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Cooperative Control of Multi-Vehicle Systems Using Cost Graphs and Optimization

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Abstract

We introduce a class of triangulated graphs for algebraic representation of formations that allows us to specify a mission cost for a group of vehicles. This representation plus the navigational information allows us to formally specify and solve tracking problems for groups of vehicles in formations using an optimization-based approach. The approach is illustrated using a collection of six underactuated vehicles that track a desired trajectory in formation.

1 Introduction

Coordination of multi-vehicle systems in a cooperative or competitive manner is a challenging problem with a variety of applications, including formation flight of unmanned air vehicles (UAVs), control of clusters of satellites and telescopes, search and rescue operations, distributed sensory networks, and control of dynamic multi-agent systems in interactive games and animation environments [14].

The use of potential functions and graph theoretic tools for coordination of multi-vehicle systems has greatly increased over the past few years. In [1], distance-based potential functions and gradient-based flows are used to perform missions for multi-vehicle systems by translational and rotational maneuvers. Coordination of a group of nonholonomic kinematic mobile robots using a graph theoretic framework is considered in [2]. A combination of graph theoretic and LMI-based frameworks are applied to control of leader–follower type architectures in [10]. A behavioral leader-follower approach is also proposed in [8].

In this paper, our objective is to develop a framework for automated decision-making for coordination of multi-vehicle systems. We achieve this objective by automatically generating the cost functions needed to pose an optimal control problem. The obtained (centralized) optimal control problem is then solved using the nonlinear trajectory generation (NTG) software package [11]. The issues regarding the distribution of this optimization problem is also discussed.

In [12], we introduced the notion of "formation graphs" and their importance in unique (and unambiguous) representation of multi-vehicle formations. These graphs are used to obtain bounded and distributed control laws for formation stabilization of vehicles with linear (i.e. double-integrator) dynamics. Later, in [13], we formalized the notion of *formations* of multiple agents/vehicles and minimal requirements, in terms of the number of edges, for uniquely specifying a formation. This is done based on the notion of "combinatorial graph rigidity" [7]. A related formalism is suggested in [4] that relies on "infinitesimal rigidity".

Here, we add specification of "foldability" [12] to the definition of a formation graph. In addition, we explicitly specify the required cost for navigation and tracking in formation. In [3], the problem of formation stabilization using Model Predictive Control (MPC) is explored. Here, we aim at developing a rather general framework for automated decision-making for cooperative execution of missions performed by multiple vehicles.

The outline of the paper is as follows. In Section 2, we define formation graphs and provide some background on graph theoretic notions used in this paper. The main optimization problem is formulated in Section 3. The method for construction of the costs for formation stabilization, collision avoidance, and tracking is given in Section 4. The dynamics of the vehicles is explained in Section 5. In Section 6, the simulation results for six vehicles are presented. Finally, concluding remarks are made in Section 7.

2 Formation Graphs and Deviation Variables

In this section, we provide some background on graph theory with application to representation and manipulation of formations of multiple vehicles. We denote a graph by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of vertices and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges of the graph. Throughout this paper, we assume all the graphs are *undirected* (unless stated otherwise) with no edges $(v_i, v_i), \forall i \in \mathcal{I}$ from a node to itself. Each edge is denoted by $e_{ij} = (v_i, v_j) \in \mathcal{E}$ or $ij \in \mathcal{E}$ for simplicity of notation where $i, j \in \mathcal{I} = \{1, \ldots, n\}$. An *orientation of the edges* of the graph, $\mathcal{E}_o \subset \mathcal{E}$, is the set of edges of the graph which contains one and only one of the two permutations of $ij \in \mathcal{E}$ (ij or ji) for all the edges $ij \in \mathcal{E}$.

