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Stability of Discrete Time Systems with Stochastically Delayed Feedback

Marcella M. Gomez[1], Gábor Orosz[2] and Richard M. Murray[3]

***Abstract*— This paper investigates the stability of linear systems with stochastic delay in discrete time. Stability of the mean and second moment of the non-deterministic system is determined by a set of deterministic discrete time equations with distributed delay. A theorem is provided that guarantees convergence of the state with convergence of the second moment, assuming the delays are identically independently distributed. The theorems are applied to a scalar equation where the stability of the equilibrium is determined.**

I. Introduction

Noise is often a source of concern for control engineers and, most recently, a concern in efforts to progress in the field of synthetic biology. This paper focuses on the uncertainty in delay in dynamical systems. Much investigation has been done on systems with uncertainty in the state matrix for linear systems [7] but little has been done on systems with uncertainty in the delay. In [6] the author provides an improved stability criterion for systems with uncertain delays in continuous time linear systems but there is still uncertainty to how conservative the criterion is.

It is important to asses the effect of uncertainty in delay since it often arises in biological systems. In genetic regulatory networks, delays arise in the transcription and translation process [8]. Often the production of a protein induced by its activating transcription factor is modeled as an instantaneous process. In fact, there is a delay in the process with some variation.

For control engineers, determining stability or designing a controller for a system with stochastic delays is often challenging. One can take the worst case scenario (e.g. largest delay) but this can lead to unnecessary conservativeness or may simply give erroneous results.

We derive a method of investigating the effects of stochastic delays in discrete time systems by deriving a set of deterministic distributed delay systems which describe the time evolution of the mean and second moment of the stochastic system. Under certain conditions, we can guarantee with probability one (w.p.1) the stability of the equilibrium. We apply this method to select examples and make some interesting observations. For example, we demonstrate that it is possible to obtain noise-induced stability in some cases.

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[1]M. M. Gomez is a graduate student in the Department of Mechanical Engineering, California Institute of Technology, Pasadena, CA 91125, USA mgomez at caltech.edu

[2]G. Orosz is with Faculty of Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109, USA orosz at umich.edu

[3]R. M. Murray is with Faculty of Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125, USA murray at cds.caltech.edu

That is, a system with a single deterministic delay may be unstable but upon introducing uncertainty in the delay, the system can become asymptotically stable.

II. Problem Set up

In this paper we consider the problem with scalar random variables and we derive the corresponding moment equations. We also demonstrate the potential stabilizing effects of stochastic delays.

We consider the scalar stochastic system

$$X(k+1) = aX(k) + bU(k), \tag{1}$$

where $X(k) \in \mathbb{R}$ is a stochastic variable at time $k \in \mathbb{Z}$ and U_k represents the uncertain delayed feedback

$$U(k) = X(k-\tau),$$

where τ takes finite positive integer values, $\tau \in [1, \ldots, N]$. Here, N is the maximum value of the delay. The initial condition includes the state values in the past N time steps. We can generalize the problem to include uncertainty in the initial condition, where $X(0), X(-1), \ldots X(-N)$ are selected from some known distributions.

We consider the following probability density function for $U(k)$

$$p_{U(k)}(u) = \int_0^\infty p_{X(k-\tau)|\tau}(u|\sigma) p_\tau(\sigma) d\sigma, \tag{2}$$

where the density function p_τ for the delay is

$$p_\tau(\sigma) = \sum_{i=1}^{N} w_i \delta(\sigma - \tau_i), \tag{3}$$

with

$$\sum_{i=1}^{N} w_i = 1.$$

The discrete stochastic variable τ has finite support, because N is a finite integer. All the possible delays are given by positive integers τ_i, and w_i represents their associated weights, or likelihood of occurring. For ease of notation we take $\tau_i = i$ and $w_i \geq 0$. The integral in (2) is considered along the positive axis because the delays are positive. If we evaluate (2) using (3) we obtain

$$p_{U(k)}(u) = \sum_{i=1}^{N} w_i p_{X(k-\tau_i)}(u).$$

With this we can proceed to analyze the statistical properties of (1).

