Consensus Seeking Using Multi-Hop Relay Protocol

Zhipu Jin and Richard M. Murray

Abstract

We consider the problem of average consensus seeking in networked multi-agent systems. Based on local information and a simple distributed algorithm, states of all agents automatically converge to the average value of the initial conditions, where the convergence speed is determined by the algebraic connectivity of the underlying communication network. In order to achieve an average consensus quickly, we propose a new type of consensus protocol, multi-hop relay protocol, in which each agent expands its knowledge by employing multi-hop communication links. We explicitly show that multi-hop relay protocol increases the convergence speed without physically changing the network topology. Moreover, accumulated delays along communication links are discussed. We show that, for multi-hop relay protocol, the faster the protocol converges, the more sensitive it is to the delay. This tradeoff is identified when we investigate the stable delay margin using frequency sweep method.

Index Terms

Networked multi-agent system, average consensus, multi-hop relay protocol, distributed algorithm, convergence speed, time delay.

I. INTRODUCTION

Consensus seeking based on simple distributed protocols in networked multi-agent systems has attracted many researchers from different disciplines. Vicsek *et al.* [1] proposes a simple and popular model for self-driven particles alignment problem in which each agent updates its heading based on the average of its own heading and its neighbors'. Based on simulation results, they show that all agents move in the same direction eventually. A theoretical explanation for

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Vicsek's model is given by Jadbabaie *et al.* in [2], where a discrete-time consensus protocol is proposed based on stochastic matrix theory. Suppose $x_i(k)$ is the state of agent i at time k. The consensus protocol can be summarized as

$$x_i(k+1) = \sum_{j \in \mathcal{N}(i) \cup \{i\}} \alpha_{ij}(k) x_j(k) \tag{1}$$

where $\mathcal{N}(i)$ represents the set of agents who can directly communicate with agent i at step k, $\alpha_{ij} \geq 0$, and $\sum_j \alpha_{ij}(k) = 1$. In other words, each agent's state is updated by a weighted average of its current value and its neighbors'. Correspondingly, a continuous-time consensus protocol is proposed by Olfati-Saber and Murray in [3] as

$$\dot{x}_i(t) = -\sum_{j \in \mathcal{N}(i)} w_{ij}(t)\big(x_i(t) - x_j(t)\big) \tag{2}$$

where $w_{ij}(t)$ are positive weights. It has been shown that, for balanced communication topologies, this protocol solves the average consensus problem, *i.e.*, the states of all agents converge to the exact average of the initial values exponentially. Moreover, consensus seeking over general directed communication topologies is discussed in [4], [5], [6], [7]. When the dynamics of agent state updating is nontrivial, the consensus behavior can be treated as the synchronization problem of interconnected dynamical systems. Different approaches are reported, such as Lyapunov's direct method in [8] and Laplacian matrix decomposition method in [9], [10]. Also, sufficient conditions for interconnected dynamical systems synchronization over general interaction topologies are discussed in [11].

The idea and technic of consensus seeking has been employed in many engineering problems. For coordinated control, consensus schemes have been applied to achieve vehicle formations [5]. In rendezvous problems, consensus seeking is used to control agents arrive at a certain location simultaneously [12]. In sensor networks, it has been used for data fusion [13] and distributed Kalman filters [14]. Other applications include spacecraft attitude alignment [15], distributed decision making [16], asynchronous peer-to-peer networks [17], and synchronization over robot networks [18].

For all of those applications, fast consensus convergence speed is important. Xiao and Boyd treat the consensus seeking process as an optimal linear iteration problem and show in [19] that, if the global topology of the communication network is known beforehand, the convergence speed can be increased by finding the optimal weights associated with communication links.

Recent work on designing the fastest averaging algorithm on arbitrary network is reported in [20], in which a distributed sub-gradient method is used. In [21], Kim and Mesbahi consider maximizing the algebraic connectivity of the state-dependent communication network by finding the optimal vertex configuration. Besides these optimization approaches, Olfati-Saber proposes a "random rewiring" procedure in [22] to change the network topology into a "small-world" graph so that the consensus process can be boosted dramatically.

It has been noticed that current consensus protocols restrict information exchange among agents inside the local connectivity and the information propagation is slow. In order to enlarge the information exchange region in a systematic way, we propose a multi-hop relay protocol based on multi-hop paths in the network. The idea is simple: each agent can get more information by passing its neighbors' states to others. The improvement in the convergence speed is given explicitly and is verified by simulation results. This protocol does not change the network topology and is easy to implement.

Furthermore, delays along communication links are considered. Previous work on consensus protocol with time delays includes [3], where a necessary and sufficient condition for the stability of the consensus protocol with homogeneous communication delays is given, and [23], where delays are only associated with neighbor's states and the final value of the consensus state is hard to predict. In this paper, we investigate delay accumulations along multi-hop communication links. By searching the stable delay margin, a tradeoff between convergence speed and delay sensitivity is identified.