A *triangulated graph* is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ with the *set of faces* $\mathcal{F} \subset \mathcal{V} \times \mathcal{V} \times V$ with elements $f_{ijk} = (v_i, v_j, v_k)$ or simply ijk ($i, j, k \in \mathcal{I}$) satisfying the *consistency condition* that for all faces $f_{ijk} = (v_i, v_j, v_k) \in \mathcal{F}$, the following holds

$$(v_i, v_j) \in \mathcal{E}, (v_j, v_k) \in \mathcal{E}, (v_k, v_i) \in \mathcal{E}. \tag{1}$$

Similarly, an *orientation of the faces* of a triangulated graph $\mathcal{G}$ is a set of faces $\mathcal{F}_o \subset F$ that contains one out of the six permutations of each face $ijk \in \mathcal{F}$. Define the *dual graph* $D(\mathcal{G})$ of a triangulated graph $\mathcal{G}$ as a graph with $|\mathcal{F}_o|$ number of nodes, one corresponding to each (oriented) face of $\mathcal{G}$. There is an edge between two distinct faces $f_1, f_2 \in \mathcal{F}_o$ if and only if f_1 and f_2 share a common edge $e \in \mathcal{E}$. A *triangulated formation graph* is a quintuple

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{D}, \mathcal{F}, \mathcal{A}), \tag{2}$$

with a connected dual graph $D(\mathcal{G})$. Let $q_i = (x_i, y_i) \in \mathbb{R}^2$ denote the position of the node v_i. Here, $\mathcal{D} = \{d_{ij} : ij \in \mathcal{E}\}$ is the set of desired distances and $\mathcal{A} = \{a_{ijk}, ijk \in \mathcal{F}\}$ is the set of desired areas of triangular faces. The *signed area of a triangle* is given by

$$h(q_i, q_j, q_k) = \det \begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix} = (q_k - q_i)^T S(q_j - q_i), \tag{3}$$

where

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

We use *Delaunay triangulation* [6] of a set of points to

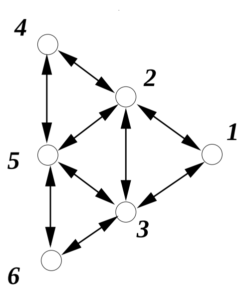

Figure 1: A triangulated six–vehicle formation.

obtain our triangulated graphs. In Figure 1 a triangulated formation of six vehicles is shown with

$$\begin{aligned} \mathcal{V} &= \{1, 2, 3, 4, 5, 6\}, \\ \mathcal{E}_o &= \{12, 13, 23, 24, 25, 35, 36, 45, 56\}, \\ \mathcal{F}_o &= \{123, 245, 253, 356\}. \end{aligned} \tag{5}$$

Fix the edge and face orientations of the triangulated graph $\mathcal{G}$ such that for all the faces $a_{ijk} \geq 0$, i.e. if for the face $ijk \in \mathcal{F}_o \subset \mathcal{F}$, $a_{ijk} < 0$ then replace the triplet $(v_i, v_j, v_k) \in \mathcal{F}_o$ by (v_j, v_i, v_k) to change the sign of the determinant in (3). The following *edge and face deviation variables* (also known as shape variables [13]) associated with the edges and faces of the triangulated graph $\mathcal{G}$ are defined, respectively, as

$$\begin{aligned} \eta_{ij} &= \|q_j - q_i\| - d_{ij}, \\ \delta_{ijk} &= q_{ik} \otimes q_{ij} = (q_k - q_i)^T S(q_j - q_i) - a_{ijk}, \end{aligned} \tag{6}$$

for edges $ij \in \mathcal{E}_o$ and faces $ijk \in \mathcal{F}_o$. In addition, $q_{rs} := q_s - q_r$ and the tensor product $\otimes$ is defined by $\alpha \otimes \beta := \alpha^T S \beta$ for $\alpha, \beta \in \mathbb{R}^2$.

