Nonlinear Control of Mechanical Systems: A Lagrangian Perspective

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Abstract. Recent advances in geometric mechanics, motivated in large part by applications in control theory, have introduced new tools for understanding and utilizing the structure present in mechanical systems. In particular, the use of geometric methods for analyzing Lagrangian systems with both symmetries and non-integrable (or nonholonomic) constraints has led to a unified formulation of the dynamics that has important implications for a wide class of mechanical control systems. This paper presents a survey of recent results in this area, focusing on the relationships between geometric phases, controllability, and curvature, and the role of trajectory generation in nonlinear controller synthesis. Examples are drawn from robotics and flight control systems, with an emphasis on motion control problems.

Key Words. Geometric mechanics, nonlinear control, Lagrangian dynamics, motion control.

1. INTRODUCTION

Mechanical systems form an important class of nonlinear control systems that have widespread application in science and industry. Although mechanical systems lie at the heart of some of the most challenging applications in control theory, many nonlinear control methodologies do not take advantage of the rich structure of these systems. Indeed, the most common technique for analyzing and controlling these systems is to convert the equations of motion from second order differential equations to first order differential equations and then linearize about an operating point. This approach destroys important geometric information which can be used to improve the global behavior of the closed loop system.

The purpose of this paper is to describe some of the special features of mechanical systems and indicate how recently developed techniques can lead to deeper understanding of the behavior of the system, and better performance from nonlinear controllers. The main emphasis is on *motion control*—movement from one point in Euclidean space to another—where the geometric nature of the nonlinearities plays a strong role. In addition, we concentrate on mechanical systems whose unforced dynamics are described by Lagrangian mechanics.

1.1. *Recent Advances in Lagrangian Mechanics*

Traditionally, work in geometric mechanics has ignored the role of constraints and external forces in the dynamics of the system. As a consequence, most current nonlinear control techniques do not make use of the extra geometric structure present in these systems. However, applications in robotics and other areas have stimulated interest in mechanical systems and there has been substantial progress over the last several years in applying techniques from dynamical systems and geometric mechanics to control problems.

A central theme in geometric mechanics is the role of symmetries in determining the evolution of the system. Given a group of transformations that leave the dynamics invariant, it is possible to make powerful statements about the global behavior of the system, and to utilize techniques such as reduction and reconstruction to simplify the description of the equations of motion. For motion control problems, symmetries associated with the group of rigid transformations play a natural role. By studying the geometric phase (or holonomy) associated with these types of symmetries, it has become possible to achieve a much more detailed understanding of the dynamics and control of mechanical systems.

One example of the application of this new theory is in the context of robotic locomotion. For a large class of land-based locomotion systems—including legged robots, snake robots, and wheeled mobile robots—it is possible to model the motion of the system using the geometric phase associated with a certain connection on a principal bundle. By modeling the locomotion process using connections, it is possible to understand more fully the behavior of the system, and in a variety of instances the analysis of the system is considerably simplified. This point of view is discussed in more detail in Section 3 and provides a starting point for studying more general classes of mechanical control systems.

A curious fact is that much of the new theory has been in terms of Lagrangian mechanics rather than Hamiltonian mechanics. One of the reasons for this may be that certain phenomena, such as constraint forces and external forces, are better dealt with in the Lagrangian framework. Nonetheless, there are many powerful techniques that have been developed from the Hamiltonian perspective which can often be applied (see, for example, (Nijmeijer and van der Schaft, 1990; Chapter 12) or (Bloch and Marsden, 1990)).

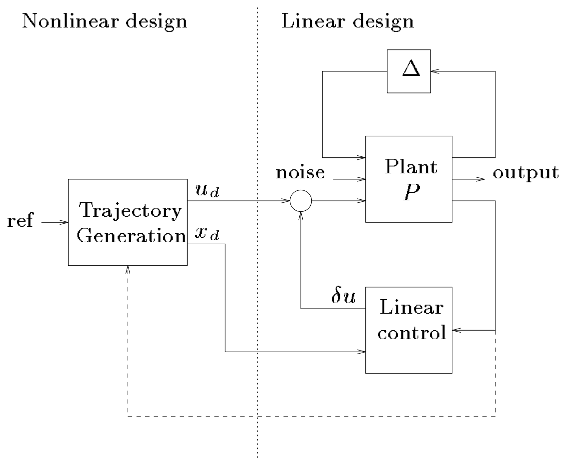

Fig. 1. Two degree of freedom controller design.

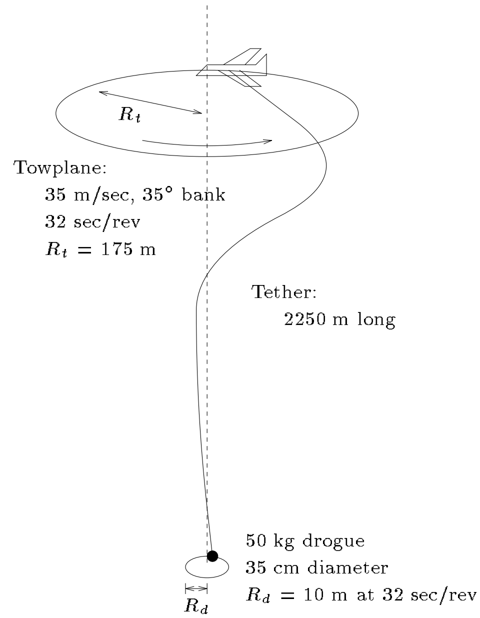

Fig. 2. Relative equilibrium of a cable towed in a circle (not to scale).

1.2. *Two Degree of Freedom Controller Design*

A large class of motion control problems consist of planning and following a trajectory in the presence of noise and uncertainty. Examples range from unmanned and remotely piloted airplanes performing surveillance and inspection tasks, to mobile robots surveying the surface of Mars, to multi-fingered robot hands performing inspection and manipulation tasks inside the human body. All of these systems are highly nonlinear and demand accurate performance. Additionally, in many of these applications the desired trajectory of the system is not known ahead of time.

One method of designing high performance controllers for this type of problem is to employ a two degree of freedom controller, as shown in Figure 1. Two degree of freedom design is a standard technique in linear control theory that separates a controller into a feedforward compensator and a feedback compensator. The feedforward compensator generates the nominal input required to track a given reference trajectory. The feedback compensator corrects for errors between the desired and actual trajectories. Many modern nonlinear control methodologies, such as feedback linearization and output regulation, can be viewed as synthesizing controllers which fall into this general framework. In those techniques, however, the feedforward and feedback problems are consider simultaneously, often leading to poor designs of the feedback portion of the controller.

The advantage of separating the trajectory generation and stabilization phases is that one can effectively exploit the geometric nature of the system to generate trajectories, while also making use of the linear structure of the error dynamics. Thus one can treat strong nonlinearities such as input saturation and global nonlinear behavior separately from the problem of robust stabilization and disturbance rejection along a reference trajectory. Modern linear techniques such as linear parameter varying analysis and synthesis (see, for example, (Packard, 1994)) appear to be particularly well-suited to this approach.

1.3. *Example: Towed Cable System*

An interesting and extremely challenging example of the need for better tools for nonlinear control of mechanical systems is a towed cable, flight control system that has been proposed for use in remote sensor applications. The system consists of an aircraft flying in a circular pattern while towing a cable with a tow body (drogue) housing a sensor system attached at the bottom. Under suitable conditions, the cable reaches a relative equilibrium in which the cable maintains its shape as it rotates. By choosing the parameters of the system appropriately, it is possible to make the radius at the bottom of the cable much smaller than the radius at the top of the cable. This is illustrated in Figure 2 and is described in more detail in (Murray, 1996).

