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TRAJECTORY GENERATION FOR A TOWED CABLE SYSTEM USING DIFFERENTIAL FLATNESS

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Abstract. In this paper we consider the problem of generating feasible motions for a towed cable, flight control system that has been proposed for use in remote sensor applications. Using the fact that the system is differentially flat, we illustrate how to construct feasible trajectories for the system and demonstrate the strengths and limitations of this approach. Simulations of the full dynamics are included to illustrate the proposed techniques. A significant limitation of the current approach is the numerical instability of the algorithm, resulting in the need for careful tuning of parameters to achieve convergence.

Keywords. Nonlinear control, flight control, trajectory planning

1. INTRODUCTION

We consider the dynamics of a system consisting consisting of an aircraft flying in a circular pattern while towing a cable with a tow body (drogue) attached at the bottom. Under suitable conditions, the cable reaches a relative equilibrium in which the cable maintains its shape as it rotates. By choosing the parameters of the system appropriately, it is possible to make the radius at the bottom of the cable much smaller than the radius at the top of the cable. This is illustrated in Figure 1. According to (Russell and Anderson 1977), the basic dynamic phenomenon of circularly towed cables was used by a missionary who used this basic method to deliver supplies.

The dynamics of this system was studied extensively in the 1970s in research projects funded in large part by the Navy. An excellent discussion of the equilibrium shapes for circularly towed cables can be found in the article by Skop and Choo (1971). They show that the equilibrium shape can be determined by analyzing the forces at the

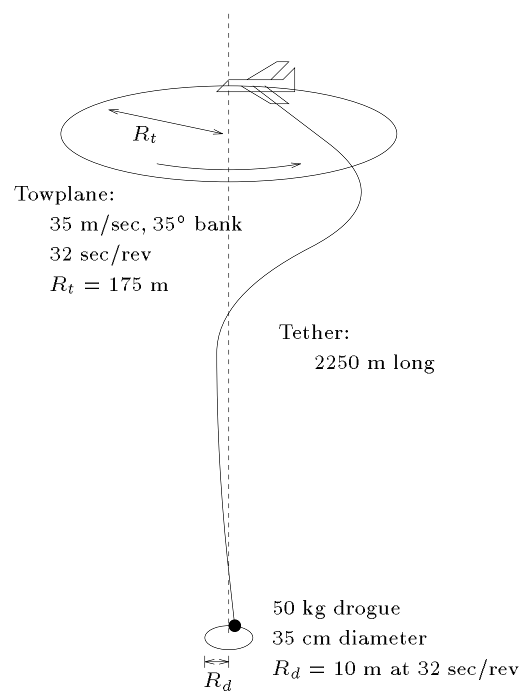

Fig. 1. Relative equilibrium of a cable towed in a circle.

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bottom of the cable, under the assumption of uniform rotation, and integrating up the cable to determine the overall shape. Russell and Anderson (1977) analyzed the stability of the resulting solutions and studied the multivalued nature of the solutions for certain values of the parameters. A more recent survey of this work can be found in the dissertation of Clifton (1992).

Our current interest is in generating trajectories for this system which allow operation away from relative equilibria as well as transition between one equilibrium point and another. Due to the high dimension of the model for the system (128 states is typical), traditional approaches to solving this problem, such as optimal control theory, cannot be easily applied. However, it can be shown that this system is differentially flat using the position of the bottom of the cable as the differentially flat output. Differential flatness is a concept introduced by Fliess et al. (1992) in which the solutions of a control problem can be determined from the motion of the flat output (and its derivatives). Thus all feasible trajectories for the system are characterized by the trajectory of the bottom of the cable (see Section 2 for details).

In principle, this allows us to reduce the problem of trajectory generation for the system to moving the bottom of the cable along a path from the initial to final configurations. In doing so, we can reduce the original optimal control problem to an optimization problem (by choosing a finite parameterization of the flat outputs). However, there are still a number of constraints on the system which complicate the problem. The chief constraint is the limited performance of the towplane. It has maximum bank angles, minimum air speeds, and limits on acceleration and deceleration which must be observed. In the flat output space, these constraints become constraints on higher order derivatives of the output and must be taken into account.