Let $p_i = \dot{q}_i$ denote the velocity of each node $v_i \in \mathcal{V}$. Then, the *edge and face deviation rate variables* (also known as shape velocities [13]) associated with the set of edges and faces of the graph $\mathcal{G}$ are defined, respectively, as follows:

$$\begin{aligned} \nu_{ij} &= \mathbf{n}_{ij}^T \cdot (p_j - p_i), \\ \xi_{ijk} &= (p_k - p_i)^T S(q_j - q_i) + (q_k - q_i)^T S(p_j - p_i), \end{aligned} \tag{7}$$

where $\nu_{ij} = \dot{\eta}_{ij}$, $\xi_{ijk} = \dot{\delta}_{ijk}$, $\mathbf{n}_{ij} = q_{ij}/\|q_{ij}\|$ for $q_i \neq q_j$. Using the notation $p_{rs} = p_s - p_r$ and $\alpha^{\perp} := S\alpha$ (thus $\alpha \otimes \beta = \alpha^T \cdot \beta^{\perp}$), we can simplify the expression for the shape velocities as

$$\begin{aligned} \nu_{ij} &:= \mathbf{n}_{ij}^T \cdot p_{ij}, \\ \xi_{ijk} &:= p_{ik} \otimes q_{ij} + q_{ik} \otimes p_{ij} = p_{ik}^T \cdot q_{ij}^{\perp} - p_{ij}^T \cdot q_{ik}^{\perp}. \end{aligned}$$

3 Formulation of the Optimization Problem

In this paper, we are interested in constructing a meaningful integrated cost function $L(x, u)$ and and terminal cost $G(x)$ for the purpose of performing a mission in a cooperative fashion using multiple vehicles with underactuated dynamics and constrained controls. More precisely, the mission of interest is *tracking in formation* for a group of underactuated hovercraft-type mobile robots and bounded control (i.e. $u_i \in \mathcal{U}, \forall i \in \mathcal{I}$ where $\mathcal{U} \ni 0$ is a compact set (i.e. $\mathcal{U} = [0 \; u_{max}]^2 \subset \mathbb{R}^2$)). The overall optimization problem can be expressed as the following:

$$\mathcal{J}(x_0) = \min_{\substack{x(0) = x_0, i \in \mathcal{I}, \\ \dot{x}_i = f(x_i, u_i), \\ u_i \in \mathcal{U}}} \int_0^T L(x, u)dt + G(x(T)), \tag{8}$$

where $x = (x_1, \ldots, x_n)$ and $G(x)$ is a control Lyapunov function (CLF) for the concatenated dynamics of all vehicles $\dot{x} = F(x, u)$ (the ith element of F is $f(x_i, u_i)$). To construct $L(x, u)$ and $G(x)$, we need to add several cost functions that each have a specific role in achieving our coordinated tracking objective. In the following, we explain how we obtain the terms that constitute $L(x, u)$ and $G(x)$. The optimal control resulting from (8) is implemented in a receding horizon fashion. A survey that details a generalized receding horizon control, or model predictive control, formulation for (nominal) asymptotic stability of nonlinear and constrained systems is given in [9]. Our implementation is akin to the formulation with no terminal constraint and a CLF terminal cost.

The optimal control problem is solved using the Nonlinear Trajectory Generation (NTG) software package, developed at Caltech. A detailed description of NTG as a real-time trajectory generation package for constrained mechanical systems is given in [11]. The package is based on finding trajectory curves in a lower dimensional space and parameterizing these curves by B-splines. Sequential quadratic programming (SQP) is used to solve for the B-spline coefficients that optimize the performance objective, while respecting dynamics and constraints. The package NPSOL [5] is used to solve the SQP problem.

4 Formation/Tracking Cost Decomposition

We demonstrate that the integrated cost for the problem of tracking in formation is constructed by decomposition of the task to formation stabilization with collision avoidance plus tracking.

