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Exact Stability Analysis of Discrete-Time Linear Systems with Stochastic Delays

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***Abstract*— This paper provides analytical results regarding the stability of linear discrete-time systems with stochastic delays. Necessary and sufficient stability conditions are derived by using the second moment dynamics which can be used to draw stability charts. The results are applied to a simple connected vehicle system where the stability regions are compared to those given by the mean dynamics. Our results reveal some fundamental limitations of connected cruise control which becomes more significant as the packet drop ratio increases.**

I. Introduction

Delays often lead to instabilities in dynamic systems which can make control design a challenging task. In addition, in systems where delays vary stochastically, the difficulty of ensuring stability increases significantly. Most methods up to date apply Lyapunov-type analysis to derive sufficient conditions for stability of equilibria. For linear systems, this leads to matrix inequalities [2,9,11], which typically provide very conservative results. However, in a recent paper [5], the authors show that stochastic delay variations can have a positive impact on stability in genetic regulatory networks. Consequently, there is demand for a method that allows the derivation of exact stability bounds for these types of problems.

This paper extends our work on mean stability in a companion paper [10]. Here we provide necessary and sufficient conditions for point-wise asymptotic stability of discrete-time systems with stochastic delays by calculating the stability of the second moment. In [3] the simplest nontrivial scalar system was studied with stochastic delay variations and some counterintuitive stability results were obtained. This inspired the authors to extend the methodology to systems with vector-valued state variables, that is, to develop a mathematical framework that can be applied to realistic physical systems. This is a challenging task because of the high-dimensional state spaces required to describe such general systems. To reduce the complexity of calculations, here, we exploit the structure of the state matrix that describes the dynamics of the second moment.

We apply the theoretical results to a practical problem of connected cruise control [8,10] where a vehicle follows a leader based on the information received through wireless vehicle-to-vehicle (V2V) communication. In this case, delay variations arise from packet drops in wireless communication. The point-wise asymptotically stable regions, which are derived using the second moment dynamics, are compared with the stable regions given by the mean dynamics. Our results show some fundamental limitations arising in connected vehicle systems when the probability of successful packet deliveries are decreased.

II. Problem Formulation

In this paper, we consider the system

$$X(k+1) = \mathbf{A}X(k) + \mathbf{B}X(k-\tau(k)), \tag{1}$$

where $X(k) \in \mathbb{R}^n$ is a vector-valued stochastic variable and $\tau(k)$ is a family of mutually independent random variables. At each k, the present delay $\tau(k)$ is selected from an identical distribution and can take positive integer values $\tau(k) \in [1, \dots, N]$. N denotes the maximum delay. The density function $p_{\tau(k)}$ for the delay is

$$p_{\tau(k)}(\sigma) = \sum_{i=1}^{N} w_i\, \delta(\sigma - i), \tag{2}$$

with condition

$$\sum_{i=1}^{N} w_i = 1. \tag{3}$$

The initial condition includes the state values in the past N time steps and it may contain uncertainty when $X(0), X(-1), \dots X(-N)$ are selected from known distributions.

Define the augmented vector as

$$\hat{X}(k) = \begin{bmatrix} X(k) \\ X(k-1) \\ X(k-2) \\ \vdots \\ X(k-N) \end{bmatrix}. \tag{4}$$

The discrete-time Markov process

$$\hat{X}(k+1) = \hat{\mathbf{A}}(k)\hat{X}(k), \tag{5}$$

is equivalent to (1) when $\hat{\mathbf{A}}(k)$ takes the values

$$\hat{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathrm{I}_i(1) & \mathbf{B}\mathrm{I}_i(2) & \cdots & \mathbf{B}\mathrm{I}_i(N) \\ \mathbf{I} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{I} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathbf{I} & 0 \end{bmatrix} \tag{6}$$

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with probabilities p_i (*cf.* (2)) for $i = 1, \ldots, N$. Here, $\mathrm{I}_i(j)$ is the indicator function such that

$$\mathrm{I}_i(j) = \begin{cases} 1 & \text{if } j = i. \\ 0 & \text{if } j \neq i, \end{cases} \tag{7}$$

and $\mathbf{I} \in \mathbb{R}^{n\times n}$ and $\mathbf{0} \in \mathbb{R}^{n\times n}$ denote the n-dimensional unit and zero matrices, respectively. The matrix $\hat{\mathbf{A}}(k) \in \mathbb{R}^{n\,(N+1)\times n\,(N+1)}$ is a stochastic variable whose probability distribution is independent of $\hat{X}(k)$. So we have

$$\begin{aligned} p_{\hat{X}(k),\hat{\mathbf{A}}(k)}(\hat{X},\hat{\mathbf{A}}) &= p_{\hat{\mathbf{A}}(k)|\hat{X}(k)}(\hat{\mathbf{A}}|\hat{X})\, p_{\hat{X}(k)}(\hat{X}) \\ &= p_{\hat{\mathbf{A}}(k)}(\hat{\mathbf{A}})\, p_{\hat{X}(k)}(\hat{X}). \end{aligned} \tag{8}$$