III. Derivation of Expressions for the Mean and Second Moment

First, we look at the stability criteria of the expected value, $\mathbb{E}[X(k)]$. Taking the expectation on both sides of (1) we obtain

$$\mathbb{E}[X(k+1)] = a\,\mathbb{E}[X(k)] + b\,\mathbb{E}[U(k)], \tag{4}$$

where the expected value of the feedback term $U(k)$ is

$$\begin{aligned}
\mathbb{E}[U(k)] &= \int_0^\infty u p_{U(k)}(u) du \\
&= \int_0^\infty u \left[\sum_{i=1}^N w_i p_{X(k-\tau_i)}(u)\right] du \\
&= \sum_{i=1}^N w_i \int_0^\infty u p_{X(k-\tau_i)}(u) du \\
&= \sum_{i=1}^N w_i \mathbb{E}[X(k-\tau_i)].
\end{aligned} \tag{5}$$

Let us define the deterministic variable

$$y(k) = \mathbb{E}[X(k)]. \tag{6}$$

Substituting this into (4) and (5), the dynamics of the expectation is described by the deterministic system with distributed delay

$$y(k+1) = a\,y(k) + b\sum_{i=1}^N w_i y(k-\tau_i). \tag{7}$$

Since the system is a discrete-time system with finite maximum delay, the state space is finite dimensional. By defining the following state vector

$$\vec{X}(k) = \begin{bmatrix} X(k) \\ X(k-1) \\ \vdots \\ X(k-N) \end{bmatrix}, \tag{8}$$

equation (1) can be rewritten as

$$\vec{X}(k+1) = \mathbf{A}(k)\vec{X}(k), \tag{9}$$

where $\mathbf{A}(k) \in \mathbb{R}^{(N+1)\times(N+1)}$ is a stochastic variable whose probability distribution is independent of $\vec{X}(k)$. So we have

$$\begin{aligned}
p_{\vec{X}(k),\mathbf{A}(k)}(\vec{X},\mathbf{A}) &= p_{\mathbf{A}(k)|\vec{X}(k)}(\mathbf{A}|\vec{X}) p_{\vec{X}(k)}(\vec{X}) \\
&= p_{\mathbf{A}(k)}(\mathbf{A}) p_{\vec{X}(k)}(\vec{X}).
\end{aligned}$$

Notice, that the sequence $\{\vec{X}(k)\}$ is a Markov chain and the sequence $\{\mathbf{A}(k)\}$ is mutually independent. Since the matrix $\mathbf{A}(k)$ can only take on a finite set of values, its probability distribution becomes

$$p_{\mathbf{A}(k)}(\mathbf{A}) = \sum_{i=1}^N w_i \delta(\mathbf{A} - \mathbf{\Lambda}_i),$$

where

$$\mathbf{\Lambda}_i = \begin{bmatrix} a & bI_i(1) & bI_i(2) & \cdots & bI_i(N) \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

with I_i being the indicator function,

$$I_i(j) = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}.$$

Indeed by taking the expected value of (9), one may also derive (7):

$$\begin{aligned}
\mathbb{E}[\vec{X}(k+1)] &= \mathbb{E}[\mathbf{A}(k)\vec{X}(k)] \\
&= \int_{\mathbb{R}^{(N+1)\times(N+1)}} \int_{\mathbb{R}^{N+1}} \mathbf{A}\vec{X} p_{\vec{X}(k),\mathbf{A}(k)}(\vec{X},\mathbf{A}) d\vec{X} d\mathbf{A} \\
&= \sum_{i=1}^N w_i \int_{\mathbb{R}^{N+1}} \mathbf{\Lambda}_i \vec{X} p_{\vec{X}(k)}(\vec{X}) d\vec{X} \\
&= \sum_{i=1}^N w_i \mathbf{\Lambda}_i \mathbb{E}[\vec{X}(k)].
\end{aligned} \tag{10}$$

Using the variable defined in (6), we define the deterministic state vector

$$\vec{y}(k) = \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-N) \end{bmatrix}, \tag{11}$$

and obtain a deterministic system with distributed delay

$$\vec{y}(k+1) = \sum_{i=1}^N w_i \mathbf{\Lambda}_i \vec{y}(k), \tag{12}$$

where the state transition matrix is given by

$$\sum_{i=1}^N w_i \mathbf{\Lambda}_i = \begin{bmatrix} a & b\,w_1 & b\,w_2 & \cdots & b\,w_N \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{13}$$

Indeed, (7) and (12,13) are equivalent.