The remainder of this paper is organized as follows: In Section II, after introducing some necessary notations, we propose a multi-hop relay protocol for fast consensus seeking. We explicitly show the improvement on the convergence speed in Section III. Section IV is devoted to investigating the stability of the protocol with homogeneous communication delays. Examples and simulation results are provided in Section V and conclusions are summarized in Section VI.

II. Multi-Hop Relay Protocol

We introduce some notation and concepts that will be used through this paper. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to represent the communication topology in a networked multi-agent system where $\mathcal{V}$ is a set of vertices, which stand for the agents, and $\mathcal{E} \subseteq \mathcal{V}^2$ is a set of edges, which stand for the communication links. Each edge in the graph is denoted by (v_i, v_j), in which we call v_i

the *head* and v_j the *tail*. A graph $\mathcal{G}$ is called *symmetric* if, whenever $(v_i, v_j) \in \mathcal{E}$, $(v_j, v_i) \in \mathcal{E}$ as well. In this paper, we focus on symmetric graphs.

In a symmetric graph, the number of edges whose head is v_i is called the *degree* of node v_i. The set of neighbors of vertex v_i is denoted by $N(v_i) = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. A *path* is a sequence of distinct vertices $[u_0, \cdots, u_r]$ such that $(u_{i-1}, u_i) \in \mathcal{E}$ for i from 1 to r. We say a path is an *m-hop path* if it has m edges. A symmetric graph is *connected* if any two vertices in the graph can be joined by a path.

An *adjacency* matrix $A = \{a_{ij}\}$ for graph $\mathcal{G}$ with n vertices is a $n \times n$ matrix defined as

$$a_{ij} = \begin{cases} 1, & (v_i, v_j) \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}$$

More generally, a *weighted adjacency* matrix $\mathcal{A} = \{\alpha_{ij}\}$ is defined as

$$\alpha_{ij} = a_{ij} \cdot w_{ij}$$

where $w_{ij} > 0$ is the weight associated with edge (v_i, v_j). For symmetric graphs, we assume that $w_{ij} = w_{ji}$. The degree of node v_i is $\sum_j \alpha_{ij}$. Let $\mathcal{D}$ be a diagonal matrix with the degree of each vertex along the diagonal and the *Laplacian* matrix $\mathcal{L}$ is defined by $\mathcal{L} = \mathcal{D} - \mathcal{A}$.

Let x_i denote the state of agent v_i. A networked multi-agent system reaches a *consensus* if $x_i = x_j$ for all v_i and $v_j \in \mathcal{V}$. This common value is called the *consensus state*, which is denoted by η. The dynamics of the whole state updating process with the continuous-time consensus protocol in Equation (2) can be represented by

$$\dot{X} = -\mathcal{L}X \tag{3}$$

where $X = [x_1, \cdots, x_n]'$. It is known that for a connected graph protocol (2) solves the average consensus problem, *i.e.*, $\lim_{t\to\infty} x_i(t) = \eta = \sum_i x_i(0)/n$ for any $v_i \in \mathcal{V}$. Moreover, eigenvalues of $\mathcal{L}$ are real and denoted by $0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leq \cdots \leq \lambda_n(\mathcal{L})$. The convergence speed is bounded by the second smallest eigenvalue $\lambda_2(\mathcal{L})$, which is called the *algebraic connectivity* of graph $\mathcal{G}$.

Clearly, information exchange is restricted between a single agent and its "immediate" neighbors in protocol (2). In order to enlarge the information exchanging region, we use those multi-hop paths in the graph. First, we introduce the two-hop relay protocol as

$$\dot{x}_i = -\sum_{j \in N(v_i)} w_{ij} \Big((x_i - x_j) + \sum_{k \in N(v_j)} w_{jk}(x_i - x_k) \Big) \tag{4}$$

where what agent v_j sends to v_i is not only its own state, but also a collection of its immediate neighbors' states. In other words, the information of agent v_k can be passed to v_i along the two-hop communication link $[v_k \to v_j \to v_i]$. Generally, we can write an m-hop relay protocol over graph $\mathcal{G}$ as

$$\dot{x}_i = \underbrace{-\sum_j w_{ij}\big((x_i - x_j) + \sum_k w_{jk}((x_i - x_k) + \cdots)\big)}_{m \text{ layers}} \tag{5}$$

where the information is passed around along all possible multi-hop paths in $\mathcal{G}$ as long as the length of the path is no larger than m.

We call Equation (5) the *multi-hop relay protocol* with the parameter m, which is the number of the layers for state updating at each agent, or the longest length of multi-hop paths the information goes through. But control the value of m, we can directly control the size of information exchange region of each agent. Note that the consensus protocol in Equation (2) is also a multi-hop relay protocol with $m = 1$, which we call the *single-hop relay protocol*.