Notice, that the sequence $\{\hat{X}(k)\}$ is a Markov chain and the sequence $\{\hat{\mathbf{A}}(k)\}$ is mutually independent. The matrix $\hat{\mathbf{A}}(k)$ can only take on a finite set of values, each of which corresponds to one of the possible delays, henceforth, its probability distribution becomes

$$p_{\hat{\mathbf{A}}(k)}(\hat{\mathbf{A}}) = \sum_{i=1}^{N} w_i\, \delta(\hat{\mathbf{A}} - \hat{\mathbf{\Lambda}}_i). \tag{9}$$

We will apply probability principles and definitions to derive expressions for the evolution of the mean and second moment dynamics of system (5,6), which is equivalent to equation (1). More results on the mean dynamics are provided in a companion paper [10], which provides necessary conditions for stability. The second moment dynamics provide necessary and sufficient conditions.

III. Stability Criteria

First, we find the expression for the evolution of the mean dynamics by taking the expected value of (5):

$$\begin{aligned} &\mathbb{E}[\hat{X}(k+1)] = \mathbb{E}[\hat{\mathbf{A}}(k)\hat{X}(k)] \\ &\quad= \int_{\mathbb{R}^{n(N+1)\times n(N+1)}} \int_{\mathbb{R}^{n(N+1)}} \hat{\mathbf{A}}\hat{X}\, p_{\hat{X}(k),\hat{\mathbf{A}}(k)}(\hat{X},\hat{\mathbf{A}})\, \mathrm{d}\hat{X}\mathrm{d}\hat{\mathbf{A}} \\ &\quad= \sum_{i=1}^{N} w_i \int_{\mathbb{R}^{n(N+1)}} \hat{\mathbf{\Lambda}}_i\hat{X}\, p_{\hat{X}(k)}(\hat{X})\, \mathrm{d}\hat{X} \\ &\quad= \sum_{i=1}^{N} w_i\hat{\mathbf{\Lambda}}_i \mathbb{E}[\hat{X}(k)]. \end{aligned} \tag{10}$$

Define the deterministic variable $\hat{Y} \doteq \mathbb{E}[\hat{X}]$. Then, the mean dynamics are given by

$$\hat{Y}(k+1) = \hat{\mathbf{\Lambda}}\,\hat{Y}(k) \tag{11}$$

where

$$\hat{\mathbf{\Lambda}} = \sum_{i=1}^{N} w_i\,\hat{\mathbf{\Lambda}}_i \tag{12}$$

$$= \begin{bmatrix} \mathbf{A} & w_1\mathbf{B} & w_2\mathbf{B} & \ldots & w_N\mathbf{B} \\ \mathbf{I} & \mathbf{0} & & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{13}$$

By following the method in the companion paper [10], the characteristic equation can be simplified as

$$0 = \det(s\hat{\mathbf{I}} - \hat{\mathbf{\Lambda}}) = \det\left(s^{N+1}\mathbf{I} - s^{N}\mathbf{A} - \sum_{i=1}^{N} s^{N-i} w_i\,\mathbf{B}\right). \tag{14}$$

If all the $n(N+1)$ roots s of this equation lie inside the unit circle in the complex plane, the mean dynamics (11,13) are asymptotically stable. We later show conditions under which the mean dynamics provide a good deterministic approximation for the stochastic system.

Now, we determine the stability of the system by the stability of second moment which implies point-wise asymptotic stability of the system (5,6). This implication does not hold in general, but holds for the problem set up in this paper [6]. The governing equations for the second moment of $\hat{X}(k)$ can be obtained from (5) by calculating

$$\hat{X}(k+1)\hat{X}^{\mathrm{T}}(k+1) = \hat{\mathbf{A}}(k)\hat{X}(k)\hat{X}^{\mathrm{T}}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k), \tag{15}$$

and then taking the expected value on both sides

$$\mathbb{E}[\hat{X}(k{+}1)\hat{X}^{\mathrm{T}}(k{+}1)] = \mathbb{E}[\hat{\mathbf{A}}(k)\hat{X}(k)\hat{X}^{\mathrm{T}}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k)], \tag{16}$$

where the expectation operator is taken element-wise and the right hand side can be evaluated as

$$\begin{aligned} &\mathbb{E}\left[\hat{\mathbf{A}}(k)\hat{X}(k)\hat{X}^{\mathrm{T}}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k)\right] \\ &= \int_{\mathbb{R}^{n\,(N+1)\times n\,(N+1)}} \int_{\mathbb{R}^{n\,(N+1)}} \hat{\mathbf{A}}\hat{X}\hat{X}^{\mathrm{T}}\hat{\mathbf{A}}^{T} p_{\hat{X}(k),\hat{\mathbf{A}}(k)}(\hat{X},\hat{\mathbf{A}})\, \mathrm{d}\hat{X}\mathrm{d}\hat{\mathbf{A}} \\ &= \sum_{i=1}^{N} w_i \int_{\mathbb{R}^{n\,(N+1)}} \hat{\mathbf{\Lambda}}_i\hat{X}\hat{X}^{\mathrm{T}}\hat{\mathbf{\Lambda}}_i^{\mathrm{T}} p_{\hat{X}(k)}(\hat{X})\, \mathrm{d}\hat{X} \\ &= \sum_{i=1}^{N} w_i\hat{\mathbf{\Lambda}}_i \int_{\mathbb{R}^{n\,(N+1)}} \hat{X}\hat{X}^{\mathrm{T}} p_{\hat{X}(k)}(\hat{X})\, \mathrm{d}\hat{X}\hat{\mathbf{\Lambda}}_i^{T} \\ &= \sum_{i=1}^{N} w_i\hat{\mathbf{\Lambda}}_i \mathbb{E}[\hat{X}(k)\hat{X}(k)^{\mathrm{T}}]\hat{\mathbf{\Lambda}}_i^{\mathrm{T}}. \end{aligned} \tag{17}$$