We now determine the governing equations for the second moment of $X(k)$. From (9) we have

$$\vec{X}(k+1)\vec{X}^T(k+1) = \mathbf{A}(k)\vec{X}(k)\vec{X}^T(k)\mathbf{A}^T(k). \tag{14}$$

Taking the expected value on both sides yields

$$\mathbb{E}[\vec{X}(k+1)\vec{X}^T(k+1)] = \mathbb{E}[\mathbf{A}(k)\vec{X}(k)\vec{X}^T(k)\mathbf{A}^T(k)], \tag{15}$$

where the expectation operator is taken element-wise, but we use the short-hand notation above. The right hand side of (15) can be evaluated to

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{A}(k)\vec{X}(k)\vec{X}^T(k)\mathbf{A}^T(k)\right] \\
&= \int_{\mathbb{R}^{(N+1)\times(N+1)}}\int_{\mathbb{R}^{N+1}} \mathbf{A}\vec{X}\vec{X}^T\mathbf{A}^T p_{\vec{X}(k),\mathbf{A}(k)}(\vec{X},\mathbf{A})d\vec{X}d\mathbf{A} \\
&= \sum_i w_i \int_{\mathbb{R}^{N+1}} \mathbf{\Lambda}_i\vec{X}\vec{X}^T\mathbf{\Lambda}_i^T p_{\vec{X}(k)}(\vec{X})d\vec{X} \\
&= \sum_{i=1}^N w_i\mathbf{\Lambda}_i \int_{\mathbb{R}^{N+1}} \vec{X}\vec{X}^T p_{\vec{X}(k)}(\vec{X})d\vec{X}\mathbf{\Lambda}_i^T \\
&= \sum_{i=1}^N w_i\mathbf{\Lambda}_i\mathbb{E}[\vec{X}(k)\vec{X}(k)^T]\mathbf{\Lambda}_i^T. \qquad (16)
\end{aligned}
$$

Defining the deterministic matrix-valued variable

$$\mathbf{P}(k) = \mathbb{E}[\vec{X}(k)\vec{X}(k)^T], \qquad (17)$$

and substituting this into (15) and (16) we obtain the deterministic system

$$\mathbf{P}(k+1) = \sum_{i=1}^N w_i\mathbf{\Lambda}_i\mathbf{P}(k)\mathbf{\Lambda}_i^T \qquad (18)$$

for the time evolution of $\mathbf{P}(k)$. Note that $p_{i,j}(k) = \mathbb{E}[X(k-i+1)X(k-j+1)]$ for $i,j = 1,\ldots,N+1$. The second moment $\mathbb{E}[X(k)^2]$ is given by the matrix element

$$p_{1,1}(k) = \mathbb{E}[X(k)^2] = \vec{C}^T\mathbf{P}(k)\vec{C}, \qquad (19)$$

where $\vec{C} = [1, 0, \cdots, 0]^T$ but the time evolution of $\mathbb{E}[X(k)^2]$ depends on other elements of the matrix $\mathbf{P}(k)$.

Exploiting that $\mathbf{P}(k)$ is a symmetric matrix, i.e.

$$p_{n,m}(k) = \mathbb{E}[X(k-n+1)X(k-m+1)] = p_{m,n}(k),$$

we carry out the matrix multiplication in (18) and obtain a set of discrete time systems that describe the time evolution of the elements of $\mathbf{P}(k)$. We obtain the distributed delay system

$$
\begin{aligned}
p_{1,1}(k+1) &= a^2\, p_{1,1}(k) + b^2\sum_{i=1}^N w_i\, p_{1,1}(k-i) \\
&\quad + 2ab\sum_{i=1}^N w_i\, p_{1,i+1}(k), \\
p_{1,j}(k+1) &= a\, p_{1,j-1}(k) + b\sum_{i=1}^{j-3} w_i\, p_{1,j-i-1}(k-i) \\
&\quad + b\sum_{i=j-2}^N w_i\, p_{1,i-j+3}(k-j+2), \qquad (20)
\end{aligned}
$$

for $j \in 2,\ldots,N+1$. If for a given j the subscript of w_j is less than one, then $w_j = 0$ is considered and if the upper value on the sum is less than the lower value, then the sum is zero.

We will now show that we can obtain a Markovian structure for the system above. We define the state vector

$$\vec{P}(k) = \begin{bmatrix} p_{1,1}(k) \\ p_{1,2}(k) \\ \vdots \\ p_{i,N+1}(k) \end{bmatrix}, \qquad (21)$$

and with this we define a super vector

$$\hat{P}(k) = \begin{bmatrix} \vec{P}(k) \\ \vec{P}(k-1) \\ \vdots \\ \vec{P}(k-N) \end{bmatrix}. \qquad (22)$$

We can now represent (20) in state space form which takes the following structure

$$\hat{P}(k+1) = \hat{\hat{\mathbf{A}}}\hat{P}(k) \qquad (23)$$

where

$$\hat{\hat{\mathbf{A}}} = \left[\begin{array}{c|cccc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_N \\ \hline \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{array}\right]. \qquad (24)$$