For the towed cable system, a major issue is generating trajectories for the system that allow transition between one equilibrium point and another with minimal settling time. Due to the high dimension of the model for the system (128 states is typical), traditional approaches to solving this problem, such as optimal control theory, cannot be easily applied. However, it can be shown that as a consequence of the mechanical structure of the system, all feasible trajectories for the system are characterized by the trajectory of the bottom of the cable. This property, called differential flatness, is described in more detail below.

In principle, this characterization of the feasible trajectories allows us to reduce the problem of trajectory generation for the system to moving the bottom of the cable along a path from the initial to final configurations. In doing so, we can reduce the original optimal control problem to an optimization problem (by choosing a finite parameterization of the flat outputs). However, there are still a number of constraints on the system which complicate the problem. The chief constraint is the limited performance of the towplane. It has maximum bank angles, minimum air speeds, and limits on acceleration and deceleration which must be observed. In the flat output space, these constraints become constraints on higher order derivatives of the output and must be taken into account. However, the problem remains algebraically and computationally tractable.

Other issues of concern are the effects of noise (for example wind and turbulence) on the performance of the system and the role of unmodeled dynamics and parameter uncertainty. All of these issues are very difficult to handle using traditional techniques since the behavior of the system is highly nonlinear and the dynamics are high dimensional. The two degree of freedom technique described above, combined with the use of differential flatness for generating trajectories, is an example of how the mechanical nature of the system can be exploited in control design.

1.4. *Overview of the Paper*

This paper is organized as follows. Section 2 gives a brief summary of some of the recent techniques developed in Lagrangian mechanics which clarify some of the important geometric features of mechanical control systems. These techniques are used in Section 3 to described controllability properties of Lagrangian systems. We show that controllability of mechanical systems can be considerably more involved than linear controllability, in part because of the second order nature of Lagrangian systems. In Section 4 we concentrate on the trajectory generation problem, focusing on the role of geometric phases and holonomy, as well as considering the special case of differentially flat systems and summarizing some known conditions for checking for flatness. Section 5 describes a few related areas of research and discusses some of the open questions in the field.

Throughout the paper we make use of a number of examples to illustrate the techniques which are available. Many of these examples are drawn from other papers by the author. Copies of these papers and more information on this general area is available via the World Wide Web at URL `http://avalon.caltech.edu/~murray/mechsys.html`.

2. LAGRANGIAN CONTROL SYSTEMS

In this section we briefly summarize some of the recent results in Lagrangian mechanics that have become available and introduce a collection of examples which will be used in the sequel.

2.1. *A Brief Review of Lagrangian Mechanics*

We begin with a short review of classical mechanics from a Lagrangian perspective to set up some of the structure needed later in this section as well as to establish notation. An introductory description of this material is available in (Rosenberg, 1977). A more abstract description of this material can be found in (Marsden and Scheurle, 1993).

Let Q be a smooth (C^∞) manifold of dimension n which describes the configuration space of the system. We write TQ for the tangent bundle of q and let $(q, v) \in T_qQ$ represent a point in the bundle. The cotangent bundle of Q is written T^*Q. A mechanical system on Q is described by the Lagrangian $L : TQ \to \mathbb{R}$ which we assume has the form of kinetic minus potential energy. In a suitable choice of local coordinates, the Lagrangian can thus be written as

$$L(q, v) = \frac{1}{2} v^T M(q) v - V(q),$$

where $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix for the system. We let $F \in T^*Q$ represent the external forces on the system, including dissipation and actuator forces. These forces are allowed to depend on the current configuration and velocity of the system.

The equations of motion for a Lagrangian system with Lagrangian $L : TQ \to \mathbb{R}$ and external forces $F : TQ \to T^*Q$ are given by

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\delta q^i = F_i(q, v)\delta q^i \qquad \delta q \in T_qQ, \tag{1}$$

where summation over repeated indices is assumed. In terms of the inertia matrix, the dynamics can be rewritten as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \frac{\partial V}{\partial q} = F \tag{2}$$

where $C(q, \dot{q})$ is the Coriolis matrix given by

$$C_{ij}(q, \dot{q}) = \left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i}\right)\dot{q}_k$$

and $F \in \mathbb{R}^n$ is the vector of external forces.

There are a number of implicit assumptions in deriving these equations which are often forgotten when applying them. The most important is that any constraints on the system are assumed to be "workless," a misnomer which is often taken as a fact rather than an assumption. A good and very complete description can be found in (Rosenberg, 1977). In essence, one assumes that the constraint forces $F_c \in T^*Q$ which force the system to evolve on Q satisfy $\langle F_c, \delta q\rangle = 0$ for all variations $\delta q \in T_qQ$. This is always the case when the configuration space is the level set of an algebraic relation on a higher dimensional manifold (a so-called holonomic constraint).

Example 1 Consider the motion of the planar, vectored thrust vehicle shown in Figure 3. This system consists of a rigid body with body fixed forces and is a simplified model for the Caltech ducted fan described in (Choi *et al.*, 1994).

Let (x, y, θ) denote the position and orientation of the center of mass of the fan. We assume that the forces acting on the fan consist of a force f_1 perpendicular to the axis of the fan acting at a distance r from the center of mass, and a force f_2 parallel to the axis of the fan. Let m be the mass of the fan, J the moment of inertia, and g the gravitational constant. The Lagrangian for this system is given by

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 - mgy$$

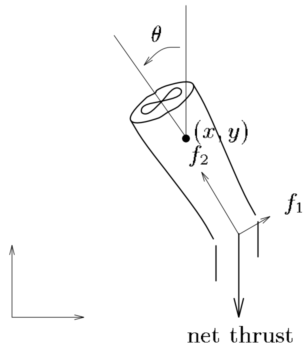

Fig. 3. Planar ducted fan engine. Thrust is vectored by moving the flaps at the end of the duct.

and Lagrange's equations yield

$$\begin{bmatrix} m\ddot{x} \\ m\ddot{y} \\ J\ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \\ r & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \quad (3)$$

2.2. *Effects of Symmetries*

One of the common features of motion control systems is the presence of symmetries. Let G be a (matrix) Lie group and let $\Phi_g : Q \to Q$ be a free left action of G on Q (see (Abraham and Marsden, 1978) for a complete definition and important details). We write $\Phi_{g*} : TQ \to TQ$ for the lifted action of G on TQ and $\xi_Q : Q \to TQ$ for the infinitesimal generator defined by

$$\xi_Q(q) = \frac{d}{ds}\Phi_{e^{\xi s}}(q)\bigg|_{s=0} \qquad \xi \in \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of G and $\exp : \mathfrak{g} \to G$ is the exponential map. A Lagrangian $L : TQ \to \mathbb{R}$ is G-invariant if

$$L(\Phi_g(q), \Phi_{g*}(v)) = L(q, v) \qquad \forall g \in G.$$

We write $\mathrm{Orb}(q)$ for the orbit of a point q under the action of G and $T_q\mathrm{Orb}$ for the tangent space of the orbit at a point q. It follows from these definitions and the definition of ξ_Q that $T_q\mathrm{Orb}(q) = \{\xi_Q(q) : \xi \in \mathfrak{g}\}$.