III. Dynamics Analysis for Multi-Hop Replay Protocol

A. Dynamics of Two-Hop Relay Protocol

Let us start with two-hop relay protocol in Equation (4). Since agent v_j sends its state and a collection of its immediate neighbors' states to v_i, it is equivalent to adding virtual edges to the original graph $\mathcal{G}$ corresponding to those two-hop paths. We define *two-hop graph* $\hat{\mathcal{G}} = (\mathcal{V}, \hat{\mathcal{E}})$ as a graph that has the same vertex set but all the edges are "two-hop" paths of $\mathcal{G}$. Figure 1 shows an example of two-hop graph.

Fig. 1. A graph and its two-hop graph

Since we focus on symmetric graphs, there exist self-loops in the two-hop graph $\hat{\mathcal{G}}$, *i.e.*, the head and tail of an edge are the same. However, according to Equation (4), those self-loops have no contribution to the dynamics since they are cancelled out in the second layer. We omit them in following discussion. Also, multiple two-hop paths may exist between the same pair of vertices. In that case, we consider them as one edge and the weight equals the sum of those paths. Thus, given $\mathcal{A}$ of $\mathcal{G}$ and $\mathcal{A}^2 = \{\beta_{ik}\}$, the adjacency matrix $\hat{\mathcal{A}} = \{\hat{\alpha}_{ik}\}$ of $\hat{\mathcal{G}}$ is

$$\hat{\alpha}_{ik} = \begin{cases} \sum_j w_{ij} w_{jk} = \beta_{ik}, & i \neq k \\ 0, & i = k. \end{cases}$$

Let the corresponding Laplacian matrices in $\mathcal{G}$ and $\hat{\mathcal{G}}$ are denoted by $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. For graph $\mathcal{G}$ with two-hop relay protocol, we consider the joint graph $\tilde{\mathcal{G}} = \mathcal{G} \bigcup \hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E} \bigcup \hat{\mathcal{E}})$ and the dynamics of the whole system is described as

$$\dot{X} = -\tilde{\mathcal{L}} X = -(\mathcal{L}_1 + \mathcal{L}_2) X. \tag{6}$$

It is not true that $\hat{\mathcal{G}}$ is always connected when $\mathcal{G}$ is connected. Figure 2 shows a simple example. The original graph on the left is connected, but the two-hop graph on the right is composed of two disconnected subgraphs. However, the following lemma states that two-hop relay protocol still solves the average consensus problem.

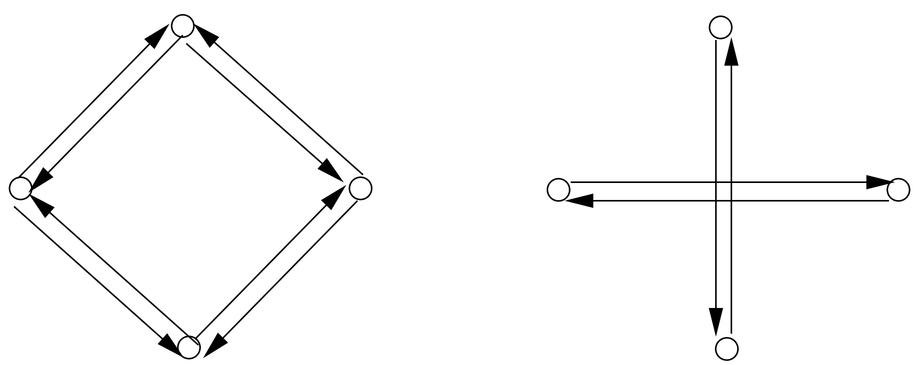

Fig. 2. An example of disconnected two-hop graph

Lemma 3.1: For a connected graph $\mathcal{G}$, Equation (6) converges to the average consensus state.

Proof: Since $\mathcal{L}_2$ is still a Laplacian matrix of a symmetric graph, thus the n-dimensional vector $\mathbf{1}_n = [1, \ldots, 1]^T$ is still an eigenvector associated with the eigenvalue 0. It is easy to show that the joint graph $\tilde{\mathcal{G}}$ is connected and the result follows. ■

The two-hop relay protocol needs extra communication bandwidth. We rewrite the single-hop relay protocol (2) as:

$$\dot{x}_i = -x_i \sum_{j\in N(v_i)} w_{ij} + \sum_{j\in N(v_i)} w_{ij}x_j \tag{7}$$

and the two-hop relay protocol (4) as

$$\begin{aligned}\dot{x}_i &= -x_i \textstyle\sum_{j\in N(v_i)} w_{ij}(1+\sum_{k\in N(v_j)} w_{jk}) \\ &\quad + \textstyle\sum_{j\in N(v_i)} w_{ij}\big(x_j + \sum_{k\in N(v_j)} w_{jk}x_k\big).\end{aligned} \tag{8}$$

For protocol (2), what link (v_i, v_j) transmits is the value of x_j. For protocol (4), what link (v_i, v_j) transmits is the value of x_j, $\sum w_{jk}x_k$, and $\sum w_{jk}$. However, for a static graph, $\sum w_{jk}$ is a constant and only needs to be transmitted once. Thus, the two-hop relay protocol needs as twice the communication bandwidth as single-hop relay protocol needs except at the very beginning.