The submatrices $\mathbf{A}, \mathbf{B}_1, \ldots, \mathbf{B}_N \in \mathbb{R}^{(N+1)\times(N+1)}$ are given by

$$\mathbf{A} = \begin{bmatrix} a^2 & 2abw_1 & 2abw_2 & \cdots & 2abw_N \\ a & bw_1 & bw_2 & \cdots & bw_N \\ 0 & a & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a & 0 \end{bmatrix}, \qquad (25)$$

$$\mathbf{B}_i = \begin{array}{cc} \begin{bmatrix} b^2w_i & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\ bw_i & bw_{i+1} & \cdots & bw_{N-1} & bw_N & 0 & \cdots & 0 \\ 0 & bw_i & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & bw_i & 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow \text{row} \\ (i+2) \\ \\ \\ \end{array} \end{array} \qquad (26)$$

and $\mathbf{I}$ is the $(N+1)$-dimensional identity matrix. Notice that $\hat{\hat{\mathbf{A}}} \in \mathbb{R}^{(N+1)^2\times(N+1)^2}$.

IV. Stability of the Mean and Second Moment

To determine stability of a deterministic discrete time system one looks at the eigenvalues of the state transition matrix. The magnitude of all eigenvalues must be less than one for the system to be stable. The stability of the mean is derived from the eigenvalues of the state transition matrix,

$\sum_{i=1}^{N} w_i \mathbf{\Lambda}_i$ in (12,13). The characteristic equation is found to be

$$(s-a) - b\sum_{i=1}^{N} w_i s^{-\tau_i} = 0. \tag{27}$$

To determine the stability boundaries in the parameter space (a, b), we evaluate the characteristic equation at the possible values s so that $|s| = 1$, in particular $s = 1$, $s = -1$ and $s = e^{\pm i\theta}$, $\theta \in (0, \pi)$. Each of these provide different stability curves [5,3]. Notice that for $s = 1$, one obtains a delay-independent condition.

Stability for the variance is determined in the same way using the state transition matrix in (24). At first glance, it appears that analyzing the stability of the second moment will involve analyzing a matrix of dimension $(N+1)^2$, but it can be reduced to analyzing an $N+1$ dimensional matrix.

We denote the submatrices delimitated by the lines in (24) as

$$\hat{\hat{\mathbf{A}}} = \left[\begin{array}{cc} \mathbf{A} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \tilde{\tilde{\mathbf{D}}} \end{array}\right], \tag{28}$$

which yields

$$\begin{aligned}\det(s\hat{\hat{\mathbf{I}}} - \hat{\hat{\mathbf{A}}}) &= \det(s\tilde{\tilde{\mathbf{I}}} - \tilde{\tilde{\mathbf{D}}})\times \\ &\quad \det\left((s\mathbf{I} - \mathbf{A}) - \tilde{\mathbf{B}}(s\tilde{\tilde{\mathbf{I}}} - \tilde{\tilde{\mathbf{D}}})^{-1}\tilde{\mathbf{C}}\right) \\ &= s^{N(N+1)} \det\left((s\mathbf{I} - \mathbf{A}) - \tilde{\mathbf{B}}(s\tilde{\tilde{\mathbf{I}}} - \tilde{\tilde{\mathbf{D}}})^{-1}\tilde{\mathbf{C}}\right),\end{aligned} \tag{29}$$

where $\tilde{\tilde{\mathbf{I}}}$ denotes the $N(N+1)$ dimensional identity matrix. We are only left with determining the remaining $(N+1)$ eigenvalues. We find

$$\det\left((s\mathbf{I} - \mathbf{A}) - \tilde{\mathbf{B}}(s\tilde{\tilde{\mathbf{I}}} - \tilde{\tilde{\mathbf{D}}})^{-1}\tilde{\mathbf{C}}\right) = \det(\mathbf{M}_1 + \mathbf{M}_2), \tag{30}$$

where

$$\mathbf{M}_1 = \begin{bmatrix} s-a^2 & -2abw_1 & -2abw_2 & & \dots & & -2abw_N \\ -a & s-bw_1 & -bw_2 & & \dots & & -bw_N \\ 0 & -a & s & 0 & \dots & & 0 \\ 0 & \frac{bw_1}{s} & -a & s & 0 & \dots & 0 \\ 0 & \frac{bw_2}{s^2} & \frac{bw_1}{s} & -a & s & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \frac{bw_{N-2}}{s^{N-2}} & \frac{bw_{N-3}}{s^{N-3}} & \dots & \frac{bw_1}{s} & -a & s \end{bmatrix} \tag{31}$$

and

$$\mathbf{M}_2 = \begin{bmatrix} b^2\sum_{i=1}^{N}\frac{w_i}{s^i} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{bw_1}{s} & \frac{bw_2}{s} & \frac{bw_3}{s} & \dots & \frac{bw_N}{s} & 0 \\ \frac{bw_2}{s^2} & \frac{bw_3}{s^2} & \dots & \frac{bw_N}{s^2} & 0 & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots & \vdots \\ \frac{bw_{N-1}}{s^{N-1}} & \frac{bw_N}{s^{N-1}} & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{32}$$

Setting (30) equal to zero, gives the characteristic equation for the second moment, whose roots are the eigenvalues of the system.