Defining the deterministic matrix-valued variable

$$\hat{\mathbf{G}}(k) = \mathbb{E}[\hat{X}(k)\hat{X}(k)^{\mathrm{T}}], \tag{18}$$

and substituting this into (16) and (17) we obtain the deterministic system

$$\hat{\mathbf{G}}(k+1) = \sum_{i=1}^{N} w_i\hat{\mathbf{\Lambda}}_i\hat{\mathbf{G}}(k)\hat{\mathbf{\Lambda}}_i^{\mathrm{T}}. \tag{19}$$

Note that $\hat{\mathbf{G}}(k)$ is symmetric. The equation for the second moment is linear but it is not trivial to determine stability as both sides are matrix valued. To resolve this problem we transform (19) into state space form where the state vector is composed of only the first n columns of $\hat{\mathbf{G}}$ stacked on top of each other and their delayed versions. We show that no other elements of $\hat{\mathbf{G}}$ need be considered. Then exploiting the structure of $\hat{\mathbf{\Lambda}}_i$ we obtain a state matrix whose eigenvalues can be calculated to determine stability. The following notation is used throughout the rest of the paper:

$[\hat{\mathbf{G}}(k)]_{i,j} \in \mathbb{R}]$ the element of the $\hat{\mathbf{G}}(k)$ matrix in the i-th row and j-th column
$[\hat{\mathbf{G}}(k)]_{:,j} \in \mathbb{R}^{n(N+1)}]$ the j-th column of the matrix $\hat{\mathbf{G}}(k)$
$[\hat{\mathbf{G}}(k)]_{l:m,p:q} \in \mathbb{R}^{(m-l+1)\times(q-p+1)}]$ the submatrix contained in rows l through m and columns p through q

and we define

$$G_m^i(k) = [\hat{\mathbf{G}}(k)]_{in+1:(i+1)n,m} \in \mathbb{R}^n. \tag{20}$$

Using index notation we find an expression for each element of the second moment matrix in the form

$$\begin{aligned}
[\hat{\mathbf{G}}(k+1)]_{p,j} &= \sum_{i=1}^{N} w_i\, [\hat{\mathbf{\Lambda}}_i\, \hat{\mathbf{G}}(k) \hat{\mathbf{\Lambda}}_i^T]_{p,j} \\
&= \sum_{i=1}^{N} w_i \sum_{m=1}^{n(N+1)} [\hat{\mathbf{\Lambda}}_i]_{p,m} \left(\sum_{k=1}^{n(N+1)} [\hat{\mathbf{\Lambda}}_i]_{j,k}\, [\hat{\mathbf{G}}(k)]_{k,m} \right).
\end{aligned} \tag{21}$$

The expression of each element can be simplified by looking at special cases for index values, given that we know the structure of $\hat{\mathbf{\Lambda}}_i$; *c.f* (6). Notice, that for $j > n$ the elements of $\hat{\mathbf{\Lambda}}_i$ are such that

$$[\hat{\mathbf{\Lambda}}_i]_{l,m} = \begin{cases} 1 & \text{if } l = m+n, \\ 0 & \text{otherwise.} \end{cases}$$

and this holds for each $i \in [1, \ldots, N]$. Applying this property for $j, p > n$, (21) implies that

$$\begin{aligned}
&[\hat{\mathbf{G}}(k+1)]_{p,j} = \\
&= \sum_{i=1}^{N} w_i \sum_{m=1}^{n(N+1)} \delta(p-(m+n)) \sum_{l=1}^{n(N+1)} \delta(j-(l+n))\, [\hat{\mathbf{G}}(k)]_{l,m} \\
&= \sum_{i=1}^{N} w_i\, [\hat{\mathbf{G}}(k)]_{l-n,p-n} = [\hat{\mathbf{G}}(k)]_{j-n,p-n} \\
&= [\hat{\mathbf{G}}(k)]_{p-n,j-n},
\end{aligned} \tag{22}$$

which yields

$$G_j^i(k+1) = G_{j-n}^{i-1}(k) \quad \text{for} \quad i \geq 1,\ j > n. \tag{23}$$