For the consensus converge speed, we have the following theorem.

Theorem 3.2: If graph $\mathcal{G}$ is connected, then

$$\lambda_2(\tilde{\mathcal{L}}) \geq \lambda_2(\mathcal{L}_1) + \lambda_2(\mathcal{L}_2). \tag{9}$$

Proof: For any vector x, it is true that

$$\begin{aligned}x^T\tilde{\mathcal{L}}x &= x^T\mathcal{L}_1x + x^T\mathcal{L}_2x \\ &= \textstyle\sum_{(v_i,v_j)\in\mathcal{E}} w_{ij}^2(x_i-x_j)^2 \\ &\quad + \textstyle\sum_{(v_i,v_j)\in\hat{\mathcal{E}}} w_{ij}^2(x_i-x_j)^2.\end{aligned}$$

Assume that x is a unit vector and orthogonal to $\mathbf{1}_n$, then

$$\frac{x^T\mathcal{L}_1x}{x^Tx} = \frac{\sum_{(v_i,v_j)\in\mathcal{E}}(x_i-x_j)^2}{\sum_{v_i\in\mathcal{V}} x_i^2} \geq \lambda_2(\mathcal{L}_1)$$

and the equality holds only when x is an eigenvector associated with $\lambda_2(\mathcal{L}_1)$.

If we take x to be a unit eigenvector of $\tilde{\mathcal{L}}$, orthogonal to $\mathbf{1}_n$, associated with eigenvalue $\lambda_2(\tilde{\mathcal{L}})$, then we have

$$\lambda_2(\tilde{\mathcal{L}}) = \frac{x^T\tilde{\mathcal{L}}x}{x^Tx} = \frac{x^T(\mathcal{L}_1+\mathcal{L}_2)x}{x^Tx} \geq \lambda_2(\mathcal{L}_1) + \frac{x^T\mathcal{L}_2x}{x^Tx}. \tag{10}$$

When $\hat{\mathcal{G}}$ is connected, $x^T\mathcal{L}_2x/x^Tx \geq \lambda_2(\mathcal{L}_2) > 0$. When $\hat{\mathcal{G}}$ is disconnected, $x^T\mathcal{L}_2x/x^Tx \geq \lambda_2(\mathcal{L}_2) = 0$. Thus, two-hop relay protocol improves the convergence speed by at least $\lambda_2(\mathcal{L}_2)$. ■

Theorem 3.2 shows that two-hop relay protocol improves the convergence speed and the improvement depends on the topology of $\hat{\mathcal{G}}$.

B. Dynamics of Multi-Hop Relay Protocol

For the multi-hop relay protocol (5) with $m > 2$, we think that the protocol adds more virtual edges to the original graph to enforce the connectivity. The dynamics of multi-hop relay protocol can be written as

$$\dot{X} = -(\mathcal{L}_1 + \mathcal{L}_2 + \cdots + \mathcal{L}_m)X \tag{11}$$

where $\mathcal{L}_i$ is the Laplacian of the i-hop graph.

Theorem 3.3: For a connected graph $\mathcal{G}$, the multi-hop relay protocol (5) solves the average consensus problem.

Proof: We have shown that, for $m = 1$ and $m = 2$, the dynamics (11) converges to the average value of the initial states. For $m > 2$, it is true that $(\mathcal{L}_1 + \mathcal{L}_2 + \cdots + \mathcal{L}_m)$ is still a Laplacian of the joint graph with all virtual edges corresponding to those multi-hop paths. Since the joint graph is connected, the result follows. ■

Following a similar argument in Theorem 3.2, it is easy to show that increasing m will increase the consensus convergence speed. However, there exist a couple of drawbacks. First, the worst case computation complexity of the m-hop relay protocol on each agent is $O(n^{m-1})$. For large scale networks, it quickly becomes infeasible. Second, at least m-times communication bandwidth are needed and the network is easy to get congested. Third, communication delays will accumulate along m-hop paths and result in instability. We will discuss the sensitivity to the communication latency in the next section.

IV. Multi-Hop Relay Protocols with Time Delays

For time delays along communication links, we consider the transfer function associated with edge (v_i, v_j) and latency τ_{ij} is $e^{-\tau_{ij}s}$. In the multi-hop relay protocol, delays can accumulate along those multi-hop paths. We study the simplest case where all delays are identical, *i.e.*, $\tau_{ij} = \tau$ for any $(v_i, v_j) \in \mathcal{E}$. Protocol (5) can be written as

$$\begin{aligned} \dot{x}_i \;\; = \;\; & -\textstyle\sum_{j\in\mathcal{N}(i)} w_{ij}\Big((x_i(t-\tau) - x_j(t-\tau)) \\ & + \textstyle\sum_{k\in\mathcal{N}(j)} w_{jk}(x_i(t-2\tau) - x_k(t-2\tau))\cdots\Big) \end{aligned} \tag{12}$$

and the dynamics is

$$\dot{X} = -\mathcal{L}_1 \cdot X(t-\tau) - \mathcal{L}_2 \cdot X(t-2\tau) - \cdots - \mathcal{L}_m \cdot X(t-m\tau). \tag{13}$$