4.1 Formation Cost

Let $\sigma(x) : \mathbb{R} \to \mathbb{R}$ be a continuous and locally Lipschitz function satisfying the following properties: i) $\sigma(0) = 0$, ii) $(x - y)(\sigma(x) - \sigma(y)) > 0, \forall x \neq y$. Then, based on ii), $x\sigma(x) > 0$ and $\phi(x) = \int_0^x \sigma(s)ds$ is a positive definite and convex function which we refer to as a *cost function*. As an example, consider

$$\sigma(x) = \frac{x}{\sqrt{x^2+1}} \implies \phi(x) = \sqrt{x^2+1} - 1. \tag{9}$$

We define the *potential-based cost* and the *kinetic-based cost* associated with the formation graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{D}, \mathcal{F}, \mathcal{A})$, respectively, as the following

$$\begin{array}{rcl} V_{\mathcal{G}}(q) & := & \sum_{ij\in\mathcal{E}_o} \phi_1(\eta_{ij}) + \sum_{ijk\in\mathcal{F}_o} \phi_2(\delta_{ijk}), \\ T_{\mathcal{G}}(q,p) & := & \sum_{ij\in\mathcal{E}_o} \phi_3(\nu_{ij}) + \sum_{ijk\in\mathcal{F}_o} \phi_4(\xi_{ijk}), \end{array} \tag{10}$$

where $\phi_i(x) = \int_0^x \sigma_i(s)ds, i = 1, 2, 3, 4$ and the σ_i's satisfy conditions i) and ii). For the special case that $\sigma_i(s) = s$, we have $\phi_i(x) = x^2/2$ and both $V_{\mathcal{G}}, T_{\mathcal{G}}$ are quadratic functions of the shape variables and velocities. Here, we use $\phi_1(x) = \phi_2(x) = \phi_3(x) = \phi_4(x) = x^2/2$ (note that the choice of $\phi_1(x) = \phi_2(x) = \sqrt{x^2+1} - 1$ is possible as well).

In general, corresponding to each edge shape variable η_{ij}, there exists a cost function $\phi_{ij}(x)$. Let $\Phi_\eta, \Phi_\nu, \Phi_\delta, \Phi_\xi$ denote the set of cost functions associated with the set of variables $\eta_{ij}, \nu_{ij}, \delta_{ijk}, \xi_{ijk}$, respectively. We refer to $\Phi_f = (\Phi_\eta, \Phi_\nu, \Phi_\delta, \Phi_\xi)$ as the set of formation costs corresponding to graph $\mathcal{G}$. The pair $(\mathcal{G}, \Phi_f)$ is called a (formation) *cost graph*. We refer to the *formation Hamiltonian* [12] given by

$$H_{\mathcal{G}}(q,p) = T_{\mathcal{G}}(q,p) + V_{\mathcal{G}}(q) \tag{11}$$

as the *formation cost* induced by the cost graph $(\mathcal{G}, \Phi_f)$.

Definition 1. (equilibrium state) We say $x^* = (q^*, p^*) \in \mathbb{R}^{4n}$ is an *equilibrium state* of the cost graph $(\mathcal{G}, \Phi_f)$ if and only if $H_{\mathcal{G}}(q^*, p^*) = 0$.

Definition 2. (orbit) Let $(R, b) \in SO(2) \times \mathbb{R}^2$ satisfy the kinematic equations

$$\left\{ \begin{array}{rcl} \dot{b} & = & v, \\ \dot{R} & = & R\hat{\omega}. \end{array} \right. \tag{12}$$

Define the elements of the vectors $\bar{q}, \bar{p}$ as

$$\left\{ \begin{array}{rcl} \bar{q}_i & = & Rq_i + b, \\ \bar{p}_i & = & Rp_i + R\hat{\omega}q_i + v. \end{array} \right. \tag{13}$$

The *orbit* of a point $x = (q, p)$ associated with $(v(\cdot), \omega(\cdot))$ is defined as $[x](t) := \mathrm{col}(\bar{q}(t), \bar{p}(t))$ where $[x](0) = x$.