Similarly, considering $p \leq n$ and $j > n$ we obtain

$$\begin{aligned}
&[\hat{\mathbf{G}}(k+1)]_{p,j} = \\
&\quad = \sum_{i=1}^{N} w_i \sum_{m=1}^{n(N+1)} [\hat{\mathbf{\Lambda}}_i]_{p,m} \sum_{k=1}^{n(N+1)} \delta(j-(k+n))\, [\hat{\mathbf{G}}(k)]_{k,m} \\
&\quad = \sum_{i=1}^{N} w_i \sum_{m=1}^{n(N+1)} [\hat{\mathbf{\Lambda}}_i]_{p,m}\, [\hat{\mathbf{G}}(k)]_{j-n,m},
\end{aligned} \tag{24}$$

which gives

$$G_j^0(k+1) = \mathbf{A}\, G_{j-n}^0(k) + \sum_{i=1}^{N} w_i\, \mathbf{B}\, G_{j-n}^i(k). \tag{25}$$

We define the new vector

$$\hat{G}_j(k) = \begin{bmatrix} G_j^0(k) \\ G_j^1(k) \\ G_j^2(k) \\ \vdots \\ G_j^N(k) \end{bmatrix} \in \mathbb{R}^{n(N+1)}, \tag{26}$$

which is essentially the j-th column of the second moment matrix $\hat{\mathbf{G}}(k)$. We combine (23) and (25) to describe the column vector update

$$\hat{G}_j(k+1) = \hat{\mathbf{\Lambda}} \hat{G}_{j-n}(k), \tag{27}$$

where $\hat{\mathbf{\Lambda}}$ is given by (13).

For $p, j \leq n$, (6) and (21) yields

$$\begin{aligned}
[\hat{\mathbf{G}}(k+1)]_{p,j} = \sum_{i=1}^{N} w_i \Bigg(&\sum_{m=1}^{n} [\mathbf{A}]_{j,m} \sum_{k=1}^{n} [\mathbf{A}]_{p,k} [\hat{\mathbf{G}}(k)]_{k,m} \\
&+ \sum_{m=1}^{n} [\mathbf{A}]_{j,m} \sum_{l=1}^{n} [\mathbf{B}]_{p,l} [\hat{\mathbf{G}}(k)]_{l+i\,n,m} \\
&+ \sum_{l=1}^{n} [\mathbf{B}]_{j,l} \sum_{k=1}^{n} [\mathbf{A}]_{p,k} [\hat{\mathbf{G}}(k)]_{k,l+i\,n} \\
&+ \sum_{l=1}^{n} [\mathbf{B}]_{j,l} \sum_{r=1}^{n} [\mathbf{B}]_{p,r} [\hat{\mathbf{G}}(k)]_{r+i\,n,l+i\,n} \Bigg),
\end{aligned} \tag{28}$$

which implies

$$\begin{aligned}
G_j^0(k+1) = \sum_{m=1}^{n} \Bigg(&[\mathbf{A}]_{j,m} \mathbf{A}\, G_m^0(k) \\
&+ [\mathbf{A}]_{j,m}\, \mathbf{B} \sum_{i=1}^{N} w_i\, G_m^i(k) \\
&+ [\mathbf{B}]_{j,m} \mathbf{A} \sum_{i=1}^{N} w_i\, G_{m+in}^0(k) \\
&+ [\mathbf{B}]_{j,m}\, \mathbf{B} \sum_{i=1}^{N} w_i\, G_{m+i\,n}^i(k) \Bigg),
\end{aligned} \tag{29}$$

for $j \in [1, 2, \ldots, n]$.

Writing (29) as a function of $\hat{G}_j$ (*cf.*(26)) and applying

(27) we get

$$
\begin{aligned}
G_j^0(k+1) &= \sum_{m=1}^{n}\Bigg([\mathbf{A}]_{j,m}\bar{\mathbf{\Lambda}}\hat{G}_m(k) \\
&\qquad +[\mathbf{B}]_{j,m}\sum_{i=1}^{N} w_i\left(\mathbf{A}\, G^0_{m+in}(k)+\mathbf{B}G^i_{m+in}(k)\right)\Bigg) \\
&= \sum_{m=1}^{n}\Bigg([\mathbf{A}]_{j,m}\bar{\mathbf{\Lambda}}\hat{G}_m(k) \\
&\qquad +[\mathbf{B}]_{j,m}\sum_{i=1}^{N} w_i\bar{\mathbf{\Lambda}}_i\hat{\mathbf{\Lambda}}^i\,\hat{G}_m(k-i)\Bigg) \\
&= \left((e_j^{\mathrm{T}}\mathbf{A})\otimes\bar{\mathbf{\Lambda}}\right)\tilde{G}(k) \\
&\quad +\sum_{i=1}^{N} w_i\left((e_j^{\mathrm{T}}\mathbf{B})\otimes(\bar{\mathbf{\Lambda}}_i\hat{\mathbf{\Lambda}}^i)\right)\tilde{G}(k-i) \qquad (30)
\end{aligned}
$$

for $j\in[1,2,\ldots,n]$. Here, $\bar{\mathbf{\Lambda}}=[\hat{\mathbf{\Lambda}}]_{1:n,:}$ and $\bar{\mathbf{\Lambda}}_i=[\hat{\mathbf{\Lambda}}_i]_{1:n,:}$ (i.e the first n rows of their respective matrices). $\hat{\mathbf{\Lambda}}^i$ denotes taking the matrix $\hat{\mathbf{\Lambda}}$ to the i-th power, $e_j\in\mathbb{R}^n$ with all elements equal to 0 except the j-th element equal to 1 and

$$
\tilde{G}(k)=\begin{bmatrix}\hat{G}_1(k)\\ \hat{G}_2(k)\\ \vdots\\ \hat{G}_n(k)\end{bmatrix}\in\mathbb{R}^{n^2(N+1)}, \qquad (31)
$$

which stacks the first n columns of the second moment matrix $\hat{\mathbf{G}}$ on top of each other; *cf.* (26). In addition, $\otimes$ denotes the Kronecker product.