Let $Z = V^{-1}X$ where

$$V^{-1} = \begin{bmatrix} 1 & \mathbf{1}_{n-1} \\ \mathbf{1}_{n-1} & -I \end{bmatrix}. \tag{14}$$

We can change Equation (13) into

$$\dot{Z} = -\sum_{i=1}^{m} V^{-1}\mathcal{L}_i V Z(t - i\tau). \tag{15}$$

It has been shown in [24] that

$$V^{-1}\ \mathcal{L}_i V = \begin{bmatrix} 0 & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1} & \mathcal{L}_{i,22} \end{bmatrix}.$$

Taking the Laplace transform of Equation (15), all states except z_1 of this autonomous system exponentially converge to 0 if and only if the following characteristic polynomial

$$p_{22}(s, e^{-\tau s}) = \det\big(sI + \sum_{i=1}^{m} \mathcal{L}_{i,22} e^{-i\tau s}\big) \tag{16}$$

has no zeros in the closed right half plane (RHP). This condition is equivalent to the case where the characteristic polynomial

$$p(s, e^{-\tau s}) = \det\big(sI + \sum_{i=1}^{m} \mathcal{L}_i e^{-i\tau s}\big) \tag{17}$$

has no zeros in RHP except a simple one at the origin. In [25], $p_{22}(s, e^{-\tau s})$ and $p(s, e^{-\tau s})$ are called real *quasipolynomials* of s. In the rest of this paper, we will consistently use this name.

One essential property of quasipolynomials is the continuity of the zeros with respect to the time delay. In other words, when τ increases continuously, zeros in the left half plane (LHP) continuously move to RHP except the zero $s = 0$. We need to find the minimum value of τ such that the first stable zero crosses the imaginary axis. Besides, the conjugate symmetry property of quasipolynomials makes it possible to calculate the value of τ and the corresponding crossing frequency.

Definition 4.1: Given initial value $X(0)$ and assumption $X(t) = 0$ for $t < 0$, the smallest value of τ such that the multi-hop relay protocol does not converge to a consensus is determined as

$$\tau^* = \min\{\tau > 0 \,|\, p(j\omega, e^{-j\tau\omega}) = 0 \ \text{ and } \omega \neq 0\}, \tag{18}$$

which is called the *delay margin* of the multi-hop relay protocol.

It is easy to show that, for any $\tau \in [0, \tau^*)$, the system (13) converges to an average consensus. For the single-hop relay protocol, based on the Schur decomposition theorem, there exists a unitary matrix T such that $U = T^{-1}\mathcal{L}T$ is upper triangular with the eigenvalues along the diagonal. Then the quasipolynomial is

$$\begin{aligned}\det(sI + \mathcal{L}_1 e^{-\tau s}) &= \det(T(sI + Ue^{-\tau s})T^{-1}) \\ &= s \cdot \Pi_{i=2}^{n}(s + \lambda_i(\mathcal{L}_1)e^{-\tau s}).\end{aligned}$$

Let ω is the crossing frequency, then $s = j\omega$ and we have at least one equation as following

$$j\omega = -e^{\tau j\omega}\lambda_i(\mathcal{L}_1). \tag{19}$$

Solving this equation gives us

$$\begin{cases} \omega = \lambda_i(\mathcal{L}_1) \neq 0 \\ \tau = \pi/2\lambda_i(\mathcal{L}_1). \end{cases} \tag{20}$$

So the delay margin

$$\tau^* = \min \frac{\pi}{2\lambda_i(\mathcal{L}_1)} = \pi/2\lambda_n(\mathcal{L}_1). \tag{21}$$

For the multi-hop relay protocol with $m > 1$, we cannot decompose multiple Laplacian matrices simultaneously. The following theorem gives the explicit result on τ^* by using frequency sweep method in [25].

Theorem 4.2: For system (13), suppose $\text{rank}(\mathcal{L}_{m,22}) = q$ and define

$$\bar{\tau}_i = \min_{1 \le k \le n-1} \theta_k^i / \omega_k^i$$

when the generalized eigenvalues $\lambda_i(G(s), H)$ satisfy the following equation:

$$\lambda_i(G(j\omega_k^i), H) = e^{-j\theta_k^i}$$

for some $\omega_k^i \in (0, \infty)$ and $\theta_k^i \in [0, 2\pi)$, where

$$G(s) = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & I \\ -sI & -\mathcal{L}_{1,22} & \cdots & -\mathcal{L}_{m-2,22} & -\mathcal{L}_{m-1,22} \end{bmatrix} \tag{22}$$