For unconstrained (or holonomically constrained) systems, the presence of a symmetry implies (via Noether's theorem) the existence of a conservation law of the form

$$\frac{d}{dt}\langle \frac{\partial L}{\partial \dot{q}}, \xi_Q \rangle = 0 \qquad \xi \in \mathfrak{g}.$$

The most common examples are conservation of linear and angular momentum, corresponding to $G = (\mathbb{R}^3, +)$ and $G = SO(3)$. It is convenient to rewrite the conservation law in terms of the *momentum map*, $J : TQ \to \mathfrak{g}^*$,

$$\langle J(q, \dot{q}), \xi \rangle = \langle \frac{\partial L}{\partial \dot{q}}, \xi_Q \rangle = \mu \text{ (constant)}.$$

This equation is interpreted as the momentum in the ξ_Q direction being constant along flows of the system.

By making use of conservation of momentum, it is possible to reduce the description of the dynamics to a lower dimensional space. Define the *locked inertia tensor* $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ as the mapping that satisfies

$$\langle \mathbb{I}\eta, \xi \rangle = \langle\langle \eta_Q, \xi_Q \rangle\rangle,$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the Riemannian metric given by the kinetic energy. We use the locked inertia tensor to define the *mechanical connection* $\mathbb{A} : TQ \to \mathfrak{g}$,

$$\mathbb{A} = \mathbb{I}^{-1} \cdot J.$$

The mechanical connection acts as a projection of the tangent space T_qQ onto the Lie algebra $\mathfrak{g}$:

$$\mathbb{A}(\xi_Q) = \xi.$$

Thus, given a velocity vector $v_q \in T_qQ$, the mechanical connection provides a means of splitting v_q into a *vertical part*, ver $v_q = (A(v_q))_Q \in T_q\mathrm{Orb}$, and a *horizontal part*, hor v_q, which satisfies $\mathbb{A}(\mathrm{hor}\ v_q) = 0$.

Using the mechanical connection, the equations of motion for an unforced Lagrangian system with symmetries can be written in the following form (Bloch *et al.*, 1996):

$$\begin{aligned} &\mathbb{A}(q)\dot{q} = \mathbb{I}^{-1}(q)\mu \\ &\dot{\mu} = 0 \\ &\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\delta q^i = 0 \qquad \mathbb{A}(q)\delta q = 0. \end{aligned} \quad (4)$$

These equations divide the dynamics into a first order set of equations in the vertical direction, a momentum equation (conservation law), and a set of second order equations in the horizontal directions. By virtue of the invariance of the Lagrangian with respect to the group action, the last equation can be dropped to $T(Q/G)$, giving a set of reduced order dynamics. The motion of the full system can be recovered from the reduced motion by integrating the first order equations in the vertical direction.

It is convenient to give a more concrete, coordinate-based description of the equations of motion. Consider the case where $Q = R \times G$ and write $q = (r, g)$ for the configuration and $\dot{q} = (\dot{r}, \xi) \in TR \times \mathfrak{g}$ for the velocity. We assume that the action of G on Q is given by $\Phi_h(r, g) = (r, hg)$ (this is always true locally with appropriate choice of R). The reduced Lagrangian in this case can be written as a matrix of the form

$$l(r, \dot{r}, \xi) = \frac{1}{2}\begin{bmatrix} \dot{r} \\ \xi \end{bmatrix}^T \begin{bmatrix} m(r) & A^T(r)I(r) \\ I(r)A(r) & I(r) \end{bmatrix} \begin{bmatrix} \dot{r} \\ \xi \end{bmatrix}. \quad (5)$$

The matrix $I(r) : \mathfrak{g} \to \mathfrak{g}^*$ is the *local locked inertia tensor* and $A(r) : TR \to \mathfrak{g}$ is the *local connection one form* for the mechanical connection.

If we define the body momentum as

$$p := \frac{\partial l}{\partial \xi} = \mathrm{Ad}_g^* J \in \mathfrak{g}^*$$

then the equations of motion can be written as

$$\begin{aligned} &\dot{g} = g(-A(r)\dot{r} + I^{-1}(r)p) \\ &\frac{d}{dt}\langle p, \eta \rangle = \langle p, [-A(r)\dot{r} + I^{-1}(r)p, \eta] \rangle + \\ &\qquad\qquad\qquad \langle F, (\mathrm{Ad}_g\eta)_Q \rangle \\ &\tilde{M}(r)\ddot{r} + \tilde{C}(r, \dot{r})\dot{r} + \tilde{N}(r, \dot{r}, p) = B(r)F \end{aligned} \quad (6)$$

where

$$\tilde{M} = m - A^T I A$$
$$\tilde{C}_{ij} = \frac{1}{2}\Big(\frac{\partial \tilde{M}_{ij}}{\partial r^k} + \frac{\partial \tilde{M}_{ik}}{\partial r^j} - \frac{\partial \tilde{M}_{kj}}{\partial r^i}\Big)\dot{r}^k$$
$$\langle \tilde{N}, \delta r\rangle = -\langle p, dA(\dot{r}, \delta r) - [A(\dot{r}), A(\delta r)] + \tfrac{1}{2}\frac{\partial (I^{-1}p)}{\partial r}(\delta r) + [I^{-1}p, A(\delta r)]\rangle. \tag{7}$$

Like equation (4), equation (6) is a splitting of the equations into a group component, a momentum equation, and the reduced base dynamics. This particular form of the equation, using body angular momentum, shows the reduction of the dynamics explicitly: in the absence of external forces, the group variables only appear in the top equation and all coupling of the base dynamics with the group dynamics is captured by the momentum, the locked inertia tensor, and the mechanical connection.

As equation (4) shows, the motion of the full system can be retrieved from the trajectory of the base variables by integrating the momentum equation and the group equation. In particular, the group equation has two terms, one linear in $\dot{r}$ and one linear in p. The term which is linear in $\dot{r}$ determines the *geometric phase* or *holonomy* associated with the path in base space. This path is independent of the time-parameterization of the base space path. The term linear in p generates the so-called *dynamic phase* and determines the motion of the system when $\dot{r} = 0$.

There are many other variations of the equations that can be used depending on the goal of the control task. Another common choice is to rewrite the equations in terms of the *spatial momentum*, μ (normally constant). In terms of μ, the equations for a mechanical system with external forces become

$$\begin{aligned} \dot{g} &= g(-A(r)\dot{r} + I^{-1}(r)\mathrm{Ad}^*_{g^{-1}}\mu) \\ \frac{d}{dt}\langle \mu, \eta\rangle &= \langle F, \eta_Q\rangle \\ \tilde{M}(r)\ddot{r} + \tilde{C}(r,\dot{r})\dot{r} &+ \tilde{N}(r, g, \dot{r}, \mu) = B(r)F. \end{aligned} \tag{8}$$

Note that this form of the equations shows directly the effects of external forces on the (nominally) conserved quantity μ, but does not decouple the group dynamics from the dynamics in the reduced space.

2.3. *Effects of Constraints*

The dynamics derived above assume that the motion of the system is restricted to a manifold Q, often given by the level set of an algebraic constraint on the system configuration. A somewhat different situation occurs when a set of velocity constraints of the form

$$\langle \omega^j(q), \dot{q}\rangle = \omega^j_i(q)\dot{q}^i = 0 \qquad j = 1, \dots, k \tag{9}$$

are present on the system. We will assume that the constraints $\omega^j(q) \in T^*_q Q$ are smooth and linearly independent. Constraints of this form are called *Pfaffian* constraints and typically occur as a consequence of ideal rolling constraints between two rigid bodies.