V. NOTIONS OF STABILITY FOR STOCHASTIC SYSTEMS

We have provided deterministic discrete time equations whose stability determine the stability of the mean and second moment for the non-deterministic system (1). However, the first and second moment converging to zero does not guarantee that the state converges to zero with probability one (w.p.1). We restate a theorem that can be found in [2]:

The following implications hold

$$\begin{array}{ccc} (X(k) \xrightarrow{w.p.1} X) & & \\ \Downarrow & & \\ (X(k) \xrightarrow{P} X) & \Rightarrow & (X(k) \xrightarrow{D} X) \\ \Uparrow & & \\ (X(k) \xrightarrow{r} X) & & \end{array}$$

for any $r \geq 1$. Also, if $r > s \geq 1$ then

$$(X(k) \xrightarrow{r} X) \Rightarrow (X(k) \xrightarrow{s} X).$$

No other implications hold in general.

Here, $X(k) \xrightarrow{r} X$ denotes that the sequence $X(k)$ converges to a constant X in r^{th} *order*, for $r \geq 1$, which holds if $\mathbb{E}[|X(k)|^r] < \infty$ for all k and

$$\mathbb{E}[|X(k) - X|^r] \to 0 \quad \text{as} \quad k \to \infty.$$

While $X(k) \xrightarrow{P} X$ and $X(k) \xrightarrow{D} X$ denote convergence in probability and distribution [2] . Notice, convergence in r^{th} *order* only guarantees convergence in probability and distribution. Finally, $X(k) \xrightarrow{w.p.1} X$ denotes convergence with probability one. Namely, $\{X(k)\}$ converges to X w.p.1 if, for every $\epsilon > 0$, $|X(k) - X| \geq \epsilon$ only finitely often. That is, for each path ω, there is a number $k(\omega)$ so that $|X(k) - X| \geq \epsilon$, for all $k > k(\omega)$, [4]. We may say that, with the exception of a set of sequences of probability zero, all sequences $\{X(k)\}$ converge to X in the usual sense.

Convergence of the second moment, $X(k)^2$, is then equivalent to convergence in 2^{nd} *order* since $X(k)^2$ is positive definite. This is why convergence of the mean is insufficient but the convergence of the second moment may be enough. Now, we would like to show the case when $\{X(k)\}$ converges with probability one (w.p.1).

Given the general vector case $\vec{X}(k+1) = \mathbf{A}(k)\vec{X}(k)$, where the $\{\mathbf{A}(k)\}$ are mutually independent random matrices, [4] provides the following theorem, using a Lyapunov function of the form $\vec{X}^T\mathbf{Q}\vec{X}$, where $\mathbf{Q}$ is positive definite i.e. $\mathbf{Q} > 0$.

Let $\mathbf{Q} > 0$, $\mathbf{C} \geq 0$ and

$$\mathbb{E}[\mathbf{A}(k)^T\mathbf{Q}\mathbf{A}(k)] - \mathbf{Q} = -\mathbf{C}. \tag{33}$$

Then $\mathbb{E}[\vec{X}(k)^T\mathbf{C}\vec{X}(k)] \to 0$ and $\vec{X}(k)^T\mathbf{C}\vec{X}(k) \to 0$ w.p.1. Let the $\mathbf{A}(k)$ be identically distributed. If $\{\vec{X}(k)\}$ is mean square stable (that is, $\mathbb{E}[\vec{X}(k)^T\vec{X}(k)] \to 0$), then for any $\mathbf{C} > 0$, there is a $\mathbf{Q} > 0$ satisfying (33).

Given this theorem, if $\{\mathbf{A}(k)\}$ are identically distributed and mutually independent, there exists a solution $\mathbf{Q}$ for (33) if we choose $\mathbf{C} = \mathbf{I}$. According to the theorem, the existence of the solution implies $\vec{X}(k)^T\vec{X}(k) \to 0$ w.p.1. This is a sufficient condition for w.p.1 stability when all possible delays are distinct and equally likely.