Last, we consider the case $p>n$ and $j\leq n$ in (21)

$$
\begin{aligned}
[\hat{\mathbf{G}}(k+1)]_{p,j} &= \sum_{i=1}^{N} w_i \sum_{k=1}^{n(N+1)}[\hat{\mathbf{\Lambda}}_i]_{j,k}\,[\hat{\mathbf{G}}(k)]_{k,p-n} \\
&= \sum_{i=1}^{N} w_i\,\hat{e}_j^T\,\hat{\mathbf{\Lambda}}_i\,\hat{G}_{p-n}(k) \\
&= \hat{e}_j^{\mathrm{T}}\,\hat{\mathbf{\Lambda}}\,\hat{G}_{p-n}(k)=e_j^{\mathrm{T}}\bar{\mathbf{\Lambda}}\hat{G}_{p-n}(k) \qquad (32)
\end{aligned}
$$

where $\hat{e}_j\in\mathbb{R}^{n(N+1)}$ with only the j-th element equal to 1. With this we can find a general expression for the vector (20), for $i\in[1,N]$ and $j\in[1,n]$, namely,

$$
\begin{aligned}
G_j^i(k+1) &= \begin{bmatrix}[\hat{\mathbf{G}}(k+1)]_{i\,n+1,j}\\ [\hat{\mathbf{G}}(k+1)]_{i\,n+2,j}\\ \vdots\\ [\hat{\mathbf{G}}(k+1)]_{(i+1)n,j}\end{bmatrix} \\
&= \begin{bmatrix} e_j^{\mathrm{T}}\,\mathbf{\Lambda}^i\hat{G}_1(k-i+1)\\ e_j^{\mathrm{T}}\,\mathbf{\Lambda}^i\hat{G}_2(k-i+1)\\ \vdots\\ e_j^{\mathrm{T}}\,\mathbf{\Lambda}^i\hat{G}_n(k-i+1)\end{bmatrix}. \qquad (33)
\end{aligned}
$$

Now we have an expression for every element of the vector $\hat{G}_j(k)$ in (26) for $j\in[1,\ldots,n]$ given by (30,33) and we can, therefore, find an expression for the time evolution of the vector $\tilde{G}(k)$ in (31) as a function of itself and its delayed values. That is, we can write an expression for the evolution of the first n columns of the second moment matrix. Some algebraic manipulation leads to:

$$
\hat{\tilde{G}}(k+1)=\hat{\tilde{\mathbf{A}}}\hat{\tilde{G}}(k) \qquad (34)
$$

where

$$
\hat{\tilde{G}}(k)=\begin{bmatrix}\tilde{G}(k)\\ \tilde{G}(k-1)\\ \vdots\\ \tilde{G}(k-N)\end{bmatrix}\in\mathbb{R}^{n^2(N+1)^2} \qquad (35)
$$

cf. (31) and

$$
\hat{\tilde{\mathbf{A}}}=\left[\begin{array}{c:cccc}
\tilde{\mathbf{A}} & \tilde{\mathbf{B}}_1 & \tilde{\mathbf{B}}_2 & \cdots & \tilde{\mathbf{B}}_N \\ \hdashline
\tilde{\mathbf{I}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \tilde{\mathbf{I}} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\mathbf{0} & \cdots & \mathbf{0} & \tilde{\mathbf{I}} & \mathbf{0}
\end{array}\right]. \qquad (36)
$$

Here we have the unit matrix $\tilde{\mathbf{I}}\in\mathbb{R}^{n^2(N+1)\times n^2(N+1)}$ and