This paper is organized as follows. In Section 2 we review the dynamics of the system and present them in a form which is suitable for subsequent calculations. Section 3 introduces the topic of differential flatness and shows that the towed cable system is differentially flat under certain (reasonable) assumptions. In Section 4 we present the results of simulations which exploit flatness to study the behavior of the system under different operating conditions.

2. SYSTEM DYNAMICS

The equations of motion for the towed cable system are simulated using a finite element model in which segments of the cable are replaced by rigid links. The forces acting on the segment are lumped and applied at the end of each rigid link. Rather than choosing a set of generalized coordinates for the system, we make explicit use of Lagrange multipliers to enforce the inelasticity constraint of the cable. This has the advantage of providing the dynamic tension on the cable, which can then be checked to insure that the cable does not snap. The model is also be easily modified to allow elastic cables to be simulated and analyzed.

We model the cable as a finite set of inextensible links connected by spherical joints. Let q_i represent the ith joint between cable elements, and $l_i = \|q_{i+1} - q_i\|$ represent the length of the ith element of the cable. The tangent at the ith element is given by

$$\tau_i = \frac{q_{i+1} - q_i}{l_i}.$$

The constraints on the system are of the form

$$\|q_{i+1} - q_i\|^2 = l_i^2 \qquad i = 0, \ldots, n-1$$

and model the fact that the length of each link must remain constant. These constraints can be differentiated to yield the Pfaffian constraint

$$W(q)\dot{q} := \begin{bmatrix} -\tau_0^T & \tau_0^T & 0 & & \\ 0 & -\tau_1^T & \tau_1^T & 0 & \\ & & \ddots & \ddots & \\ & & & -\tau_{n-1}^T & \tau_{n-1}^T \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = 0.$$

The equations of motion can be computed using Lagrange's equations with Lagrange multipliers added to account for the constraint forces. The equations of motion satisfy

$$\begin{aligned} M\ddot{q} + W^T\lambda &= f \\ W\ddot{q} + \dot{W}\dot{q} &= 0, \end{aligned} \tag{1}$$

where

$$M = \begin{bmatrix} m_0 I_3 & & & \\ & m_1 I_3 & & \\ & & \ddots & \\ & & & m_n I_3 \end{bmatrix}$$

and f represents the vector of external forces applied to each link. The second equation in (1) comes from differentiating the constraint equation. The vector of Lagrange multipliers λ corresponds to the tensions in the links of the cable and can be solved from equation (1):

$$\lambda = (WM^{-1}W^T)^{-1}(WM^{-1}f + \dot{W}\dot{q}). \tag{2}$$

We will make several assumptions to simplify the analysis. We model each link as a point mass, concentrated at the *end* of the link. A closer approximation would be to place the point mass at the center of the link, but for a large number of links this difference is small. Similarly, we lump all aerodynamic and other forces applied across a link at the end of the link.

The external forces acting on the system consist of gravity forces and aerodynamic drag. Thus we have

$$f = f_{\text{weight}} + f_{\text{drag}}.$$

The gravity forces have the form

$$f_{\text{weight}} = -mg\hat{\mathbf{z}}$$

where $\hat{\mathbf{z}}$ is a unit vector in the vertical direction. The drag forces on the cable have the form

$$f_{\text{drag}} = -\tfrac{1}{2}\rho C_{\text{drag}} \|v^{\perp}\| v^{\perp},$$

where $v^{\perp}$ is the velocity of the perpendicular to the direction of the cable at a given point, relative to any wind.

In addition to the forces on the cable, we must also consider the forces on the drogue and the towplane. The drogue is modeled as a sphere and essentially acts as a mass attached to the last link of the cable, so that the forces acting on it are included in the cable dynamics. The external forces on the drogue again consist of gravity and aerodynamic drag. The drag forces on the drogue are slightly different than those on the cable since the drogue is modeled as a sphere instead of a cylinder. The towplane is attached to the top of the cable (at q_0) and is subject to drag, gravity, and the force of the attached cable. For simplicity, we simply model the towplane as a pure force applied at the top of the cable.