and

$$H = \text{diag}(I, \ldots, I, \mathcal{L}_{m,22}). \tag{23}$$

Then the consensus delay margin of (13) is

$$\tau^* = \min_{1 \le i \le (n-1)(m-1)+q} \bar{\tau}_i.$$

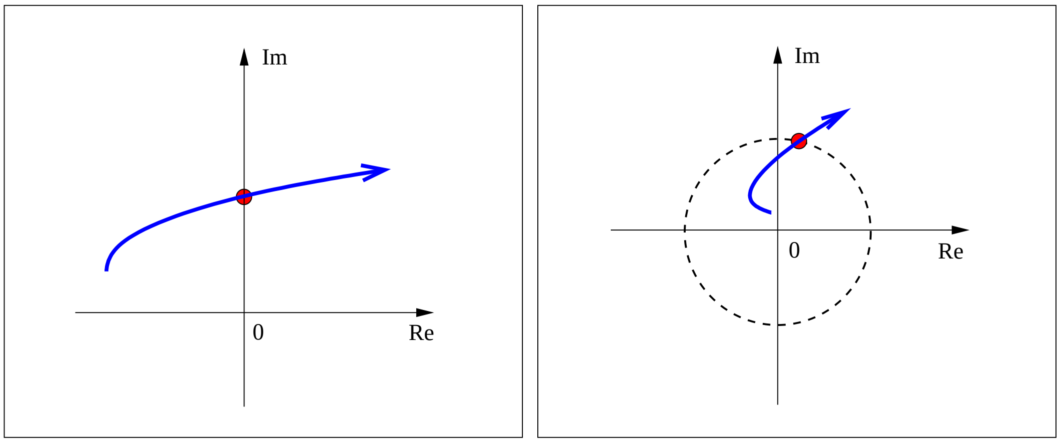

Fig. 3. Locus of the zero of the quasipolynomial and the generalized eigenvalue

Proof: *Generalized eigenvalue* for matrix pair (A, B) is defined as a scalar $\lambda(A, B)$ that satisfies $Ay = \lambda(A, B) \cdot By$ for an nonzero vector y. The vector y is called the *generalized eigenvector*. When $B = I$, $\lambda(A, B) = \lambda(A)$. It is a well-known fact that the number of finite generalized eigenvalues for (A, B) is at most equal to rank(B). Also, if rank(B) is constant, $\lambda(A, B)$ is continuous with respect to the elements of A.

Based on the aforementioned similarity transform, system (13) converges to a consensus if the following system

$$\begin{bmatrix} \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = -\sum_{i=1}^{m} \mathcal{L}_{i,22} \begin{bmatrix} z_2(t - i\tau) \\ \vdots \\ z_n(t - i\tau) \end{bmatrix} \tag{24}$$

is stable. Based on Schur determinant complement, we have

$$\det\left(sI + \sum_{i=1}^{m} \mathcal{L}_{i,22} e^{-i\tau s}\right) = (-1)^{(n-1)m} \det(G(s) - e^{-\tau s} H) \tag{25}$$

where G and H have the format as in Equation (22) and (23), respectively. Then, the quasipolynomial with multiple delay terms transfers to a new equation with a single delay term.

Since $\tau \in \mathcal{R}$, whenever a zero is located on the imaginary axis, there exists $s = j\omega$ so that $e^{-j\omega\tau}$ is a generalized eigenvalue of $(G(s), H)$. Figure 3 shows this correspondence by plotting the zero locus of the quasipolynomial and the eigenvalue locus. Thus, we can transfer the problem of finding τ so that the quasipolynomial has zeros with pure imaginary parts to the problem of finding a ω so that $(G(s), H)$ has a generalized eigenvalue with magnitude 1.

Since rank$(H) = q + (m-1)(n-1)$, there are at most $q + (m-1)(n-1)$ generalized eigenvalues of $(G(s), H)$. When s moves along the imaginary axis from 0 to $j\infty$, there exists at most $n-1$ frequency ω_k^i so that $\|\lambda_i(G(j\omega_k^i), H)\|_2 = \|e^{-j\theta_k^i}\|_2 = 1$. Thus, the delay margin τ^* is the minimum value of all possible $\bar{\tau}_k^i = \theta_k^i/\omega_k^i$. ■

Because there exist many efficient algorithms for generalized eigenvalue searching, it is much easier for us to find $\lambda(G(s), H)$ than to solve the quasipolynomial directly. However, for large scale graphs, it is still a difficult problem due to the sizes of G and H.

V. Examples and Simulation Results

In order to verify the efficiency of the multi-hop relay protocol, we test it on three different networks listed in Figure 4, denoted as $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ from left to right. Topology $\mathcal{G}_1$ is a 2-regular graph, $\mathcal{G}_2$ is a net in which each vertex connects to other vertices located inside a certain range, and $\mathcal{G}_3$ is a complete graph. All of them have ten vertices. They are all symmetric and connected. For simplicity, we assume that $w_{ij} = 1$ for any edge.