$$
\tilde{\mathbf{A}}=\left[\begin{array}{c}
(e_1^{\mathrm{T}}\mathbf{A})\otimes\bar{\mathbf{\Lambda}}\\
\mathbf{I}\otimes(e_1^{\mathrm{T}}\bar{\mathbf{\Lambda}})\\
\mathbf{0}\\ \vdots\\ \mathbf{0}\\ \hdashline
(e_2^{\mathrm{T}}\mathbf{A})\otimes\bar{\mathbf{\Lambda}}\\
\mathbf{I}\otimes(e_2^{\mathrm{T}}\bar{\mathbf{\Lambda}})\\
\mathbf{0}\\ \vdots\\ \mathbf{0}\\ \hdashline
\vdots\\ \hdashline
(e_n^{\mathrm{T}}\mathbf{A})\otimes\bar{\mathbf{\Lambda}}\\
\mathbf{I}\otimes(e_n^{\mathrm{T}}\bar{\mathbf{\Lambda}})\\
\mathbf{0}\\ \vdots\\ \mathbf{0}
\end{array}\right],\quad
\tilde{\mathbf{B}}_i=\left[\begin{array}{c}
w_i\,(e_1^{\mathrm{T}}\mathbf{B})\otimes(\bar{\mathbf{\Lambda}}_i\,\hat{\mathbf{\Lambda}}^i)\\
\mathbf{0}\\ \vdots\\ \mathbf{0}\\
\mathbf{I}\otimes(\hat{e}_1^{\mathrm{T}}\hat{\mathbf{\Lambda}}^{i+1})\\
\mathbf{0}\\ \vdots\\ \mathbf{0}\\ \hdashline
w_i\,(e_2^{\mathrm{T}}\mathbf{B})\otimes(\bar{\mathbf{\Lambda}}_i\,\hat{\mathbf{\Lambda}}^i)\\
\mathbf{0}\\ \vdots\\ \mathbf{0}\\
\mathbf{I}\otimes(\hat{e}_2^{\mathrm{T}}\hat{\mathbf{\Lambda}}^{i+1})\\
\mathbf{0}\\ \vdots\\ \mathbf{0}\\ \hdashline
\vdots\\ \hdashline
w_i\,(e_n^{\mathrm{T}}\mathbf{B})\otimes(\bar{\mathbf{\Lambda}}_i\,\hat{\mathbf{\Lambda}}^i)\\
\mathbf{0}\\ \vdots\\ \mathbf{0}\\
\mathbf{I}\otimes(\hat{e}_n^{\mathrm{T}}\hat{\mathbf{\Lambda}}^{i+1})\\
\mathbf{0}\\ \vdots\\ \mathbf{0}
\end{array}\right], \qquad (37)
$$

for $i = 1, \dots, N-1$ and

$$\tilde{\mathbf{B}}_N = \begin{bmatrix} w_i\,(e_1^{\mathrm{T}}\mathbf{B}) \otimes (\bar{\mathbf{\Lambda}}_N\,\hat{\mathbf{\Lambda}}^i N \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hdashline w_i\,(e_2^{\mathrm{T}}\mathbf{B}) \otimes (\bar{\mathbf{\Lambda}}_N\,\hat{\mathbf{\Lambda}}^N) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hdashline \vdots \\ \hdashline w_i\,(e_n^{\mathrm{T}}\mathbf{B}) \otimes (\bar{\mathbf{\Lambda}}_N\,\hat{\mathbf{\Lambda}}^N) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}. \tag{38}$$

The matrix $\mathbf{I} \otimes (\hat{e}_j^{\mathrm{T}}\,\hat{\mathbf{\Lambda}}^i) \in \mathbb{R}^{n \times n^2(N+1)}$ has $\hat{e}_j^{\mathrm{T}}\,\hat{\mathbf{\Lambda}}^i \in \mathbb{R}^{1 \times n(N+1)}$ along the "diagonal". Each of the sections delimitated by the dashed line is of dimension $n^2(N+1) \times n^2(N+1)$ and for $\tilde{\mathbf{B}}_i$ the matrix $\mathbf{I} \otimes (\hat{e}_j^{\mathrm{T}}\,\hat{\mathbf{\Lambda}}^{i+1})$ begins in $((i+1)\,n+1)$-th row of each section. The characteristic equation of (34,36,37,38) is given by

$$0 = \det(s\hat{\hat{\mathbf{I}}} - \hat{\hat{\mathbf{A}}}) = \det\Big(s^{N+1}\tilde{\mathbf{I}} - s^N\tilde{\mathbf{A}} - \sum_{m=1}^{N} s^{N-m}\tilde{\mathbf{B}}_m\Big). \tag{39}$$

We are now ready to state the following theorem.

Theorem III.1. *The stochastically delayed system (1) is point-wise asymptotically stable iff all $n^2(N+1)^2$ roots of (39) lie within the unit circle in the complex plane.*

Proof: The eigenvalues being within the unit circle imply stability of the system in (34) [4,7], which in turn implies second moment stability in (19). Finally, stability of the second moment implies point-wise asymptotic stability of the stochastic system (1).

We will apply this result to a simple connected vehicle system in the next section.

IV. Connected Vehicle Application

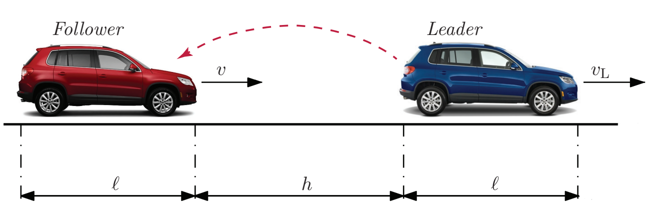

Fig. 2. Simple car-following configuration.

Wireless vehicle-to-vehicle (V2V) communication has a large potential to increase safety and mobility of road transportation. Such technology may allow vehicles to obtain information about the motion of nearby vehicles without using expensive sensors. For example, such signals may be used to control the longitudinal motions of vehicles following a leader, leading to the concept of connected cruise control (CCC). However, intermittencies and packet drops introduce stochastic time delays into the communication that has to be taken into account when designing such controllers.