Under the assumption that the forces of constraint satisfy $\langle F_c, \delta q\rangle = 0$ for all δq such that $\langle \omega^j(q), \delta q\rangle = 0$, $j = 1, \dots, k$ (d'Alembert's principle), the equations of motion for a Lagrangian system with Pfaffian constraints are given by

$$\begin{aligned} \Big(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\Big)\delta q^i &= F_i(q, v)\delta q^i \qquad \delta q \in \mathcal{D}_q \\ \langle \omega^j(q), \dot{q}\rangle &= 0 \qquad j = 1, \dots, k, \end{aligned} \tag{10}$$

where $\mathcal{D}_q = \{v_q : \langle \omega^j(q), v_q\rangle = 0,\ j = 1, \dots, k\}$. We call these equations the *Lagrange-d'Alembert* equations. Note that the only change in the presence of constraints is to restrict the variations δq to those that satisfy the constraints and append the constraint equations to the system description. If the distribution $\mathcal{D}$ defined by the constraints is integrable, the constraint equations define a foliation of Q (under suitable regularity conditions) and it is easy to show that the Lagrange-d'Alembert equations simplify to Lagrange's equations on a leaf determined by the initial conditions.

When both symmetries and nonholonomic constraints are present, the interaction between the two can introduce subtle effects and, in general, momentum will *not* be conserved, even in the absence of external forces. A full derivation of the dynamics in this case is beyond the scope of this paper; see (Bloch *et al.*, 1996) for a detailed treatment. A special case are so-called *Chaplygin constraints*, where the kinematic constraints provide a connection on the principal bundle $Q = R \times G$. This requires, in particular, that *both* the Lagrangian and the constraint distribution be invariant under the action of G.

For the case of Chaplygin constraints, the dynamics can be derived by replacing the conservation law with the constraints. If the constraints are G invariant and the distribution of feasible velocities, $\mathcal{D}_q$, is transverse to $T_q\mathrm{Orb}$, then the constraints can be written as

$$\Gamma_{\mathrm{kin}} = \mathrm{Ad}_g(g^{-1}\dot{g} + A_{\mathrm{kin}}(r)\dot{r}) = 0, \tag{11}$$

where $\Gamma_{\mathrm{kin}} : TQ \to \mathfrak{g}$ is the *kinematic connection* and $A_{\mathrm{kin}} : TR \to \mathfrak{g}$ is the associated local connection one form. It can be shown that the full equations of motion have the form

$$\begin{aligned} \dot{g} &= g(-A_{\mathrm{kin}}(r)\dot{r}) \\ \tilde{M}(r)\ddot{r} + \tilde{C}(r,\dot{r})\dot{r} &= B(r)F. \end{aligned} \tag{12}$$

The first equation is the kinematic constraints and the second the reduced dynamics. The exact expression for the base dynamics is given in (Bloch *et al.*, 1996), but can be derived using the Lagrange-d'Alembert equations without difficulty.

Example 2 Consider the dynamics of a two-wheeled mobile robot which is able to drive in the direction in which it points and spin about its center, as shown in Figure 4. Let ψ_1 and ψ_2 denote the angles of rotation of the two wheels (with respect to arbitrary initial states). The position of the robot is given by the xy location of its center and the heading angle θ. Balance is maintained by a small castor whose effect we shall otherwise ignore. Thus $q = (\psi_1, \psi_2, x, y, \theta)$ denotes the configuration of the system.

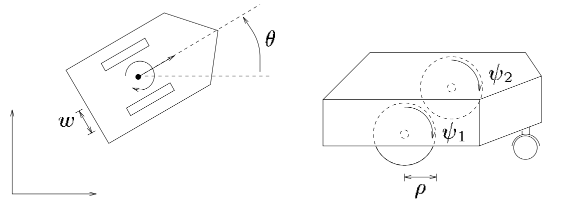

Fig. 4. Two-wheeled planar mobile robot.

We model each contact using a *pure rolling* assumption: each wheel can roll in the direction in which it points and spin about its vertical axis, but cannot slide. A simple calculation shows that this assumption yields constraints

$$\begin{aligned} \omega^1(q)\dot{q} &= \dot{x}\cos\theta + \dot{y}\sin\theta - \frac{\rho}{2}(\dot{\psi}_1 + \dot{\psi}_2) = 0 \\ \omega^2(q)\dot{q} &= -\dot{x}\sin\theta + \dot{y}\cos\theta = 0 \\ \omega^3(q)\dot{q} &= \dot{\theta} - \frac{\rho}{2w}(\dot{\psi}_1 - \dot{\psi}_2) = 0. \end{aligned} \tag{13}$$

The Lagrangian for the system is given by

$$L(q,\dot{q}) = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}J_w(\dot{\psi}_1^2 + \dot{\psi}_2^2),$$

where m is the mass of the robot, J its inertia, and J_w the inertia of the wheels. (We have assumed for simplicity's sake that the center of mass of the robot lies on the line between the two drive wheels.)

Using the Lagrange-d'Alembert equations, the dynamics of the system can be shown to satisfy

$$\begin{bmatrix} \dot{x}\cos\theta + \dot{y}\sin\theta \\ -\dot{x}\sin\theta + \dot{y}\cos\theta \\ \theta \end{bmatrix} = \begin{bmatrix} \rho/2 & \rho/2 \\ 0 & 0 \\ \rho/(2w) & -\rho/(2w) \end{bmatrix} \begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{m\rho^2}{4} + \frac{J\rho^2}{4w^2} + J_w & \frac{m\rho^2}{4} - \frac{J\rho^2}{4w^2} \\ \frac{m\rho^2}{4} - \frac{J\rho^2}{4w^2} & \frac{m\rho^2}{4} + \frac{J\rho^2}{4w^2} + J_w \end{bmatrix} \begin{bmatrix} \ddot{\psi}_1 \\ \ddot{\psi}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix},$$

where τ_1 and τ_2 are the torques applied to the drive wheels. Notice that the first three equations are the constraints and that, given ψ_1 and ψ_2 as functions of time, we can solve them uniquely for $\dot{x}$, $\dot{y}$, and $\dot{\theta}$. The second equation describes the reduced dynamics for the internal "shape" of the system.

3. CONTROLLABILITY

A nonlinear control system is said to be *controllable* if there exists a feasible path for the system between any two states of the system. For mechanical systems, this definition is often more restrictive than necessary. For example, a mechanical system with symmetries can never be controllable under this definition: the symmetries imply a conservation law which has the form of a set of constraints on the states (configurations plus velocities). In this section we consider several alternative definitions of controllability for mechanical systems and give some initial results on characterizing these notions of controllability.

The results in this section rely on the use of the Lie algebra rank condition (see (Isidori, 1989) for a description) to determine controllability by using the involutive closure of the drift and input vector fields. However, we exploit the explicit structure of Lagrangian dynamical systems to extract out only the bracket expressions which are needed to determine different types of controllability. In so doing, we are able to gain deeper insight into the geometric structure of Lagrangian control systems. Our eventual goal is to understand not only the ramifications of different types of controllability for mechanical systems, but also to understand how the different geometric objects which define the dynamics interact with each other to achieve controllability. The work of (Bloch and Crouch, 1992) and (San Martin and Crouch, 1984) provides a starting point for some of the ideas presented here.