For the ith link in the chain, the system equations can be written in a simple way by making use of the tension in the cable, $T_i = \lambda_i \tau_i$. The equations governing the motion of the ith mass are then

$$m_i \ddot{q}_i + T_i - T_{i-1} = -\tfrac{1}{2}\rho C_{\text{drag}} \|v_i^{\perp}\| v_i^{\perp} - m_i g\hat{\mathbf{z}} \tag{3}$$

The motion of the drogue and the towplane are obtained in a similar fashion and have the same basic form.

3. FLATNESS ANALYSIS AND TRAJECTORY GENERATION

The model for towed cable system proposed above is differentially flat using the position of the drogue as the flat output. To see this, suppose that we know the motion of the drogue at the bottom of the cable. Given the mass of the drogue and its shape, we can determine all forces which must be acting on the drogue from gravity and drag (even in the presence of wind, as long as the wind direction is known). Using Newton's second law, we can determine the direction and tension of the cable, since this is the only undetermined force in the system. Given the direction of the cable, we can integrate back one link (from q_n to q_{n-1}) and determine the motion of the top of the link segment.

Proceeding inductively, assume we know the position, velocity, and acceleration of the ith link along the cable and the tension due to the links further down the cable. Using equation (3), we can solve for the tension in the previous link, T_{i-1}, and hence determine the direction of the cable at that point. Using the (fixed) length of the link, we can compute the motion of the $(i-1)$th link and continue to work our way back up the cable until we reach the towplane. In this way, we see that the system is differentially flat, even in the presence of aerodynamic drag and (known) wind.

Knowing that the system is differentially flat, the trajectory generation problem reduces to searching for trajectories of the drogue which give an appropriately bounded motion for the towplane. Since it is the motion of the drogue that we are most interested in, this point of view can be more insightful than commanding the motion of the top of the cable and then simulating to determine the eventual motion of the drogue.

To illustrate how trajectory generation proceeds, consider the dynamics of the system in the absence of aerodynamic drag. We also make the simplification that the tension on the cable is primarily due to gravity and not to accelerations and drag and that the lengths and masses of each link in the discrete approximation are the same. Again, the only results of these simplifications is to minimize computations and highlight the geometry of the problem in a concise way. The numerical simulations of the next section make use of the complete equations for the system.

With these simplifying assumptions in place, the equations of motion for the system can be shown to satisfy

$$q_0 = q_n + nl + \sum_{i=1}^{n} a_i^n \left(\frac{l}{g}\right)^i q_n^{(2i)} \tag{4}$$

$$T_0 = nmg + nm\ddot{q}_n + m \sum_{i=1}^{n} b_i^n \left(\frac{l}{g}\right)^i q_n^{(2i+2)} \tag{5}$$

where the coefficients a_i^n and b_i^n satisfy the recursive relations

$$a_1^1 = 1 \quad a_1^{n+1} = a_1^n + 1$$

$$a_i^1 = 0 \quad a_i^{n+1} = a_i^n + \frac{b_{i-1}^n}{n} \quad i = 2, \ldots, n+1$$

$$b_1^1 = 1 \quad b_1^{n+1} = a_1^{n+1} + b_1^n$$

$$b_i^1 = 0 \quad b_i^{n+1} = a_i^n + b_i^n + \frac{b_{i-1}^n}{n} \quad i = 2, \ldots, n+1$$

We see from the basic form of equation (4) that the system is flat (everything depends only on q_n and its derivatives) and that given a desired motion of the drogue, we can easily solve for the required motion of the towplane. Furthermore, the coefficients in front of the derivatives of the flat output determine the rate at which the flat output can change while satisfying the input constraints imposed by the towplane.

4. SIMULATIONS

We now perform a series of simulations to illustrate the utility of flatness in analyzing the behavior of the system. All of the simulations in this section were performed by coding the equations above (with all aerodynamic terms included) and numerically solving for the feasible trajectories of the system give the trajectory of the drogue. To avoid the use of symbolic expressions for the derivatives, the past two values of the position of each link were stored and used to compute the velocity and acceleration of the link. This numerical procedure was found to work well for some situations, but had some severe limitations which limited overall performance of the algorithm (these limitations are discussed at the end of this section).