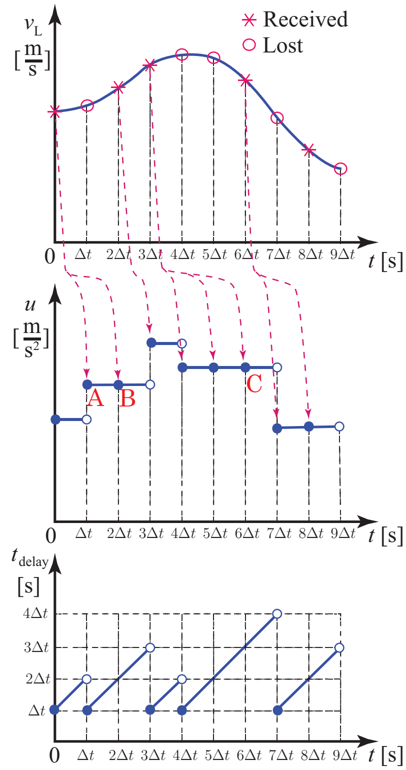

Fig. 3. Top: Sampling a continuous signal using wireless communication. Middle: A zero-order-hold control signal based on the received packages. Bottom: Stochastic delay variations caused by zero-order-hold and packet drops.

Consider the simple car-following scenario shown in Fig. 2, where the leader's and follower's velocity is denoted by v_{L} and v, while the bumper-to-bumper distance (called the headway) is denoted by h. We consider the car-following system subject to digital control that is presented in detail in [10]:

$$\begin{cases} \dot{h}(t) = v_{\mathrm{L}}(t) - v(t)\,, \\ \dot{v}(t) = u(t_{k-\tau(k)})\,, \quad t \in [t_k, t_{k+1}), \end{cases} \tag{40}$$

where $t_k = k\Delta t$ and

$$u = K_{\mathrm{p}}\big(V(h) - v\big) + K_{\mathrm{v}}\big(W(v_{\mathrm{L}}) - v\big). \tag{41}$$

Here, $V(h) - v$ represents the error between the desired distance-dependent velocity $V(h)$ and the actual velocity v of the vehicle. The function $V(h)$ is called the range policy and defined as

$$V(h) = \begin{cases} 0 & \text{if} \quad h < h_{\mathrm{st}}\,, \\ \dfrac{v_{\max}}{2}\Big(1 - \cos\big(\pi\dfrac{h - h_{\mathrm{st}}}{h_{\mathrm{go}} - h_{\mathrm{st}}}\big)\Big) & \text{if} \quad h_{\mathrm{st}} \le h \le h_{\mathrm{go}}\,, \\ v_{\max} & \text{if} \quad h > h_{\mathrm{go}}\,, \end{cases} \tag{42}$$