VI. Examples

Now that we have set up our conditions for mean and second moment stability, we will give some examples where we utilize this method to analyze systems with different delay distributions. We will use distributions that satisfy the conditions for the theorem stated in [4], wherein, stability of the second moment guarantees w.p.1 stability. Figure 1 shows the discrete probability function for a uniform delay distribution while Fig. 2 is for a distribution with two equally probably delays which we refer to as a toggle distribution.

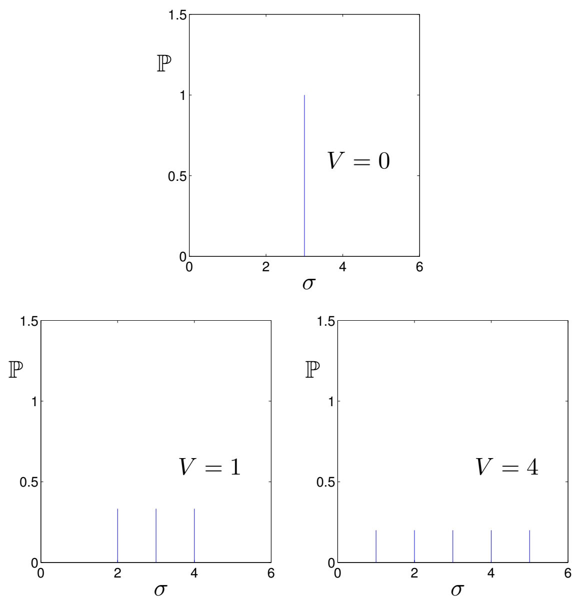

Fig. 1. Discrete probability function for uniform distribution with $E = 3$.

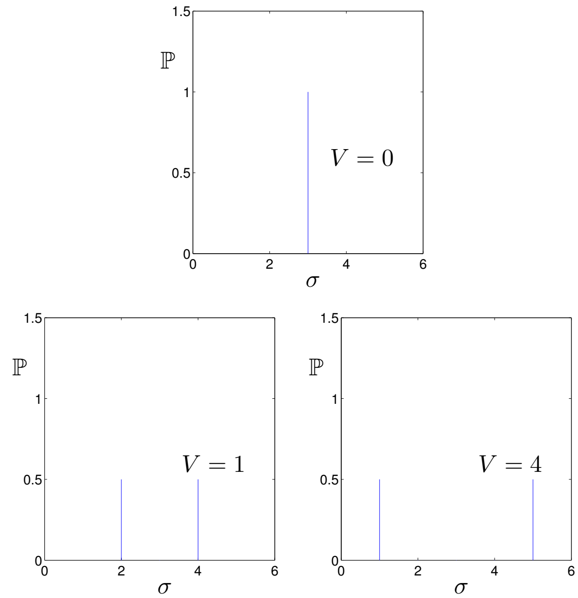

Fig. 2. Discrete probability function for a toggle distribution with $E = 3$.

Although stability of the second moment implies stability of the mean, it is interesting to take a look at the region of stability for the mean since it provides necessary conditions for stability. In [1] we show how introducing additional delays to an already delayed continuous system can stabilize an unstable system. It is interesting to see that a similar result can be obtained for the discrete time system.

Figure 3 shows the stability region for the system (7) with uniform distribution. E is the expected delay value of the distribution and V is the variance. The black and red curves indicate eigenvalue crossings of the unit circle on the complex plane at 1 and -1. The cyan curve indicates complex conjugate eigenvalues crossing the unit circle. One can see that as the variance is increased, the region of stability (cyan region) increases. It is important to point out the region of stability for a single delay is not contained in the region of stability for the distributed delay case.

Next, we look at w.p.1 stability region. Recall that a system with identically independent distributed delays is stable w.p.1. if the second moment is stable. We first consider such systems with uniform distribution, then look at systems where the delay toggles between two values, each with equal probability.

Figure 4 shows the stability boundaries for the mean of the non-deterministic system with uniform delay distribution. For the curves the same notation is used as in Fig. 3 but here the cyan region indicates the region of w.p.1 stability. The cyan bound was found by sweeping across the state space of (a, b) and checking the eigenvalues of the system (23,24) for the second moment.

In Fig. 5 we show the stability when the delay toggles between two values with equal probability. Again, we plot the mean stability curves and indicate the second moment stability region by shading, which implies w.p.1 stability. From the toggle cases investigated here, the w.p.1 stability region for the uncertain delay is dominated by the region of stability for the mean of the system.

The introduction of uncertainty in the delay distorts the stability region for the single deterministic delay as can be seen in Fig. 6. Since some of the w.p.1 stability regions extend outside the stability bounds for the deterministic system we can stabilize the system by introducing uncertainty in the delay. We demonstrate this by numerical simulation in Fig. 7 where the parameters correspond to the mark "$\times$" in Fig. 6.