The parameters used for the simulations are those shown in Figure 1, with 20 links used to model the cable. The air density was allowed to vary as a function of altitude. The normal velocity to the cable, $v^{\perp}$, was initially computed using the tension of the current link, T_i. However, to get better agreement with the simulation model used for verifying the results, an iterative procedure was implemented which allowed T_{i-1} to be used to estimate $v^{\perp}$.

4.1 *Circular motion of the drogue*

To check the accuracy of the numerical simulation, the first scenario which was tested was that of a relative equilibria. For this motion, an exact solution can be computed, allowing the flatness based solution to be compared to the analytical one (which is also numerically computed, but using a method which relies on the system being at a relative equilibrium). Figure 2 shows the comparison between the flatness-based solution and the analytical solution. The input data for this simulation was a command that the drogue move in a circle of radius 10 meters. The flatness-based solution shows very good agreement with the analytical solution, resulting in an error of 27.5 m in the towplane location for a cable length of 2,325 m.

4.2 *Circular motion in the presence of steady wind*

A second simulation was run to understand the effects of wind on the motion of the system. The effect of wind on

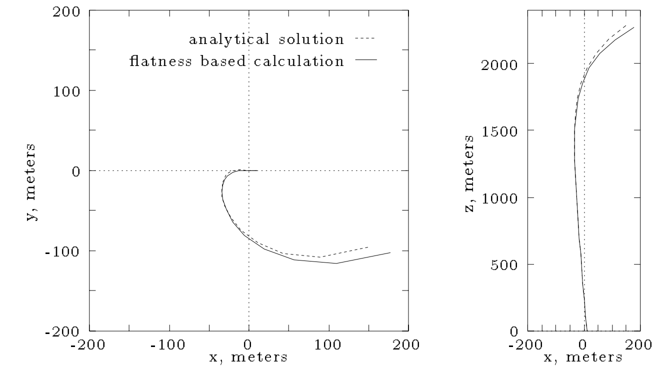

Fig. 2. Comparison of flatness-based solution with analytical solution for a relative equilibrium.

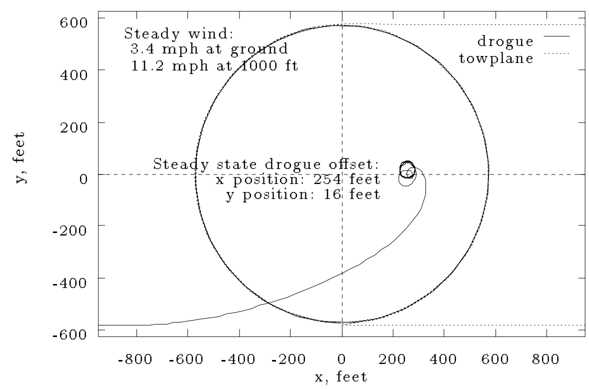

Fig. 3. Effect of wind on the xy position of the drogue.

the motion of the cable is quite interesting and has apparently never been discussed in the literature. Simulation studies indicate that ground winds have the largest effect on the location of the drogue. This is to be expected since these winds directly affect the forces acting on the drogue. Ground winds cause a steady state offset in the xy position of the drogue and an oscillation in the z position. This oscillation occurs because the distance from the plane to the drogue changes during the flight path. Since the cable is practically vertical, this change in distance causes the drogue to be lifted and dropped as the plane revolves.

A sample simulation showing the behavior of the system is shown in Figures 3 and 4. The simulated wind corresponds to "light" wind conditions with ground winds of approximately 1.5 m/s, increasing to 8 m/s at the top of the cable. Note that even though the ground winds are purely in the x direction, the drogue is offset in both the x and y directions. This is due to the overall system dynamics interacting with the disturbance created by the wind.