3.1. *Equilibrium Controllability*

A natural question for motion control systems is that of *equilibrium controllability:* when is it possible to find forces which steer a mechanical system from one equilibrium point (or relative equilibrium point) to another? This captures the notion of pointing a satellite at a target or moving the towed cable system described in the introduction from one surveillance point to another.

Definition 1 (Lewis, 1995) A mechanical control system is *equilibrium controllable* if for any two equilibrium points $q_0 \in Q$ and $q_1 \in Q$ there exists a time T and set of forces $F : [0,T] \to T^*Q$ which drives the system from q_0 to q_1.

For general nonlinear control systems, proving controllability turns out to be quite difficult. Instead, one checks for *accessibility*, which measures the ability of a control system to reach an open subset of the state space, but not necessarily a neighborhood of an equilibrium point. Let $\mathcal{R}_Q(q_0, \leq T)$ denote the set of configurations which can be reached at *some* velocity in time $t \leq T$ starting from an equilibrium point q_0 and by some choice of inputs on $[0,T]$.

Definition 2 (Lewis and Murray, 1995) A mechanical control system is *configuration accessible* if there exists a time T such that $\mathcal{R}_Q(q_0, \leq T)$ contains an open subset of Q.

In order to derive conditions for equilibrium controllability and configuration accessibility, we rewrite the equations of motion in terms of the covariant derivative on Q with Riemannian metric given by the inertia tensor for the system. We restrict ourselves to the case of purely kinetic energy, so that the Lagrangian is given by

$$L(q,v) = \frac{1}{2}M_{ij}v^i v^j.$$

The covariant derivative with respect to the metric M is given by

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial q_j}X^j + \Gamma^i_{jk}X^j Y^k\right)\frac{\partial}{\partial q^i}$$

where X and Y are smooth vector fields on Q with components X^i, Y^i and Γ^i_{jk} are the Christoffel symbols,

$$\Gamma^i_{jk} = \frac{1}{2}(M^{-1})^{il}\left(\frac{\partial M_{lj}}{\partial q_k} + \frac{\partial M_{lk}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_l}\right).$$

In terms of these definitions, the equations of motion have the form

$$\nabla_{\dot{q}}\dot{q} = M^{-1}F$$

where F are the external forces and $M^{-1}F$ defines a tangent vector on Q. In particular, if the external forces are all control forces, then we write the dynamics as

$$\nabla_{\dot{q}}\dot{q} = Y_1(q)u^1 + \cdots + Y_m(q)u^m, \tag{14}$$

where each Y_i is a vector field on Q. Notice that the Riemannian metric, and hence the Lagrangian, enter the definition of the control vector fields $Y_1, \ldots, Y_m$.

In addition to the usual Lie brackets between the vector fields Y_i, we also make use of the *symmetric product*,

$$\langle Y_i : Y_j \rangle = \nabla_{Y_i} Y_j + \nabla_{Y_j} Y_i.$$

This bracket arises in the computation of certain high order Lie brackets involving the full dynamics. Its importance is that it is expressed only in terms of quantities defined on Q (namely, the vector fields Y_i and Y_j, and the Riemannian metric). We define the following sequence of collections of vector fields:

$$\begin{aligned}
\mathcal{G}^{(1)} &= \{Y_1, \ldots, Y_m\} \\
\mathcal{G}^{(i)} &= \{\nabla_Y X + \nabla_X Y : X \in \mathcal{G}^{(j)},\ Y \in \mathcal{G}^{(k)},\ i = j+k\} \\
\mathcal{G}^{(\infty)} &= \bigcup_{i=1}^{\infty} \mathcal{G}^{(i)}.
\end{aligned}$$

By applying Chow's theorem to this special class of vector fields, it is possible to arrive at the following result, originally given in (Lewis and Murray, 1995):

Theorem 1 Suppose that the involutive closure of the vector fields $\mathcal{G}^{(\infty)}$ spans T_qQ for each $q \in Q$. Then the system (14) is locally configuration accessible at each $q \in Q$.

Further extensions of this result are possible by making use of the characterizations of small time local controllability described by Sussmann (Sussmann, 1987). In order to do so, we need to define "good" and "bad" symmetric products. We say that an element of $\mathcal{G}^{(\infty)}$ is "bad" if it contains an even number of copies of Y_i for each $i = 1, \ldots, m$. Otherwise we say the symmetric product is "good".

Theorem 2 Suppose that every bad symmetric product in $\mathcal{G}^{(\infty)}$ evaluated at an equilibrium point has the property that it can be written as a linear combination of good symmetric products of lower order. Then the system is locally equilibrium controllable.

These results show that it is possible to determine special types of controllability for mechanical systems by performing computations using objects that are defined only on the configuration manifold Q and not the entire state space TQ. They also provide a first step towards developing a control theory for general mechanical systems, but they still ignore many important factors. Dissipative forces such as friction and drag are present in almost all mechanical systems of engineering interest, but their effects are not accounted for in the limited treatment presented here. Additionally, the effects of nonholonomic constraints are not taken into account and will likely lead to much more intricate controllability structure. More details can be found in (Lewis, 1995).

Example 3 Consider the planar ducted fan example considered earlier, where for simplicity we ignore gravity (thus making the system a planar rigid body with two body fixed forces). The control vector fields are given by

$$\begin{aligned}
Y_1 &= \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y} + \frac{r}{J}\frac{\partial}{\partial\theta} \\
Y_2 &= -\frac{\sin\theta}{m}\frac{\partial}{\partial x} + \frac{\cos\theta}{m}\frac{\partial}{\partial y}.
\end{aligned}$$

The Lie bracket between Y_1 and Y_2 is

$$[Y_1, Y_2] = -\frac{r\cos\theta}{mJ}\frac{\partial}{\partial x} - \frac{r\sin\theta}{mJ}\frac{\partial}{\partial y}$$

and the interesting covariant derivatives are

$$\begin{aligned}
2\nabla_{Y_1} Y_1 &= -\frac{2r\sin\theta}{mJ}\frac{\partial}{\partial x} + \frac{2r\cos\theta}{mJ}\frac{\partial}{\partial y} \\
\nabla_{Y_1} Y_2 + \nabla_{Y_2} Y_1 &= -\frac{r\cos\theta}{mJ}\frac{\partial}{\partial x} - \frac{r\sin\theta}{mJ}\frac{\partial}{\partial y} \\
2\nabla_{Y_2} Y_2 &= 0.
\end{aligned}$$

We see that the vector fields $\{Y_1, Y_2, [Y_1, Y_2]\}$ generate all directions on TQ so the system is locally configuration accessible with these inputs. Furthermore, the bad covariant derivative $2\nabla_{Y_1} Y_1$ is a multiple of Y_2 so the system is equilibrium controllable as well.

It is also possible to study the controllability properties when only the sideways force, Y_1, is available. In this case Y_1, $2\nabla_{Y_1} Y_1$, and

$$[Y_1, 2\nabla_{Y_1} Y_1] = -\frac{2r^2\sin\theta}{mJ^2}\frac{\partial}{\partial x} - \frac{2r^2\sin\theta}{mJ^2}\frac{\partial}{\partial y}$$

span TQ and hence the system is configuration accessible. However, the bad covariant derivative cannot be written in terms of Y_1 and so the system does not satisfy the sufficient conditions for equilibrium controllability.

