega)\right]$ and $\mathrm{Im}\left[V(j\omega)\right]$, which is given by

$$\begin{aligned}\underset{s=j\omega,\omega\in(0,+\infty)}{\Delta\arg}\left[V(s)\right]=&\frac{\pi}{2}\mathrm{sgn}\left[V(0)\right]\cdot\left\{\mathrm{sgn}\left[\mathrm{Re}\left(V(j\omega_0)\right)\right]-2\,\mathrm{sgn}\left[\mathrm{Re}\left(V(j\omega_1)\right)\right]\right.\\&\left.+2\,\mathrm{sgn}\left[\mathrm{Re}\left(V(j\omega_2)\right)\right]+\cdots+(-1)^e 2\,\mathrm{sgn}\left[\mathrm{Re}\left(V(j\omega_e)\right)\right]\right\}\quad\text{for } v \text{ odd}\end{aligned}$$

or

$$\begin{aligned}\underset{s=j\omega,\omega\in(0,+\infty)}{\Delta\arg}\left[V(s)\right]=&\frac{\pi}{2}\mathrm{sgn}\left[V(0)\right]\cdot\left\{2\,\mathrm{sgn}\left[\mathrm{Im}\left(V(j\omega_0)\right)\right]-2\,\mathrm{sgn}\left[\mathrm{Im}\left(V(j\omega_1)\right)\right]\right.\\&\left.+2\,\mathrm{sgn}\left[\mathrm{Im}\left(V(j\omega_2)\right)\right]+\cdots+(-1)^e 2\,\mathrm{sgn}\left[\mathrm{Im}\left(V(j\omega_e)\right)\right]\right\}\quad\text{for } v \text{ even}\end{aligned}$$

Here, $\omega_0<\omega_1<\omega_2<\cdots<\omega_e$ are the zeros of $\mathrm{Im}\left[V(j\omega)\right]$ for $v$ odd or those of $\mathrm{Re}\left[V(j\omega)\right]$ for $v$ even.

## V. Delay-Margin Maximum and Local Controller Design of Consensus Network

In order to implement the local controller easily, the simple first-order controller is considered.

$$C(s)=\frac{g}{s+f} \tag{29}$$

Under the constrait that the outputs of the system with the local controller converge to the same value as those of the original system, the following equation has to be satisfied.

$$\left.\frac{P(s)C(s)}{1+P(s)C(s)}\right|_{s=0}=\left.P(s)\right|_{s=0} \tag{30}$$

Substituting (1) and (29) into (30), we have

$$g=\frac{\mu_0}{\mu_0-\upsilon_0}f \tag{31}$$

For the special case $\mu_0=\upsilon_0$, the controller $C(s)$ can be defined as $C(s)=g/s$. If $\mu_0=0$, the relation between $g$ and $f$ can be obtained approximately by giving $\mu_0$ a sufficient small positive number.

Substituting (1) and (29) into (11) yields

$$\begin{aligned}\delta(s)=&sD(s)R(s)+f\left\{D(s)R(s)\right.\\&\left.+\frac{\mu_0}{\mu_0-\upsilon_0}\left[D(s)Q(s)-D(s)R(s)+N(s)R(s)e^{-\theta s}\right]\right\}\end{aligned} \tag{32}$$

It is seen that Equation (32) has the same form as the quasipolynomial (12). Let

$$K=f, \tag{33}$$

$$U(s)=sD(s)R(s), \tag{34}$$

and

$$V(s)=D(s)R(s)+\frac{\mu_0}{\mu_0-\upsilon_0}\left[D(s)Q(s)-D(s)R(s)+N(s)R(s)e^{-\theta s}\right] \tag{35}$$

Substituting (34) and (35) into (13), we have

$$F(s)=\frac{1}{s}\left\{1+\frac{\mu_0}{\mu_0-\upsilon_0}\left[\frac{Q(s)}{R(s)}+\frac{N(s)}{D(s)}e^{-\theta s}-1\right]\right\} \tag{36}$$

If $L$, $\tau$ and $P(s)$ are given, the frequency response data of $V(s)$ and $F(s)$ can be obtained. Thus, the set of $f$ ensuring the stability of (32), i.e. the stability of the whole consensus network, can be derived based on Theorem 1. For a given controlled agent, the algorithm of determining the delay margin and the corresponding low-order controller is presented as follows:

**Step 1:** Let $\tau=(\eta-1)h$, where $\eta$ is a integer starting from 0 and $h$ is a small value denoting the step length, and increase the value of $\eta$ by 1;

**Step 2:** Determine the value of $\Delta\arg\left[V(j\omega)\right]$ for $\omega$ from 0 to $+\infty$ in terms of Remark 2, where $V(s)$ is given in (35);

**Step 3:** Obtain the frequency response data of $F(s)$ in (36);

**Step 4:** Derive the feasible set of $f$ based on Theorem 1;

**Step 5:** Go to Step 1 until the intersection of all the resultant feasible sets of $f$ is empty.

**Step 6:** It can be derived that the delay margin $\tau^*$ is $(\eta^*-1)h$, where $\eta^*$ is the first value that causes the empty set of $f$. The value of the control parameter chosen in the intersection of all the resultant stabilizing sets of $f$ for $\tau\in[0,\tau^*]$ can guarantee the stability of the system when the delay $\tau$ changes in the range $[0,\tau^*]$.

The maximal delay margin and the corresponding feasible set of the controlled parameters can be obtained by combining all the resultant delay margins for each controlled agent.

***Remark 3***: For the control parameter chosen in the intersection of all the resultant stabilizing set of $f$ for $\tau\in[0,\tau^*]$, the zero exclusion criterion for interval quasipolynomial families in [23] can be further employed to examine whether there exist some values of the delay in $[0,\tau^*]$ causing the instability of the system.

**Theorem 2** (Zero exclusion criterion): Denote the quasipolynomial family as $W=\{\delta(s,\tau):0\le\tau\le\tau^*\}$ and define $\chi_{\omega,\tau}=\{\delta(j\omega,\tau):0\le\tau\le\tau^*\}$. Suppose that at lease one $\delta\in W$ is stable, and $0\notin\chi_{0,\tau}$. The quasipolynomial family $W$ is stable if and only if $0\notin\chi_{\omega,\tau}$ for each real $\omega\in(0,\omega^*]$ and $\tau\in[0,\tau^*]$, where $\omega^*$ is a sufficiently large value.

Define that $|\chi_{\omega,\tau}|=\{|\delta(j\omega,\tau)|:0\le\tau\le\tau^*\}$. It is seen that $0\notin\chi_{\omega,\tau}$ is equivalent that $0\notin|\chi_{\omega,\tau}|$. For each real $\omega$, $|\chi_{\omega,\tau}|$ is a continuous function with respect to $\tau$ and the minimum of $|\chi_{\omega,\tau}|$, denoted as $|\chi_\omega|$, can be derived for $\tau\in[0,\tau^*]$. Thus, the quasipolynomial family $W$ is stable if and only if at lease one $\delta\in W$ is stable and the minimum of $|\chi_\omega|$ in $[0,\omega^*]$ is larger than zero.

***Remark 4***: The objective of introducing the local controller is to improve the robustness of the communication delay. If the form of the local controller is fixed, in some cases it may not be guaranteed that the delay margin is enlarged in comparison with that of the original system. In order to solve the problem, the controller with more tuning parameters can be employed or more local controllers are introduced to other agents. Such the problem will be investigated in the future study.

## VI. NUMERICAL EXAMPLE

***Example 1*** Consider a consensus network of six agents described by

$$P(s)=\frac{1}{s+2}$$

The interconnection topology is sketched in Fig. 2 .

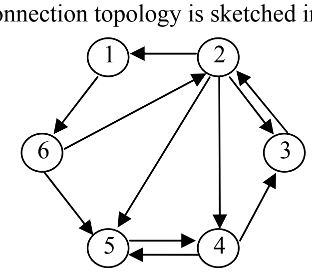



Fig. 2. Communication topology

From Fig.2, it is known that the Laplacian matrix is

$$L=\begin{bmatrix}1&0&0&0&0&-1\\-0.25&1&-0.25&-0.25&-0.25&0\\0&-1&1&0&0&0\\0&0&-0.5&1&-0.5&0\\0&0&0&-1&1&0\\0&-0.5&0&0&-0.5&1\end{bmatrix}$$