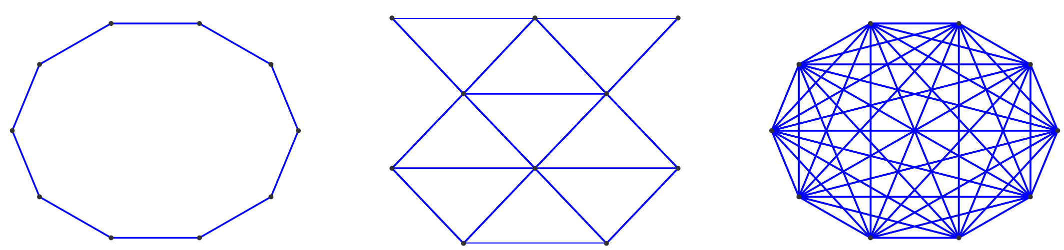

Fig. 4. Three different topologies: $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$

Figure 5, 6, and 7 show the simulation results of $\mathcal{G}_2$ with the same initial conditions and different delays. It is clear that, two-hop relay protocol converges much faster, but only tolerates much smaller time delays than single-hop relay protocol. Note that, even though the system becomes unstable, the sum of all states is still constant.

Table I shows the values of convergence speed and delay margin for all three graphs with different multi-hop relay protocols. Delay margins with single-hop relay protocol are calculated

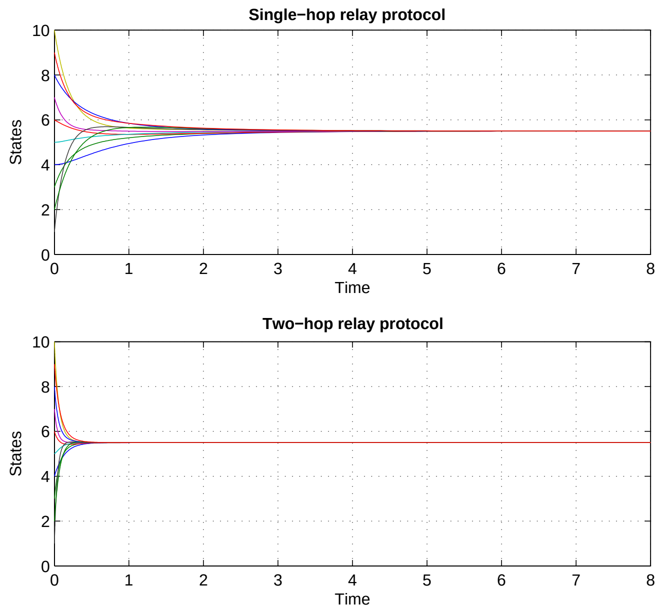

Fig. 5. Consensus convergence for graph $\mathcal{G}_2$ with no delay.

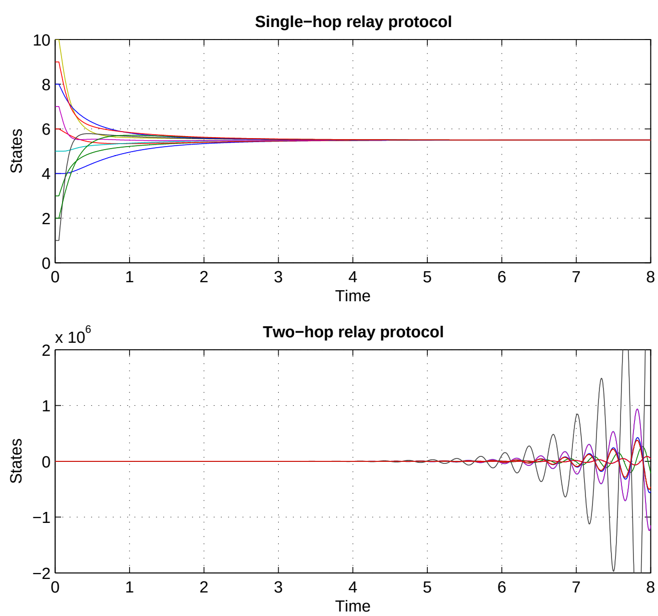

Fig. 6. Consensus convergence for graph $\mathcal{G}_2$ with delay $\tau = 0.05$.