Proposition 1. *The formation cost $H_{\mathcal{G}}$ is invariant along any orbit of the point $x = (q, p) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$.*

Proof. The proof is by direct calculation (the property $RS = SR$ is the key in this proof). □

4.2 Collision Avoidance Cost

One of the main challenges in formation stabilization for multiple vehicles, regardless of tracking, is collision avoidance between vehicles that get too close to each other. Let q_i, q_j be two vehicles that are not necessarily neighbors in graph $\mathcal{G}$. Imagine a circular *protection zone* with radius r_0 around each vehicle. Define the *safety variable* between any two arbitrary vehicles v_i and v_j as

$$\mu_{ij} = \|q_j - q_i\| - r_0. \tag{14}$$

Apparently, two vehicles collide if $\mu_{ij} = -r_0$. The objective is to satisfy $\mu_{ij} > -r_0$. Furthermore, if two vehicles are already apart by r_0, or $\mu_{ij} \geq 0$, there is no need to worry about collision avoidance. Since no vehicle can physically apply an infinite force to avoid another vehicle, it is not reasonable to use potential (or barrier) functions of the type $V(q_i, q_j) = -\log(\|q_i - q_j\|)$ or $V(q_i, q_j) = 1/\|q_i - q_j\|$ between two vehicles with $\mu_{ij} < 0$. A numerically feasible alternative to applying forces with singularities is to use a constant repelling force between two vehicles with $\mu_{ij} < 0$ and applying no force when $\mu_{ij} \geq 0$, i.e. we could use the potential function $V(q_i, q_j) = \psi(\|\mu_{ij}\|)$ where

$$\psi(x) := -\min\{0, x\} = \frac{-x + |x|}{2}. \tag{15}$$

A smooth approximation of this function is given by

$$\psi_\epsilon(x) : \frac{-x + \sqrt{x^2 + \epsilon^2}}{2}, \quad 0 < \epsilon \ll 1 \tag{16}$$

Both $\psi(x), \psi_\epsilon(x)$ are depicted in Figure 2. Notice that

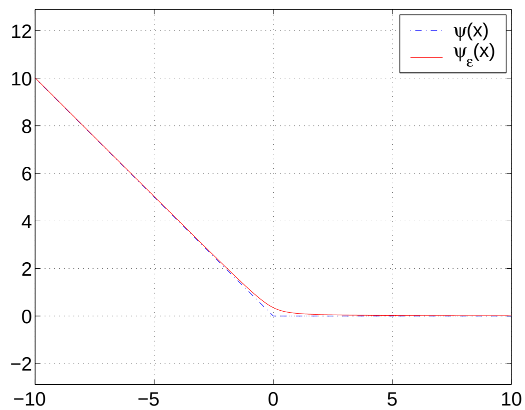

Figure 2: The function $\psi(x)$ and its smooth approximation $\psi_\epsilon(x)$.

$$\begin{aligned} f_\epsilon(x) &= -\nabla\psi_\epsilon(x) = \frac{1}{2}(1 - \frac{x}{\sqrt{x^2+\epsilon^2}}) \\ &= \frac{1}{2}(1 - \sigma(\frac{x}{\epsilon})) > 0, \forall x. \end{aligned} \tag{17}$$

This means that for $x > \epsilon_0 = 5\epsilon$, $f_\epsilon(x) \approx 0$. Now, define the following continuous approximation of $f_\epsilon(x)$

$$\tilde{f}_\epsilon(x) := \begin{cases} f_\epsilon(x) - f_\epsilon(\epsilon_0), & x \leq \epsilon_0 \\ 0, & x > \epsilon_0, \end{cases} \tag{18}$$

which satisfies the property $\tilde{f}_\epsilon(x) \geq 0, \forall x$. Define the positive semidefinite function

$$\tilde{\psi}_\epsilon(x) := -\int_{\epsilon_0}^{x} \tilde{f}_\epsilon(s) ds, \tag{19}$$

then, we have

$$\tilde{\psi}_\epsilon(x) = 0, \forall x \geq \epsilon_0, \ \tilde{\psi}_\epsilon(x) > 0, \forall x < \epsilon_0. \tag{20}$$

Let $\mathcal{N}_i$ denote all the *spatial neighbors* of vehicle v_i defined as follows:

$$\mathcal{N}_i := \{j \in \mathcal{I} : \|q_j - q_i\| \leq r_0 + \epsilon_0\} \tag{21}$$

By definition, we have $\mathcal{N}_i := \{j \in \mathcal{I} : \mu_{ij} \leq \epsilon_1\}$. Let us define the following *collision avoidance cost*

$$V_{col}(q) = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{N}_i} \psi_\epsilon(\mu_{ij}). \tag{22}$$

If no two vehicles are spatial neighbors of each other, i.e. $\mu_{ij} > \epsilon_0, \forall i < j \in I$, then the collision avoidance cost is zero ($V_{col}(q) = 0$).