3.2. *Controllability of Locomotion Systems*

The results of the previous subsection apply to general Lagrangian systems with externally controlled forces. When symmetries are present in the system, somewhat more detailed answers to the question of controllability can be given. To illustrate this, we consider the special case of robotic *locomotion systems*, roughly defined as systems for which the shape of the robot is used to control the position of the robot. The means of locomotion can be due to constraints, conservation laws, or both.

More concretely, assume the configuration space of the system is given by a trivial principal bundle $Q = R \times G$ and the base dynamics are fully actuated. For simplicity we consider only the purely kinematic case, where the dynamics have the form

$$\begin{aligned}
\dot{g} &= g(-A(r)\dot{r}) \\
M(r)\ddot{r} + C(r,\dot{r})\dot{r} &= \tau.
\end{aligned} \tag{15}$$

The local connection form A is to be regarded as a mapping $A : TR \to \mathfrak{g}$ which describes how the shape of the robot interacts with its environment; it can come from either the kinematic connection or the mechanical connection (assuming $\mu = 0$). Our goal is to study the controllability of the system described by equation (15) in terms of the properties of the local connection form $A(r)$. This has the advantage of properly taking into account the role of the Lie group G and thus avoids the use of local coordinates, which can lead to extremely messy formulas.

We now make precise the notion of controllability for a locomotion system. There are two cases to consider: controllability on the entire space $Q = R \times G$ and controllability only on the fibers.

Definition 3 A locomotion system is said to be *totally controllable* if, for any $q_0 = (r_0, g_0)$ and $q_f = (r_f, g_f)$, there exists a time $T > 0$ and a curve $r(\cdot)$ connecting r_0 and r_f in the base space such that the solution curve for (15) passing through q_0 satisfies $q(0) = q_0$ and $q(T) = q_f$.

Definition 4 A locomotion system is said to be *fiber controllable* if, for any initial position $g_0 \in G$, final position $g_f \in G$, and initial shape $r_0 \in R$, there exists a time $T > 0$ and a base space curve $r(\cdot)$ satisfying $r(0) = r_0$ such that the solution curve for (15) passing through (r_0, g_0) satisfies $q(0) = q_0$ and $q(T) = (r(T), g_f)$ for some $r(T) \in R$.

In order to determine controllability, we make use of the *curvature* of the connection form $A : TR \to \mathfrak{g}$.

Definition 5 Given a local connection form A on Q, we define the corresponding local *curvature form* to be the $\mathfrak{g}$-valued differential two form DA satisfying

$$DA(v_1, v_2) = dA(v_1, v_2) - [A(v_1), A(v_2)], \qquad (16)$$

where $[\cdot,\cdot]$ denotes the Lie bracket on $\mathfrak{g}$.

The curvature form measures the integrability of the kinematic constraints. In particular, if the constraints are holonomic then the curvature is identically zero.

There are two basic ways to determine controllability for a locomotion system whose dynamics are modeled using a principal connection. The first is to apply the Ambrose-Singer theorem, which characterizes controllability in terms of the curvature of the connection evaluated at all *reachable* points of a system. Since it is precisely the reachable points that we are trying to determine, it is more constructive to appeal again to Chow's theorem and check controllability by computing Lie brackets. Using the left-invariance properties of the connection, the structure of the equations of motion, and the general identity for the curvature of a differential form given by

$$d\omega(X, Y) = L_X\omega(Y) - L_Y\omega(X) - \omega([X, Y])$$

it is possible to rewrite these conditions in terms of the curvature. Define the following sequence of subspaces of the Lie algebra $\mathfrak{g}$ at a fixed point $x \in M$:

$$\begin{aligned}
\mathfrak{h}_1 &= \operatorname{span}\{A(X) : X \in T_rR\} \\
\mathfrak{h}_2 &= \operatorname{span}\{DA(X,Y) : X, Y \in T_rR\} \\
\mathfrak{h}_3 &= \operatorname{span}\{L_Z DA(X,Y) - [A(Z), DA(X,Y)], \\
&\qquad [DA(X,Y), DA(W,Z)] : \\
&\qquad W, X, Y, Z \in T_rR\} \\
&\;\;\vdots \\
\mathfrak{h}_k &= \operatorname{span}\{L_X\xi - [A(X), \xi], [\eta, \xi] : X \in T_rR, \\
&\qquad \xi \in \mathfrak{h}_{k-1}, \eta \in \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_{k-1}\}.
\end{aligned} \qquad (17)$$

Theorem 3 (Kelly and Murray, 1995) A system defined on a trivial principal bundle Q over R with structure group G and local connection $A(r)$ is *locally fiber controllable* near $q \in Q$ if and only if

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots .$$

The system is *locally totally controllable* if and only if

$$\mathfrak{g} = \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots .$$

This proposition again highlights the mechanical nature of the system and exploits the extra structure provided by the Lie group. More details and additional examples, including snakelike and legged locomotion systems, are provided in (Kelly and Murray, 1995). Control of locomotion systems that rely on the interaction between constraints and symmetries are described in (Ostrowski and Burdick, 1995).

Example 4 Consider again the mobile robot introduced in Example 2. If we redefine the base coordinates as $\phi_1 = \psi_1 + \psi_2$ and $\phi_2 = \psi_1 - \psi_2$, then the local connection form for the system is

$$A = \begin{bmatrix} -a\,d\phi_1 \\ 0 \\ -b\,d\phi_2 \end{bmatrix},$$

and the local curvature form is

$$DA(r) = dA - [A, A] = 0 - \begin{bmatrix} 0 \\ ab\,d\phi_1 \wedge d\phi_2 \\ 0 \end{bmatrix}.$$

The vectors

$$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix}$$

span $\mathfrak{h}_1$, and the vector

$$DA \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -ab \\ 0 \end{bmatrix}$$

spans $\mathfrak{h}_2$. Since these three vectors are linearly independent, the system is fiber controllable. Since these are all constant vectors and the third component of the Lie bracket of any two elements of $se(2)$ is zero, no element of $\mathfrak{h}_k$, $k = 3, 4, \ldots$, will have a nonzero third component. The system cannot, therefore, be totally controllable. The reason for this is clear: the angle of the robot directly determines the difference between the wheel angles and hence we always have $\theta = \rho\phi_2$. Of course, since we usually are not interested in specifying the internal angles of the wheels, the fact that the system is not totally controllable is of no practical consequence.

4. TRAJECTORY GENERATION

We now return to the problem of generating trajectories for nonlinear control systems. Here again, by using the structure of mechanical systems, it is possible to understand more fully the geometry of trajectory generation. We consider two different ways of computing trajectories, both based on reducing the description of the feasible trajectories of the system to a smaller space where computations can be carried out more easily.

4.1. *Holonomy*

For most motion control problems, the position of the system is specified as an element of the group of Euclidean motions $SE(3)$ or one of its proper subgroups. Motion of the system often involves motion between group elements and hence we can consider the trajectory generation problem as one of finding a path for the reduced dynamics which gives a desired holonomy. In particular, in the special case of either zero (conserved) momentum or Chapylgin kinematic constraints, the group dynamics reduce to

$$\dot{g} = g^{-1}(-A(r)\dot{r})$$

and the trajectory generation problem becomes that of finding a path for the reduced variables r which moves from an initial position g_0 to a final position g_f. Depending on the particular system, there may also be constraints on the initial and final reduced variables.