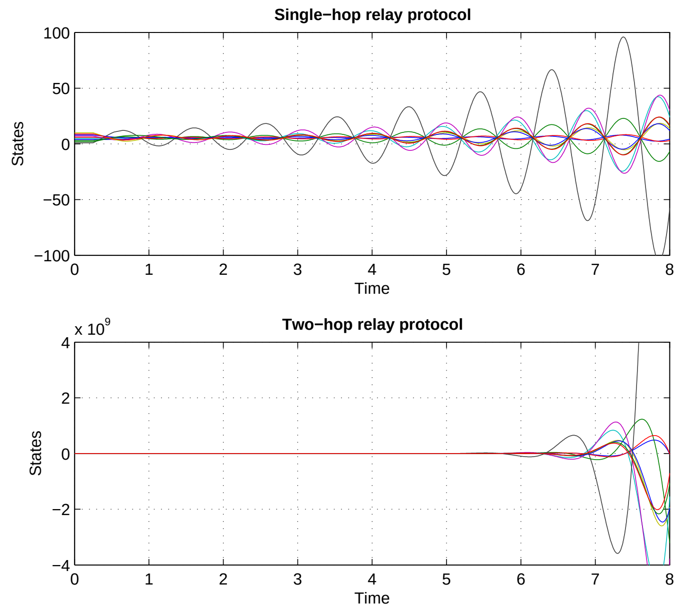

Fig. 7. Consensus convergence for graph $\mathcal{G}_2$ with delay $\tau = 0.25$.

according to Equation (21). Delay margins for other multi-hop relay protocols are computed based on Theorem 4.2. Note that magnitudes of generalized eigenvalues inevitably exceed 1 after a certain value of ω. The computation is done only over a finite frequency interval. We actually run the computation twice. The first time we try to find an appropriate frequency interval using larger frequency steps. The second time we use much smaller frequency step over the interval to find an accurate value of the delay margin.

For each network, the multi-hop relay protocol improves the convergence speed by increasing m. However, the robustness against time delays is impaired due to the delay accumulation along multi-hop paths. Moreover, along the columns of the table, we can tell that convergence speed increases and delay margin decreases when the graph includes more links. We put these data in Figure 8, which clearly marks that multi-hop relay protocol actually boosts convergence speed while sacrificing the robustness.

VI. Conclusions

In this paper, we propose a multi-hop relay protocol for fast consensus seeking over networked multi-agent systems, which efficiently improves the convergence speed without physically chang-

TABLE I

PERFORMANCE VS. ROBUSTNESS FOR RELAY PROTOCOLS

	Convergence speed λ_2				**Delay margin** τ^*			
	single-hop	two-hop	three-hop	four-hop	single-hop	two-hop	three-hop	four-hop
$\mathcal{G}_1$	0.382	1.7639	5.5279	14.674	0.3927	0.1796	0.0332	0.0176
$\mathcal{G}_2$	0.9118	7.3846	40.245	178.18	0.2167	0.0396	0.0051	9.731×10^{-4}
$\mathcal{G}_3$	10	90	820	7380	0.1571	0.0095	0.00068	5.545×10^{-5}

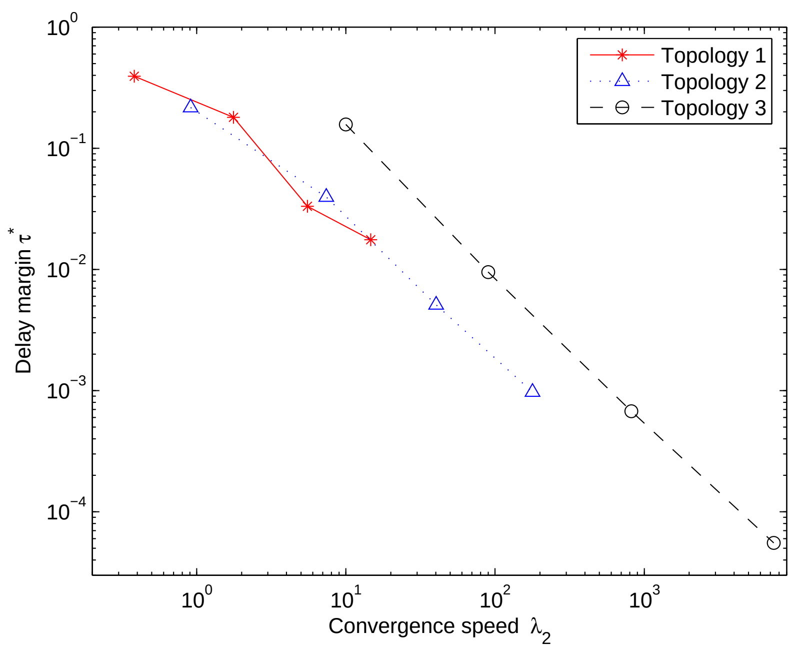

Fig. 8. Tradeoff between convergence speed λ_2 and delay margin τ^*.

ing the network topology. By employing multi-hop paths, we enlarge the information exchanging region of each agent in a systematical way. The cost we need to pay are extra communication bandwidth and local computation load.

Due to delay accumulation along multi-hop communication links, the multi-hop relay protocol may becomes unstable. Necessary and sufficient conditions for the stability are listed by explicitly giving the delay margin with homogeneous time delays. A tradeoff between the performance and the robustness is identified. It is true that, the larger m is, the faster the convergence speed is, and the more sensitive the protocol is to time delays. Also, with the same multi-hop relay protocol, the topology with more edges has faster convergence speed and smaller delay margin.

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