To compensate for oscillation in height which occurs when winds are present, the trajectory of the towplane can be modified appropriately. The basic idea is to modify the trajectory of the plane so that the distance be-

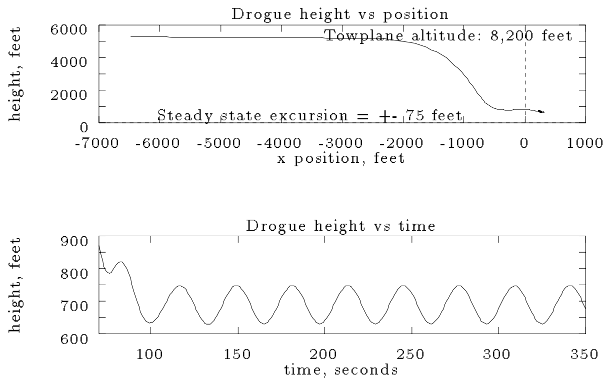

Fig. 4. Effect of wind on the z position of the drogue.

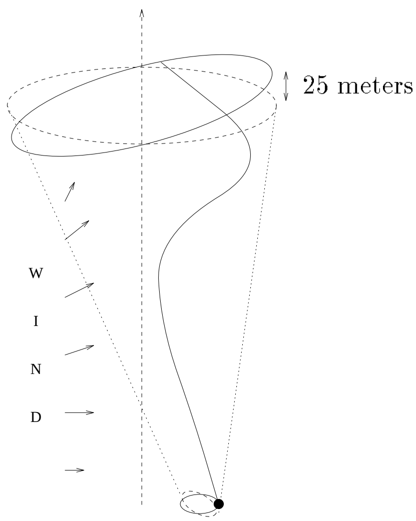

Fig. 5. Modification of towplane trajectory to account for wind.

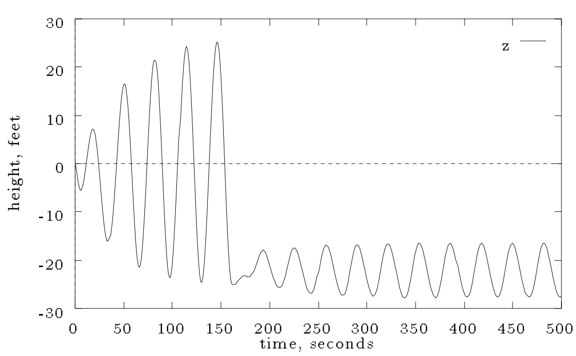

Fig. 6. Variation in drogue height with compensation at $t = 160$.

tween the drogue and the top of the cable is more nearly constant, as shown in Figure 5. The resulting behavior of the system is shown in Figure 6. The modified towplane trajectory is followed after $t = 160$, resulting in a considerable improvement in the vertical excursion of the drogue.

An interesting question which flatness is ideally posed to

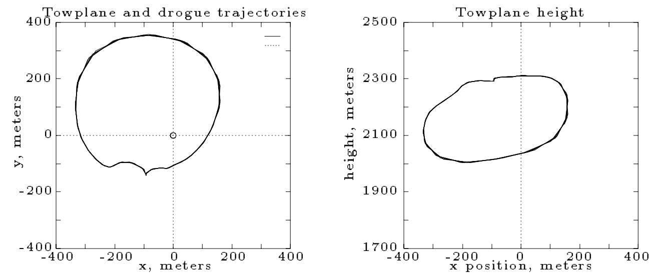

Fig. 7. Towplane trajectory required in order for the drogue to move in a perfect circle in the xy plane.

answer is the following: what type of trajectory would it take to completely eliminate the vertical motion of the drogue due to wind. A partial answer to this question is shown in Figure 7, which shows the required motion of the towplane to obtain a perfect circle at the bottom of the cable. This trajectory was obtained by explicitly solving for the solution of the towed cable dynamics in the presence of a steady "light" wind. The computed towplane trajectory is not achievable for current aircraft and clearly indicates that some tradeoff between the desired performance of the drogue and the limited abilities of the towplane is required.

Notice that the trajectory in Figure 7 is consistent with the modified trajectory depicted in Figure 5. The towplane must follow a tilted, roughly circular orbit in order to maintain the position of the drogue.