The problem of generating a given holonomy can be simplified in a number of special cases. If the group G is Abelian then computing the holonomy for a given path can be reduced to evaluation of the so-called *area rule*:

$$\begin{aligned} g(T) &= \exp\left(-\int_C A(r)\right) g(0) \\ &= \exp\left(-\iint_S dA(r)\right) g(0), \end{aligned} \tag{18}$$

where C is a (closed) curve in the base space and S is any surface such that $C = \partial S$. Thus, we can compute the holonomy by simply integrating the weighted area enclosed by a path in base space.

If the group G is not Abelian then the area rule cannot be utilized and one is left with solving an ordinary differential equation to determine the motion in the group variables. If the group G is nilpotent (meaning that brackets of some fixed, possibly high, order are always zero) then it can be shown that in a suitable set of coordinates the group equation has a triangular structure:

$$\begin{aligned} \dot{x}_1 &= a_1(r, \dot{r}) \\ \dot{x}_2 &= a_2(x_1, r, \dot{r}) \\ &\vdots \\ \dot{x}_n &= a_n(x_1, \ldots, x_{n-1}, r, \dot{r}). \end{aligned}$$

It follows that the motion in the group variables can be determined by simple integration. This allows, for example, the use of a parameterized set of base curves and the use of symbolic integration to get a closed form formula for the motion in the group variables.

In the case of a non-Abelian, non-nilpotent group, the full group equations must be integrated numerically. This is still considerably simpler than solving the full control problem because the dimension of the group dynamics can be much smaller than the dimension of the full system. This is particularly true when the dynamics in the base are fully actuated, such as in the case of locomotion systems.

It is also possible to take drift terms into account using holonomy, for example when there is a nonzero momentum term present. In this case, the net motion in the group variables is divided into a *geometric phase*, due to changes in the shape space, and a *dynamic phase*, due to the momentum term. Constructive controllability is much more difficult in this case but similar techniques can be applied as a starting point for obtaining solutions to the problem.

4.2. *Differentially Flat Systems*

One of the classes of systems for which trajectory generation is particularly easy are so-called differentially flat systems. Roughly speaking, a control system is differentially flat if we can find a set of outputs (equal in number to the number of inputs) such that all states and inputs can be determined from these outputs without integration. More precisely, if the system has states $x \in \mathbb{R}^n$, and inputs $u \in \mathbb{R}^m$, then the system is flat if we can find outputs $y \in \mathbb{R}^m$ of the form

$$y = y(x, u, \dot{u}, \ldots, u^{(p)}) \tag{19}$$

such that

$$\begin{aligned} x &= x(y, \dot{y}, \ldots, y^{(q)}) \\ u &= u(y, \dot{y}, \ldots, y^{(q)}). \end{aligned} \tag{20}$$

Differentially flat systems were originally studied by Fliess and coworkers in the context of differential algebra (Fliess *et al.*, 1992; Martin, 1992) and later using Lie-Bäcklund transformations (Fliess *et al.*, 1993). In (van Nieuwstadt *et al.*, 1994), flatness was reinterpreted in a differential geometric setting. It has been shown that differential flatness is a generalization of dynamic feedback linearizability and that, in an open and dense set, all differentially flat systems are dynamic feedback linearizable. However, the point of view in flatness is more closely tied to two degree of freedom design and concentrates on trajectory generation rather than feedback transformation to a given normal form.

There are several existing results which indicate that differential flatness for mechanical systems may be

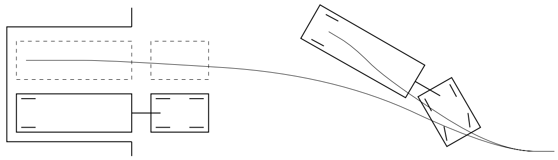

Fig. 5. Control of a truck with attached trailer.

easier to determine than flatness for general nonlinear systems. For example, it can be shown that all of the following systems are differentially flat: a car pulling N trailers with the hitch of the ith trailer attached at the axle of the preceding vehicle in the chain (Rouchon *et al.*, 1992; Tilbury *et al.*, 1995); any planar rigid body with forces whose lines of action do not intersect at the center of mass (Martin *et al.*, 1994); an airplane towing a cable with a rigid body attached at the end of the cable; and a satellite system with three control torques and a single thruster whose line of action intersects the center of mass. In all of these examples, the differentially flat output is not an arbitrary combination of the configuration variables and velocities of the system, but rather consists of a set of points and angles. We conjecture that this special fact is due to second order nature of the system combined with symmetry relations, when present, and the structure of the inertia tensor for the system. Some initial results in this direction are available in (Rathinam and Murray, 1996).

We illustrate the utility of flatness for motion control problems in the context of two examples.

Example 5 Consider first the problem of controlling a truck with trailer, as shown in Figure 5. The task we wish to perform is to generate a trajectory for the truck to move it from a given initial position to the loading dock. We ignore the role of obstacles and concentrate on generation of feasible trajectories.

The dynamics of the system can be determined using the Lagrange-d'Alembert equations. For simplicity, we consider only the kinematic portion of the dynamics here. Under the assumption of pure rolling, the equations of motion are given by

$$\begin{aligned}\dot{x} &= \cos\theta\, u_1\\ \dot{y} &= \sin\theta\, u_1\\ \dot{\phi} &= u_2\\ \dot{\theta} &= \frac{1}{l}\tan\phi\, u_1\\ \dot{\theta}_1 &= \frac{1}{d}\cos(\theta-\theta_1)\sin(\theta-\theta_1)u_1,\end{aligned} \tag{21}$$

where (x, y, θ) is the position and orientation of the truck, ϕ is the angle of the steering wheels, θ_1 is the angle of the trailer, and l and d are the length of the truck and trailer.

It can be shown that the position of the center of the rear wheels of the trailer is a flat output for the system. Rather than working through the equations in detail, it is considerably more instructive to prove the flatness of this system by the following geometric argument. First, given a feasible motion for the truck/trailer system, there is clearly a well defined and unique trajectory for the center of the rear wheels. Conversely, suppose we know the trajectory of the center of the rear wheels. This defines a curve in the plane and the tangent to that curve clearly determines the angle of the trailer with respect to horizontal. Furthermore, it is not difficult to see that the curvature of the curve determines the angle of the truck with respect to the trailer and the next higher derivative of the curve determines the angle of the front wheels. (If you don't believe this verbal argument, an algebraic argument is given in (Rouchon *et al.*, 1992).) So, given the curve followed by the rear wheels of the trailer, the motion of the complete truck with trailer can be uniquely determined. Thus the system is differentially flat.

Using flatness it is possible to solve the problem shown in Figure 5 as follows: given a starting configuration of the system, we can determine the initial point, tangent vector, curvature, and derivative of curvature of the flat outputs that correspond to the initial condition. Similarly, the final configuration determines a final point, tangent, curvature, and derivative of curvature. Therefore, in the flat output space, we must find a curve which connects a starting point with given tangent, curvature and derivative of curvature to an ending point with given tangent, curvature and derivative of curvature. This is a standard problem using splines or Bezier curves and is easily solved in a number of ways. Any solution to this simple problem corresponds to a feasible trajectory for the full system. Thus the problem of backing a trailer into a loading dock is easily solved by commanding the trajectory of the back of the trailer. One such solution is shown in the figure.