Remark 1. The method that is presented here for construction of $V_{col}(q)$ is a *distributed way of defining a collision avoidance cost for a multi-vehicle system* as compared to a centralized approach. In a centralized cost, there is a pair-wise potential function $\psi_\epsilon(\mu_{ij})$ between any two vehicles that will never vanish no matter how far the vehicles are from each other. The cost $V_{col}(q)$ can be calculated in a distributed manner because each vehicle needs to know about its local spatial neighborhood $\mathcal{N}_i$ and not all other vehicles.

4.3 Tracking Cost

To perform the tracking, we choose a subgroup of vehicles called the *core vehicles* to be in charge of doing the navigation and tracking in addition to trying to stay in formation with *follower vehicles*, i.e. the vehicles that are not among the core vehicles. For doing so, we could choose a subgraph of the triangulated graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{D}, \mathcal{F}, \mathcal{A})$ called $\mathcal{G}_c$ which contains a subset of the faces in $\mathcal{F}$ with their corresponding edges and vertices. Clearly, $\mathcal{G}_c$ is a rigid and unfoldable formation graph. In its simplest form, $\mathcal{G}_c$ consists of a single triangle with the set of vertices $\mathcal{V}_c = \{v_1, v_2, v_3\}$. For simplicity of notation, we assume all vehicles $v_i, i > 3$ are the followers. In general, the index set of the set of the core vehicles and the followers are denoted by J_c and J_f, respectively. Notice that $J_c \cup J_f = \mathcal{I}, J_c \cap J_f = \emptyset$. Among the core vehicles, we choose one of the vehicles to be the *attitude leader* v_{j^*} [13], i.e. $j^* \in J_c$. Define the *position and attitude* of the core formation as

$$\begin{aligned} q_c &= \frac{1}{|J_c|} \textstyle\sum_{j \in J_c} q_j, \\ r_c &= \frac{q_c - q_{j^*}}{\|q_c - q_{j^*}\|}, \end{aligned} \tag{23}$$

where $|J_c|$ denotes the number of members of the set J_c. Let $R_c = [r_c | r_c^\perp] \in SO(2)$, then $\dot{R}_c = R_c \hat{\omega}_c$ defines ω_c satisfying $\dot{\theta}_c = \omega_c$. Here, θ_c is the angle of r_c with the horizontal axis of the reference frame. Similarly, we can define the velocity of $p_c = \dot{q}_c$

$$p_c = \frac{1}{|J_c|} \sum_{j \in J_c} p_j \tag{24}$$

This allows us to view the whole set of core vehicles as a single vehicle called the *aggregated vehicle* with the following dynamics:

$$\text{aggregated vehicle:} \begin{cases} \dot{q}_c &= p_c, \\ \dot{p}_c &= u_c, \\ \dot{\theta}_c &= \omega_c, \\ \dot{\omega}_c &= \tau_c, \end{cases} \tag{25}$$

where $x_c = (q_c, p_c, \theta_c, \omega_c) \in \mathbb{R}^6$, $u_c \in \mathbb{R}^2$, and $\tau_c \in \mathbb{R}$. Both u_c and τ_c can be directly calculated from the vehicles dynamics and the definition of q_c, θ_c. Now, define a virtual vehicle called the *navigator* with the same dynamics as the aggregated vehicle:

$$\text{navigator:} \left\{ \begin{array}{rcl} \dot{q}_d & = & p_d, \\ \dot{p}_d & = & u_d, \\ \dot{\theta}_d & = & \omega_d, \\ \dot{\omega}_d & = & \tau_d, \end{array} \right. \tag{26}$$