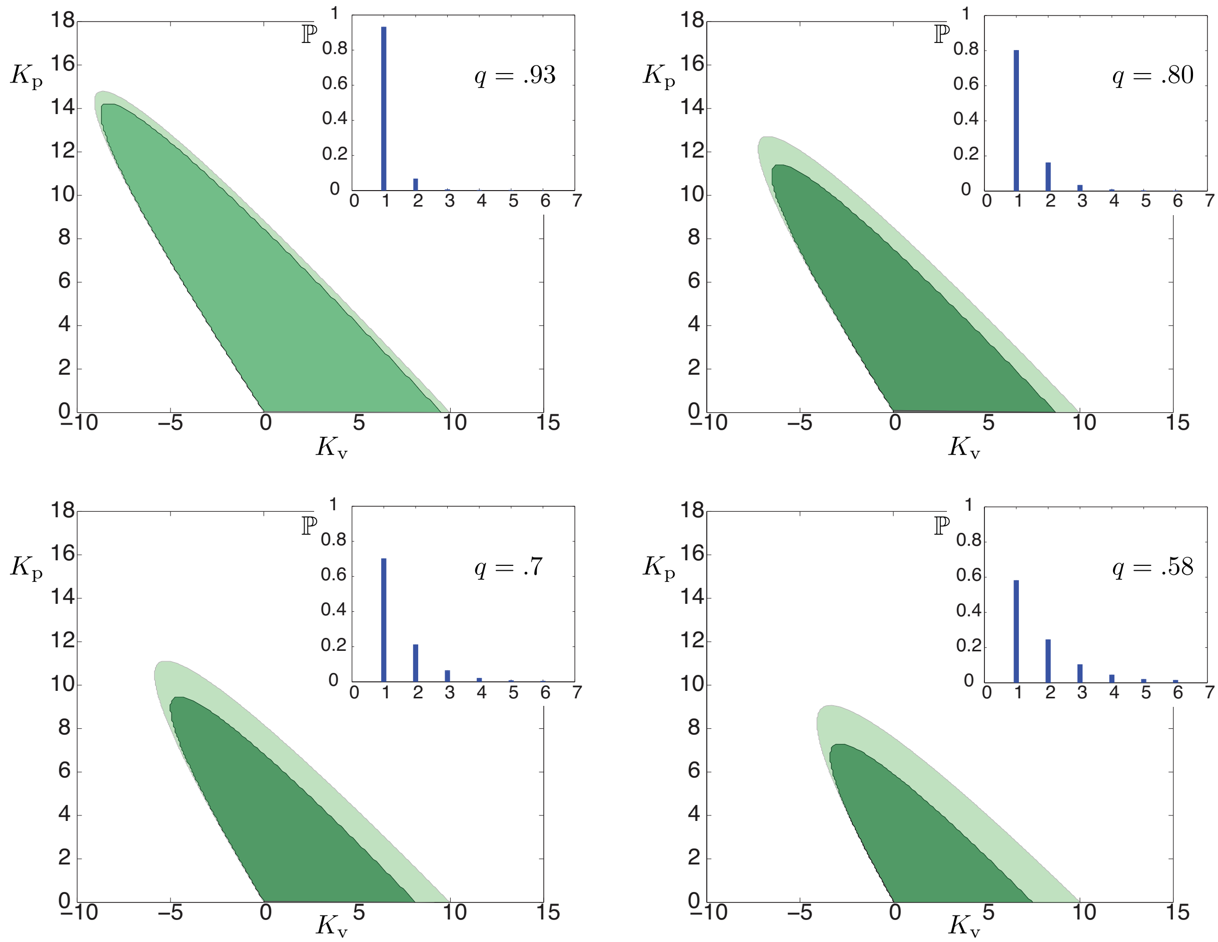

Fig. 1. Stability charts for different packet delivery ratios q as indicated (and shown by inlets). The outer curve envelops the mean stable region, while within the dark shaded domain point-wise asymptotic stability is achieved. The other parameters used here are $v_{\max} = 30\mathrm{m/s}$, $h_{\mathrm{st}} = 5\mathrm{m}$, $h_{\mathrm{go}} = 35\mathrm{m}$, $v^* = 15\mathrm{m/s}$; *cf.* (42).

while the saturation function is given by

$$W(v_\mathrm{L}) = \begin{cases} v_\mathrm{L} & \text{if} \quad v_\mathrm{L} < v_{\max}\,, \\ v_{\max} & \text{if} \quad v_\mathrm{L} > v_{\max}\,. \end{cases} \tag{43}$$

Information about the leader's motion (velocity and GPS coordinate) is transmitted in every Δt time via wireless vehicle-to-vehicle communication. Stochasticity in delay arises in the probability of packages being dropped. Figure 3 depicts how the signal is affected by packet drops. The state at time $3\,\Delta t$ is dependent on the state in the previous two discrete-time steps, whereas the state at $4\,\Delta t$ is dependent on the state in the previous time step only. Let $\tau(k) - 1$ denote the number of packet drops. At each time step Δt, $\tau(k)$ is selected from a (truncated) geometric distribution with finite support which assigns the following weight values

$$\left\{ \begin{aligned} & w_r = q(1-q)^{r-1} \quad \text{for } r = 1,\, 2,\, \cdots,\, N-1, \\ & w_N = 1 - \sum_{r=1}^{N-1} w_r, \end{aligned} \right. \tag{44}$$

to the distribution in (2). We consider $.58 \leq q \leq .93$. The bounds on q and the distribution are chosen based on real data from [1].

After linearizing the system about the equilibrium point $v_\mathrm{L}^* = v^* = V(h^*)$ and solving the obtained equations between $k\,\Delta t$ and $(k+1)\,\Delta t$, we obtain the discrete-time map

$$X(k+1) = \mathbf{A}X(k) + \mathbf{B}X(k - \tau(k)) \tag{45}$$

where

$$X(k) = \begin{bmatrix} \tilde{h}(k) \\ \tilde{v}(k) \end{bmatrix}, \tag{46}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix}, \tag{47}$$

$$\mathbf{B} = \begin{bmatrix} -\frac{1}{2}\Delta t^2\, K_\mathrm{p}\, N_* & \frac{1}{2}\Delta t^2\,(K_\mathrm{p} + K_\mathrm{v}) \\ \Delta t\, K_\mathrm{p}\, N_* & -\Delta t\,(K_\mathrm{p} + K_\mathrm{v}) \end{bmatrix}. \tag{48}$$

The results from the previous section are applied to (45,47,48) and the results shown in Fig. 1 for different values of the packet delivery ratio q. The union of the light and dark shaded regions indicate the mean stable region (given by parameters in which all eigenvalues in (14) are located inside the unit circle), while dark regions indicate stability of the second moment (given by parameters in which all eigenvalues in (39) lie within the unit circle). Notice that both regions shrink with decreasing q, which shows that connected cruise control has to be designed carefully for high packet drop ratios. Note that as q is increased the asymptotically stable region appears to approach the mean stable region. That is, for $q \approx 1$ the average dynamics provide a good approximation of stability.

V. CONCLUSIONS

A method was presented to determine exact stability bounds of a linear discrete-time system with stochastic delays. We motivated the work through a connected vehicle example, showing that as the probability of packet drop increases, the stochasticity of the delay impacts the performance significantly. Future work includes extending stability analysis for periodically forced systems and investigating the effects of intentional packet drops as a potential design tool.

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