Suppose that the local controller is imposed on Agent 1. The non-zero eigenvalues of $L$ are 0.4859, 1, 2.428, 3 and 5.0861, and those of $\bar{L}$ in (7) are 0.382,1,1, 2.618 and 3. From (10) and (32), the characteristic equation of the system is given by

$$\begin{aligned}\delta(s)&=s(s+2)(s+2+0.382e^{-\tau s})(s+2+e^{-\tau s})(s+2+2.618e^{-\tau s})\\&+f\big[2(s+2)(s+2+0.4859e^{-\tau s})(s+2+2.428e^{-\tau s})(s+2+5.0861e^{-\tau s})\\&-s(s+2+0.382e^{-\tau s})(s+2+e^{-\tau s})(s+2+2.618e^{-\tau s})\big]\end{aligned} \quad (37)$$

Let

$$U(s)=s(s+2)(s+2+0.382e^{-\tau s})(s+2+e^{-\tau s})(s+2+2.618e^{-\tau s})$$

$$\begin{aligned}V(s)&=2(s+2)(s+2+0.4859e^{-\tau s})(s+2+2.428e^{-\tau s})\\&\cdot(s+2+5.0861e^{-\tau s})-s(s+2+0.382e^{-\tau s})(s+2+e^{-\tau s})\\&\cdot(s+2+2.618e^{-\tau s})\end{aligned}$$

We first consider the case for $\tau=0.8$ and determine the stabilizing set of the control parameter $f$. The highest order of $V(s)$ is 4 and the zeros of $\text{Re}[V(j\omega)]$ are 6.646 and 10.082. From Remark 2, it can be derived that

$$\underset{s=j\omega,\omega\in(0,+\infty)}{\Delta\arg}[V(s)]=0.$$

The curve of $\text{Im}[F(j\omega)]$ with respect to $\omega$ is shown in Fig. 3. From Fig. 3, it is seen that the zeros of $\text{Im}[F(j\omega)]$ are 0, 1.979 and 3.571. Based on the sufficient and necessary condition in Theorem 1, the quasipolynomial (37) is stable if and only if

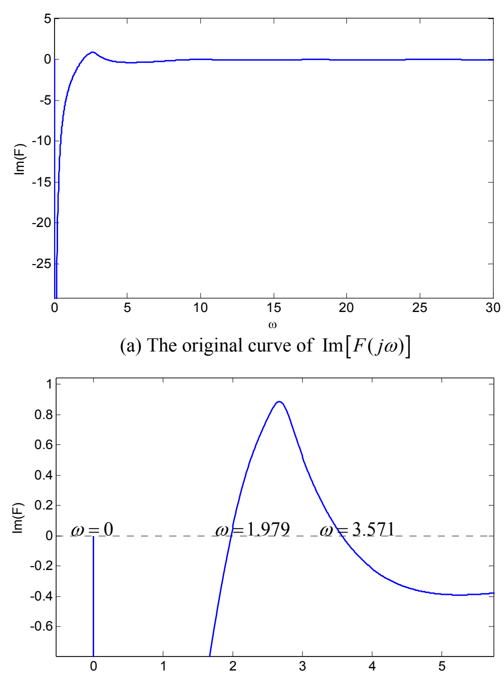
$$i_0-2i_1+2i_2=5$$

Thus, $i_0=i_2=1$ and $i_1=-1$. By using (17), we have

$$0<f<0.5485.$$

The agents can agree on a coordinate using the local controller if $0<f<0.5485$. For example, the value of $f$ is chosen as 0.3 and the local controller is

$$C(s)=\frac{0.6}{s+0.3}$$




(a) The original curve of $\text{Im}[F(j\omega)]$

(b) The magnified image of the curve in (a)

Fig. 3. The curve of $\text{Im}[F(j\omega)]$ with respect to $\omega$

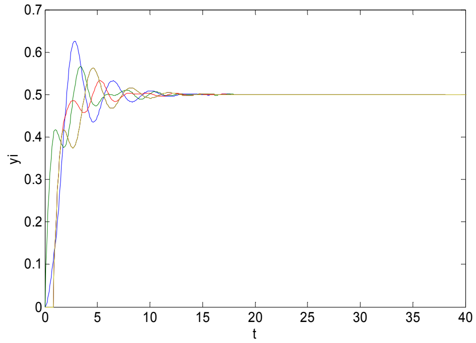



Fig. 4. Simulation results of the interconnected system with the local system for $\tau = 0.8$

When the step signal with magnitude 1 is added to the input of each agent, the outputs of the consensus network with the local controller, which are shown in Fig. 4, are presented. From Fig.4, it is seen that the whole multi-agent system is stable and the outputs of each agent can arrive at the same value.