with *the initial conditions that are identical to the initial conditions of the aggregate vehicle.* Let $y_r(t) = (q_r(t), \theta_r(t)) : \mathbb{R} \to \mathbb{R}^3$ be a reference trajectory for the purpose of tracking. The navigator applies the following control input to achieve asymptotic output tracking of y_r:

$$\left\{ \begin{array}{rcl} u_d & = & -c_1(q_d - q_r) - c_2(p_d - \dot{q}_r) + \ddot{q}_r \\ \tau_d & = & -c_1(\theta_d - \theta_r) - c_2(\omega_d - \dot{\theta}_r) + \ddot{\theta}_r \end{array} \right. \tag{27}$$

with $c_1, c_2 > 0$. With the idea of trying to force the aggregated vehicle to follow the navigator, we define the following *tracking cost function*:

$$\begin{array}{rcl} H_{tr}(x_c, x_d) & := & \phi_5(q_c - q_d) + \phi_6(p_c - p_d) \\ & + & \phi_7(\theta_c - \theta_d) + \phi_8(\omega_c - \omega_d), \end{array} \tag{28}$$

where $x_d = (q_c, p_c, \theta_c, \omega_c)$, $\phi_5(x) = \phi_6(x) = \|x\|^2/2$, and $\phi_7(x) = \phi_8(x) = x^2/2$. We take

$$\begin{array}{rcl} L(x, x_d, u) & := & H_{\mathcal{G}}(q, p) + H_{tr}(x_c, x_d) + V_{col}(q) \\ G(x, x_d) & := & H_{\mathcal{G}}(q, p) + H_{tr}(x_c, x_d) \end{array} \tag{29}$$

The reason that the terminal cost $G(x, x_d)$ does not contain any collision avoidance cost is that it is assumed that *in the final desired formation no vehicle is in the protection zone of any other vehicle*, i.e. $\mathcal{N}_i = \emptyset, \forall i \in \mathcal{I}$.

Definition 3. (tracking in formation) We say a group of vehicles asymptotically achieve tracking in formation if and only if the following conditions hold:

i) The formation of the vehicles asymptotically converges to the desired formation which is the equilibrium state of the cost graph $(\mathcal{G}, \Phi_f)$.

ii) The output $y_c = (q_c, \theta_c)$ of the aggregated vehicle of the set of core vehicles asymptotically tracks the output y_r.

Remark 2. (distributed optimal control) The cost specification procedure detailed in the previous sections is beneficial not only because it is scalable to an arbitrary number of vehicles, but it also lends itself to a distributed implementation. The formation cost is additive, and consequently separable, with respect to the (edge and face) deviation and deviation rate variables. Note that even a single deviation variable is non-separable with respect to the positions of the end points of an edge. Assuming an agent is locally solving an optimization for the purpose of computing its own control, each such local optimization must account for the state of the neighbors of that agent on the graph. We intend to formally address the problem of distributing an optimal control problem for multi-vehicle system in the future.

5 Underactuated Robot Dynamics

In this paper, we assume each vehicle is a hovercraft mobile robot with the following dynamics:

$$\left\{ \begin{array}{rcl} \dot{q}_i & = & p_i \\ m\dot{p}_i & = & \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} (u_i^1 + u_i^2) - k_1 \cdot p_i \\ \dot{\theta}_i & = & \omega_i \\ I_0\dot{\omega}_i & = & r_0(u_i^1 - u_i^2) - k_2 \cdot \omega_i \end{array} \right. \tag{30}$$

where $0 \leq k_1/m, k_2/I_0 \ll 1$ and $m, I_0, r_0 > 0$ are physical parameters of the vehicle dynamics. In addition, the *control inputs of each vehicle are positive and bounded* or $0 \leq u_i^1, u_i^2, \leq u_{max}$. In other words, the control $u_i = (u_i^1, u_i^2)$ belongs to a compact set $\mathcal{U}$ as

$$u_i \in \mathcal{U} = [0, u_{max}] \times [0, u_{max}] \tag{31}$$

For simplicity of notation, we represent the dynamics of each vehicle in (30) as the following

$$\dot{x}_i = f(x_i, u_i) = Ax_i + B(\theta_i)u_i, \quad u_i \in \mathcal{U}, i \in \mathcal{I} \tag{32}$$

with the state $x_i = \text{col}(q_i, p_i, \theta_i, \omega_i) \in \mathbb{R}^6$ and obvious definitions of A and $B(\theta_i)$. Clearly, the system in (30) is an *underactuated system* with 3 degrees of freedom (DOF) and 2 control inputs.