4.3 *Limitations of the algorithm*

Our primary interest in studying the system is generation of trajectories from one operating point to another in a manner which accounts for the performance limits of the towplane (e.g., available thrust and bank angle limits). Although the flatness-based algorithm used above was able to compute feasible trajectories for certain nonconstant motions (the circular trajectories have derivatives of all order, for example), it did not perform well when given trajectories that had small discontinuities in the higher derivatives of the drogue motion.

The limitations of the algorithm are illustrated in Figure 8, which shows the results of commanding the drogue to move from a circular orbit of radius 20 meters to a circular orbit of radius 10 meters in approximately 100 seconds. This trajectory was implemented by commanding the radius of the drogue to vary according to the formula

$$r(t) = r_0 - \frac{r_f - r_0}{2}\Big(1 - \cos(\frac{\pi t}{T_f - T_0})\Big)$$

where r_0 is the radius at time T_0 and r_f is the desired radius at time T_f. The cosine function is used to obtain a smooth (C^1) transition of the radius.

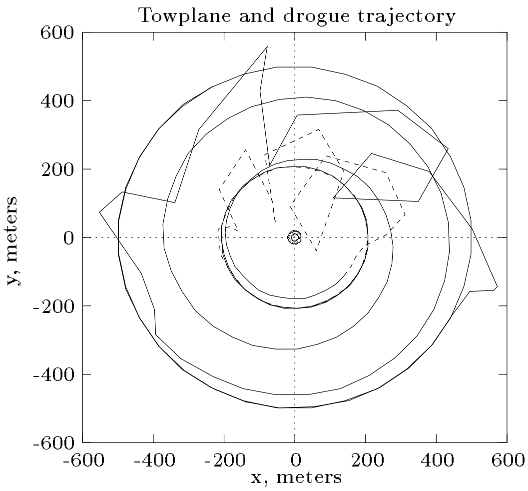

Fig. 8. Response of the algorithm to a spiral change in radius of the drogue. The nonsmooth sections of the twoplane path correspond to instabilities in the numerical algorithm.

The simulation results of Figure 8 clearly show the extreme sensitivity of the algorithm at the transition into and out of the spiral pattern. This simulation is consistent with other simulations performed by the author and is a serious limitation of the current algorithm. It is interesting to note that during the spiral motion the system is never at either an equilibrium point or a relative equilibrium. However, it is only during the transitions to and from the spiral that the instability of the numerical algorithm is a problem.

Other indications of the numerical instability of the algorithm where the long transient response at the beginning of all simulations (not shown in the figures included here) and the fact that decreasing the time step in the algorithm often led to worse performance instead of better. For example, a time step of 1 second was used for all of the trajectories presented here, corresponding to about 1/30 of a revolution of the drogue. This time step could be decreased to approximately 0.25 seconds before the algorithm failed to converge. This lack of convergence was exhibited by the top links of the cable failing to stay at a fixed angle between one time step and the next.

There are a number of potential solutions to the instability of the algorithm. Currently a causal computation is used to determine the velocity and acceleration of each link. This results in a triangular structure which can be exploited to obtain an efficient algorithm for computing trajectories. Local smoothing of the velocity and acceleration computations was implemented, but this seemed to have only a minor effect and resulted in time skewing as the computation proceeded up the cable. A potentially more robust approach would be to compute the entire trajectory history of a given link and then smooth the trajectory to remove high frequency numerical noise before computing the trajectory of the next link. This approach and other modifications are the subject of current work.

5. CONCLUSIONS

The use of differential flatness for trajectory generation gives important and powerful guidance for generating nonlinear controls for strongly nonlinear systems. In the instance of the towed cable system considered here, it reduces the problem of trajectory generation from a high dimensional, dynamic optimization problem to a low dimensional, nonlinear, static optimization problem. The main trade-off is that the low-dimensional problem is highly nonlinear. However, the fact that it is essentially algebraic leads to a more tractable problem in a number of instances.

A major difficulty in the current approach is the apparent numerical instability of the algorithm. Although it is clear that some instability is inherent in the nature of the problem (since it requires very high derivatives of the flat output), further work on developing numerically robust methods for computing flatness based trajectories should yield improvement over the results reported here.

Future work will explore the use of more complicated models of the system combined with optimization based approaches for solving the trajectory generation problem with realistic input constraints.

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