By employing the above-mentioned algorithm for the values of $\tau$ increasing from 0, it can be derived that the delay margin of $\tau$ is 1.445. For the values of $\tau$ in the interval $[0,1.445]$, the lower bound of the stabilizing set of $f$ is 0. Its upper bound in the interval $[0,0.194)$ is $+\infty$, and in $(0.194,1.445]$, it is described by the curve in Fig. 5. If $\tau > 1.445$, the stabilizing set of $f$ is empty. The upper bound of the stabilizing set of $f$ corresponding to $\tau = 1.445$ is 0.2929. From Fig.5, it is known that the stabilizing value of the control parameter $f$ has to be chosen in the range $(0,0.2929)$ if the delay $\tau$ changes in the interval $(0,1.445]$. Recall the case for $\tau = 0.8$. From Fig.5, it is also known that if $f = 0.3$, the local controller can make the system stable for $\tau$ in $(0,0.8]$. Then, based on Theorem 2, we further examine whether there exist the delay values in $(0,0.8]$ that cause the system instability. It can be computed that if $\omega^*$ is chosen as 50, the minimum of $|\chi_\omega|$ in $[0,\omega^*]$ is 711.62, which is larger than zero, and thus the system is stable for all $\tau$ in $(0,0.8]$.

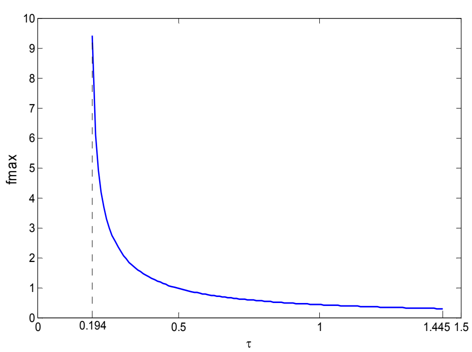



Fig. 5 The upper bound of the stabilizing set of $f$ with respect to $\tau$

Subsequently, the cases for each controlled agent except Agent 1 are all considered. By employing the above-mentioned algorithm, it can be derived that the delay margin of $\tau$ are 0.459, 0.425, 0.446, 0.437 and 0.437 for the choice of Agent 2, Agent 3, Agent 4, Agent 5 and Agent 6, respectively. The results show that, Agent 1 is the best choice, which leads to maximal delay margin 1.445. The corresponding feasible set of the controller parameter $f$ that tolerates the change of the communication in [0,1.445] is $(0,0.2929)$.

For the original multi-agent system without the local controller, the system is stable if and only if the zeros of all the following equations lie in the left-hand complex plane.

$$\delta_1(s) = s + 2 + 0.4859e^{-\tau s}, \ \delta_2(s) = s + 2 + 2.428e^{-\tau s}$$

$$\delta_3(s) = s + 2 + e^{-\tau s}, \ \delta_4(s) = s + 2 + 5.0861e^{-\tau s}$$

In terms of the results in [15], it can be derived that the minimal value among the delay margins of the above-mentioned equations is 0.4223. Thus, the original multi-agent system becomes unstable if $\tau > 0.4223$. In comparison with the delay margin of original multi-agent system, the delay margin is enlarged by imposing the local controller on the first agent. Hence, the local controller in this example improves the robustness of the communication delay.

## VII. Conclusions

The systematic design approach of a local low-order controller for the multi-agent system is studied in this paper to improve the robustness with respect to the uniform communication delay. Different from the existing work using centralized controller of the consensus dynamic of the multi-agent system, only the local information is implemented to regulate the performance of the entire multi-agent system. The stability problem of the multi-agent system with the local low-order controller is reduced to the stabilization problem of a low-order controller for the SISO system with complicated dynamics. By employing the frequency response data, such the stabilization problem is well solved based on the argument principle and the generalized Nyquist criterion. The algorithm determining the maximal delay margin of the multi-agent system is presented and simultaneously a local low-order controller is designed to tolerate the perturbation of the communication delay in a certain range. The proposed method is suitable for the cases of the agents with arbitrary LTI dynamics and complex directed or indirect interconnection.

In the future work, the best choice of the controlled agent will be presented by analyzing the property of the networked topology and the agents, and the local controller design for the improvement of other performance criteria will be considered.

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