VII. Conclusion

In this paper we defined a notion of stability and derived a method of stability analysis for a class of linear system with stochastic delay. We investigated stability regions for scalar stochastically delayed feedback systems and made some interesting observations. We looked at two different types of delay distributions and found they had very different effects on the region of stability. In the case of the uniformly

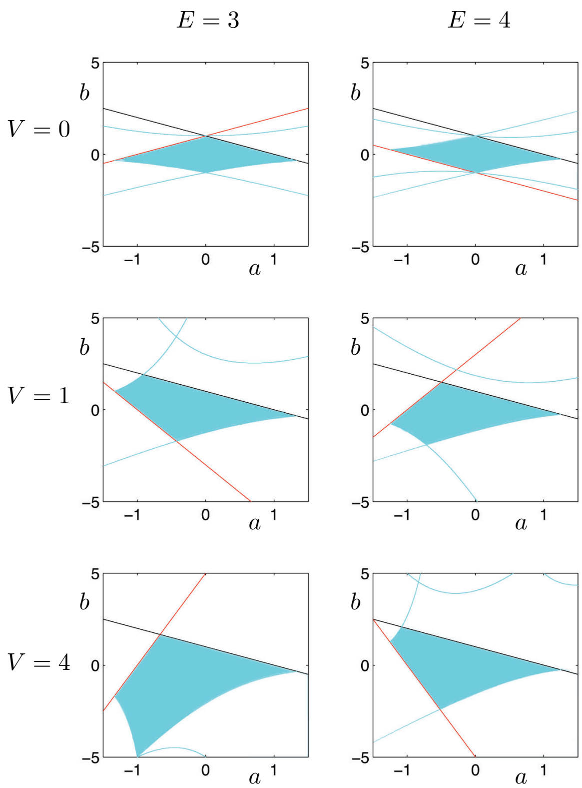

Fig. 3. Shaded region denotes the mean stability region for a uniform distribution. When crossing a black curve an eigenvalue crosses the unit circle at 1, while crossing a red curve (from stable to unstable) indicates that an eigenvalue crosses the unit circle at -1. Crossing a cyan curve indicates that a pair of complex conjugate eigenvalues cross the unit circle.

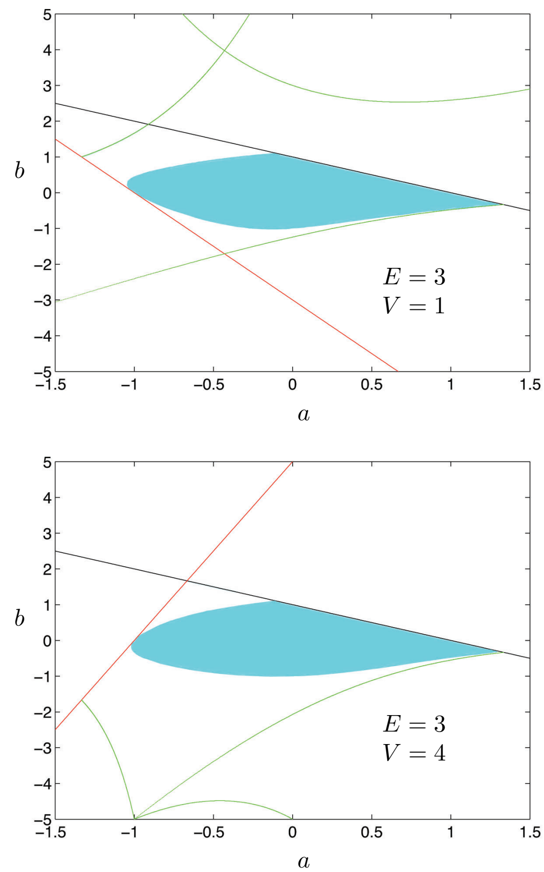

Fig. 4. Stability boundaries for uniform distribution given by the second moment. The curves represent the stability losses of the mean as in Fig.3, while the shaded cyan region indicates w.p.1 stability.

distributed delays, a worst case scenario would certainly be conservative. However, for two equally probably delays, the stability region of the mean seems to provide a good approximation of the stability bounds. We also found that introducing stochasticity in the delay may lead to stabilization of the system. Future work includes generalizing results for higher dimensional systems and w.p.1 stability conditions for general delay distributions.