6 Simulation Results

In this section we present the simulation results for tracking in formation for a group of six underactuated mobile robots with dynamics given in (30). The graph for the desired formation is shown in Figure 1 (each edge is of length 1). We choose $J_c = \{2, 3, 5\}$ as the set of indices of the core vehicles.

In the simulations that follow, the horizon time is 6 seconds and the update time is 1 second. Figure 3 (a) shows a 6-vehicle formation tracking a reference moving with constant velocity and heading. The vehicles are multi-colored and the reference vehicle is clear (white). The trajectories of each vehicle are denoted by lines with "x" marks for each receding horizon update.

The vehicles are initially lined up with a velocity of 1 m/s in the horizontal direction, equal to the reference velocity. Note that without the collision avoidance cost term, vehicles 4 and 3 collide around 2.5 seconds. Figure 3 (b) shows snapshots of the evolution of the formation for the first 5 seconds of tracking.

7 Conclusion

In this paper, a framework is developed for automated decision-making for multi-vehicle systems to perform coordinated tasks. Graph theoretic tools were introduced for algebraic specification of a formation in an

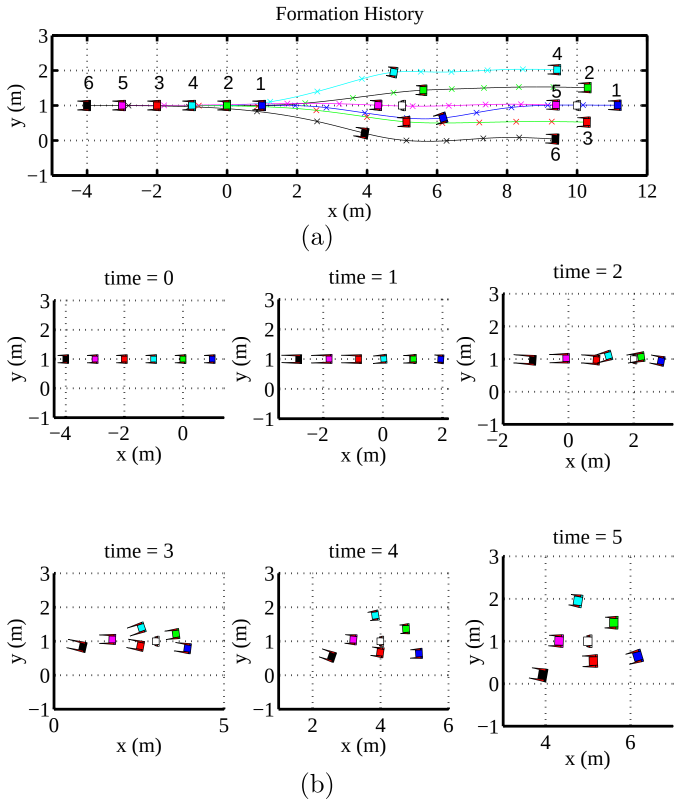

Figure 3: Tracking and formation stabilization for a six-vehicle formation: (a) the evolution and the path of the formation, (b) snapshots of the evolution of the formation (note: the two cones at the sides of each vehicle show the magnitudes of the control inputs).

unambiguous way based on the notion of "formation graphs". This specification allowed us to construct cost functions for formation stabilization, collision avoidance, and tracking which constitute three terms of the integrated cost and terminal cost of a finite horizon optimal control problem. We presented simulation results for tracking of six underactuated mobile robots moving in formation. The obtained optimal control problem is solved using the NTG software package. We also discussed the issues regarding the distribution of the main optimization problem in this paper.

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