From this starting point it is possible to add on a number of important considerations. First, we must worry about state space constraints and input constraints since these will limit our feasible trajectories. For this system, it turns out that limits on the angle between the truck and the trailer are easily converted to limits on the curvature of the flat output curve. Similarly, steering wheel limits give bounds on the maximum derivative of curvature. These are easily incorporated into the problem by requiring that the spline which connects starting point to end point observe these constraints on the geometric properties of the curve.

Example 6 As a second, more dynamic example, consider again the ducted fan whose equations of motion were derived in Section 2. The dynamics for the system are

$$\begin{aligned}m\ddot{x} &= f_1\cos\theta - f_2\sin\theta\\ m\ddot{y} &= f_1\sin\theta + f_2\cos\theta - mg\\ J\ddot{\theta} &= r f_1.\end{aligned} \tag{22}$$

(Martin *et al.*, 1994) showed that this system is flat and that one set of flat outputs is given by

$$\begin{aligned}z_1 &= x - (J/mr)\sin\theta\\ z_2 &= y + (J/mr)\cos\theta.\end{aligned} \tag{23}$$

Using the system dynamics, it can be shown that

$$\ddot{z}_1\cos\theta + (\ddot{z}_2 + g)\sin\theta = 0. \tag{24}$$

and thus given $z_1(t)$ and $z_2(t)$ we can find $\theta(t)$ except for an ambiguity of π and away from the singularity

$\ddot{z}_1 = \ddot{z}_2 + g = 0$. The forces $f_1(t)$ and $f_2(t)$ can then be obtained from the dynamic equations.

Having determined that the system is flat, it follows that all feasible trajectories for the system are characterized by the evolution of the flat outputs. Thus, we can employ the same techniques as in the previous example to move the system from one hover point to another: we construct a curve with specified initial and final derivatives that joins the two configurations. Experimental results that exploit this structure can be found in (van Nieuwstadt and Murray, 1996).

Observe that the flat outputs correspond to the coordinates of a body fixed point, a point that is on the line joining the center of mass G and point of intersection P of the forces. This point is distance J/mr from G on the other side of P. This point is historically known as *center of oscillation* and becomes important in the study of planar rigid pendulums.

4.3. *Combining Holonomy and Differential Flatness*

Both reduction and differential flatness provide methods of simplifying the trajectory generation problem by reducing the order of the control problem. In the case of reduction, this is done in a way in which the reduced system is itself Lagrangian and trajectory generation consists in part of finding a curve in the base space whose integral (the holonomy) has a given value. For differentially flat systems, the reduced system is the trivial system (no dynamics) and trajectory generation is performed by finding curves which connect two points plus have a given, finite number of derivatives specified at the endpoints.

Given a choice, it is much more convenient to work with differential flatness than with geometric phases. Unfortunately, at the current time, general conditions for determining if a given system is differentially flat are not known, although a number of special cases are understood for nonlinear control systems which are affine in the inputs (for example, all static and dynamic feedback linearizable systems are flat). Some specialized results are also available for Lagrangian control systems are available (Rathinam and Murray, 1996), but many known examples of differentially flat Lagrangian systems are not included in the class of systems to which the results apply. It is also clear that, despite the large collection of example systems that are differentially flat, many engineering systems are not obviously flat. For example, if aerodynamic forces (drag and lift) are added to the ducted fan example, the center of oscillation is no longer the flat output and it is not clear how to properly modify the outputs. It is interesting to note, however, that a conventional aircraft *is* flat (Martin, 1992; Martin, 1996).

The situation for reduction and reconstruction in the presence of symmetries is much more understood. Conditions under which reduction can be performed are well characterized and the effects of constraints and external forces have been incorporated into the problem formulation. However, generating a given holonomy is relatively simple only when the base space is fully actuated. If this is not the case, then only certain paths can be followed in the base space and so the trajectory generation problem again becomes complicated (though still of reduced order).

A more general paradigm for trajectory generation for mechanical systems is to combine differential flatness and symmetries. Consider the following diagram for the reduction of the dynamics of a mechanical system:

$$\begin{array}{rcl} & TQ & \leftarrow \text{full system} \\ \text{reduction} & \downarrow\ \uparrow & \text{reconstruction} \\ & T(Q/G) \times \mathfrak{g} & \leftarrow \text{reduced system} \\ \text{Cartan reduction} & \downarrow\ \uparrow & \text{Cartan prologation} \\ & M & \leftarrow \text{flat system} \end{array}$$

We refer to the procedure of reducing the dynamics of a differentially flat system to the (trivial) dynamics of its flat outputs as Cartan reduction and the process of lifting that path as Cartan prolongation (see (van Nieuwstadt *et al.*, 1994) for a discussion). The idea is to try to find systems in which we can perform a reduction of the dynamics (via symmetries) to a system that is differentially flat. The trajectory generation problem then becomes one of finding a path in the flat output space which has given derivatives at the endpoints and whose integral (in some generalized sense) has a given value.

Allowing reduction by symmetries can increase the class of systems to which differential flatness techniques can be applied. This is essentially the technique that we have already used in planning for locomotion systems, where the flat outputs are the base space configurations (since the system is fully actuated). Applications of this idea to other classes of systems is the subject of current work, but initial examples indicate that reduction by symmetries combined with flatness of the reduced system can be utilized to find effective methods for trajectory generation.

5. DISCUSSION AND OPEN PROBLEMS

In this paper we have surveyed some of the recent techniques which have been developed for nonlinear control of mechanical systems. The emphasis has been on trajectory generation techniques as part of an overall two degree of freedom control design paradigm. By utilizing the special structure of mechanical systems whose unforced dynamics satisfy Lagrange's equations, it is possible to obtain a more detailed understanding of the dynamics of the system in a way which is compatible with modern control theory.

There are numerous other areas in control of mechanical systems that we have not discussed here due to space limitations. Problems in stabilization of mechanical systems, especially using Hamiltonian structure, have received considerable interest as have problems in stabilization of mechanical systems with nonholonomic constraints and conservation laws. This latter case is particularly interesting since it can be shown that the system cannot be stabilized using continuous static state feedback and hence more advanced techniques are required. A good review of the Hamiltonian stablization problem can be found in (Nijmeijer and van der Schaft, 1990); see also (Bloch and Marsden, 1990) for the use of energy-Casimir methods to analyze stability. In addition, there has been substantial work on the problem of control on Lie groups which arise naturally in mechanical systems with symmetry. We refer to (Leonard and Krishnaprasad, 1995) for a recent survey of the literature.

There are a host of open problems in nonlinear control of mechanical systems that must be better understood in order to develop effective tools for control design. Necessary and sufficient conditions for differential flatness of Lagrangian systems are not yet available and the role of symmetries in determining the flat outputs is not yet clear. Necessary and sufficient conditions for equilibrium controllability, particularly in the presence of dissipation and other non-potential forces, and a description of the connection between the symmetric product and constructive controllability, are also needed. The use of other nonlinear normal forms for use in Cartan reduction could help to solve problems that are not differentially flat, and lead to new insights in the control of strongly nonlinear systems. Finally, approximation techniques for differential flatness and bounds on the performance penalties due to different approximations will almost certainly be needed before these methods can be applied to large engineering systems for which analytic models are rarely available.

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