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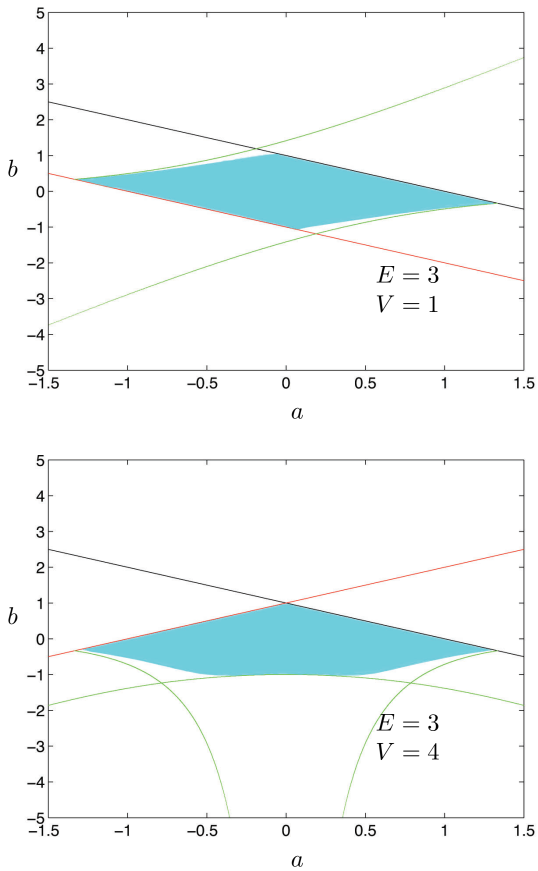

Fig. 5. Stability boundaries for a toggle distribution. The curves represent the stability losses of the mean while cyan shading indicates the w.p.1 stability region given by the second moment.

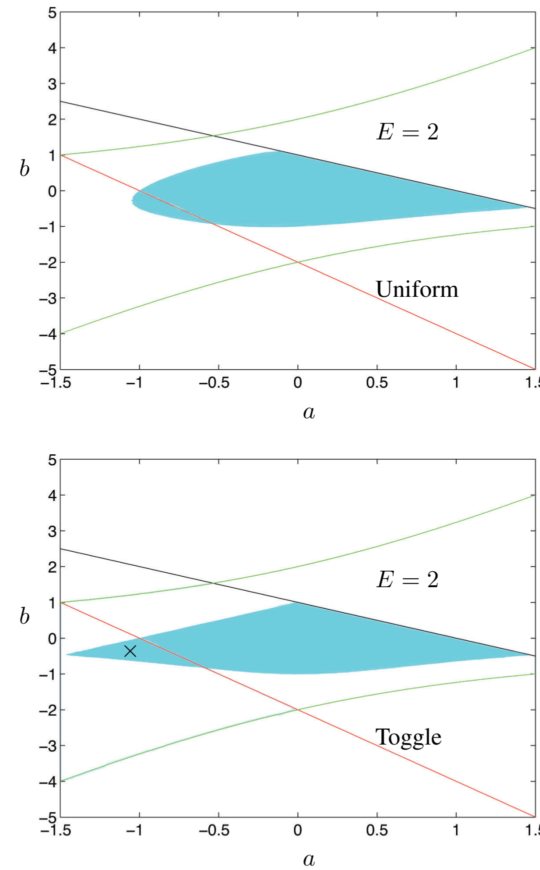

Fig. 6. Comparing stability for the case of a single delay (curves) with w.p.1 stability region (shaded cyan). For the discrete delay $V = 0$, while for the stochastic delay $V = 1$ is used.

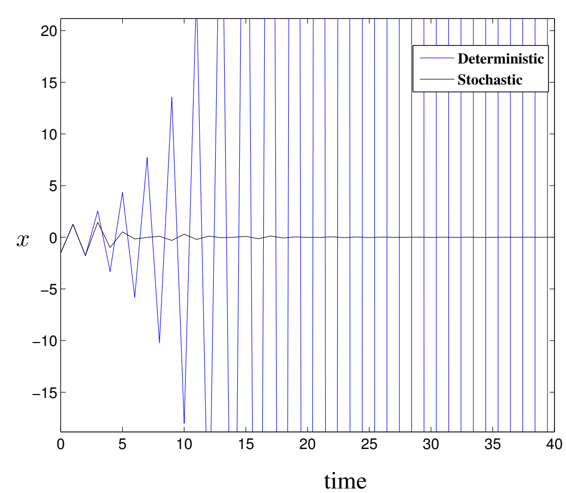

Fig. 7. Simulation comparing the cases deterministic and stochastic delay with $a = -1.1$ and $b = -.4$ as indicated by "$\times$" in Fig. 6.