2012 IFAC Workshop on Time Delay Systems
http://www.cds.caltech.edu/~murray/papers/2012a_gm12-ifac.html

Stabilization of Feedback Systems via Distribution of Delays

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Abstract: This paper investigates the results of distributing the delay of a single feedback system. To distribute the delayed feedback, the single delay is replaced by the sum of two distinct delays with the same effective delay. The statistical properties of the new distribution function in the feedback, namely the sum of two delta functions, are used to quantify the effectiveness of delay distribution. We show that the distribution is effective in reducing the magnitude of the open loop transfer function, thereby, decreasing the gain-crossover frequency and improving the phase margin. Finally, using these results, we explain the stabilizing effects of a delayed controller proposed in another publication.

Keywords: Delays, Biological Systems, Stability, Control, Distribution Functions

1. INTRODUCTION

There have been seemingly contradictory results on the role of delays in stability. Delays in control theory are attributed to instability while biology inspired papers have demonstrated their effectiveness in stabilizing, otherwise, unstable systems as shown in Lavaei et al. (2010) and Orosz et al. (2010). We aim to illustrate that both concepts are correct. In this paper, we show how an unstable single-input single-output (SISO) system with delayed feedback can be stabilized by distributing the delay about an effective mean. A system with a single finite delayed feedback can become unstable for a large enough delay but simply distributing the delay around a mean can lead to a robustly stable system.

Prior work such as Bernard et al. (2001) shows conditions for which distributed delay systems are stable and suggests distributed delays are preferable to single finite delays. Thiel et al. (2003) gives examples of biological systems with delays and then compares results of simulations with finite delays and distributed delays, the latter shown to be more stable.

The contributions of this paper are formal results using control theory tools to show the stabilizing effects of two delayed feedbacks versus one. We show that a system with two distinct delayed feedbacks has a larger stability region than a single delayed feedback system with the same effective mean. We show how these results can potentially explain the stabilizing effects of a delayed controller design proposed in Orosz et al. (2010) for the repressilator [Elowitz and Leibler (2000)], a genetic regulatory network. Last, we show how these results can potentially be applied to real systems.

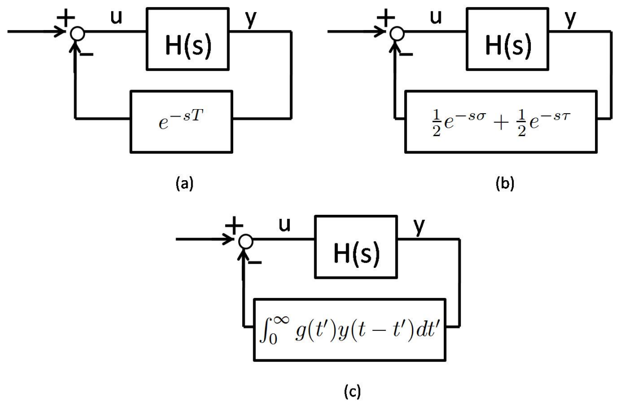

Fig. 1. (a) Single delay feedback system. (a) Two delay feedback system. (c) General distributed delay feedback system.

2. DISTRIBUTING A SINGLE FINITE DELAY

Suppose we have a system such as that shown in Figure 1(a) with a single delayed feedback. The phase margin of the linear system $H(s)$ is θ_{PM}. For simple transfer functions $H(s)$ it is often that case that if the delay, T, is greater than $\frac{\theta_{PM}}{\omega_\beta}$,where ω_β is the gain-crossover frequency, then we will have instability. We aim to improve the phase margin of the system by adding more delayed feedbacks. Figure 1(c) shows the most general form of a distributed delayed feedback, however, we would like to maintain the feedback as a sum of finite delays assuming they are inherent in the system and cannot be removed.

Now we will replace the delay e^{-sT} with the distributed delay

$$G(s) = \frac{1}{2}e^{-s\sigma} + \frac{1}{2}e^{-s\tau}$$

which has the corresponding distribution function

$$g(t) = \frac{1}{2}\delta(t-\sigma) + \frac{1}{2}\delta(t-\tau).$$

We put equal weights on both delays. The factor of $\frac{1}{2}$ is necessary in order to keep from adding a gain to the signal in the feedback. With $\int_0^\infty g(t)dt = 1$ we get unity gain. Convolving the output $y(t)$ with the distribution function gives $\frac{y(t-\sigma)+y(t-\tau)}{2}$ so the input is the average of the feedbacks and not the sum.

The system dynamics are deterministic but we use the statistical properties of the distribution functions as parameters. In Anderson (1993), the author shows that stability of a delayed system can be investigated with only the statistical properties of the time delay known. Accordingly, we expect that the mean and variance for symmetric distributions directly influence the stability of a system. From now on we will refer to the mean of the distribution as the effective delay. The effective delay of the new distribution function is

$$T = \frac{\sigma+\tau}{2}$$

and the variance, V, of the new distribution function is

$$V = \frac{(\sigma-\tau)^2}{4}.$$

We have chosen, without loss of generality, $\sigma \geq \tau$.

This new system is shown in Figure 1(b). We set the effective delay to be equal to the original single delay to maintain the concept of distributing the delay about the original one. Furthermore, the distribution is symmetric so that higher moments of the distribution are zero. In Bernard et al. (2001)there has been evidence to suggest that higher moments, when not zero, may play a role in stability in addition to mean and variance.

The permissable delay for a single feedback system is determined by the phase margin of the open loop transfer function. We will now derive a complex function for the distributed delay system mentioned above that will allow us to apply the same method used to determine phase margin in the single delayed feedback system. We assume that $H(s)$ is a stable linear system. By the Nyquist criterion, any encirclement of -1 by the Nyquist plot of the loop transfer function will indicate an unstable system [Åström and Murray (2008)].

The closed loop transfer function is

$$H_{cl} = \frac{H(s)}{1 + H(s)(e^{-s\sigma} + e^{-s\tau})/2} \tag{1}$$

and, by the Nyquist criterion, at the boundary of instability we have that the loop transfer function

$$H(jw)(e^{-jw\sigma} + e^{-jw\tau})/2 = -1, \tag{2}$$

for some ω, σ and τ. With some algebraic manipulation we will show the loop transfer function to take the form

$$Ge^{-sT},$$

where, G is a complex function. In this form the permissable effective delays of the distributed system is determined by the plot of G on the complex plane. The effective delay term T will add a clockwise rotation until Equation (2) is satisfied. Let us find what G is.

The loop transfer function is

$$\begin{aligned} L(s) &= H(s)(e^{-s\sigma} + e^{-s\tau})/2 \\ &= H(s)(1 + e^{-s(\sigma-\tau)})e^{-s\tau}/2 \\ &= H(s)\frac{(e^{s\frac{\sigma-\tau}{2}} + e^{-s\frac{\sigma-\tau}{2}})}{2}e^{-s\frac{\sigma-\tau}{2}}e^{-s\tau} \\ &= H(s)\frac{(e^{s\frac{\sigma-\tau}{2}} + e^{-s\frac{\sigma-\tau}{2}})}{2}e^{-s\frac{\sigma+\tau}{2}}. \end{aligned} \tag{3}$$

If we evaluate the transfer function at $s = j\omega$ we can further simplify the equation to

$$\begin{aligned} L(s)|_{s=j\omega} &= H(j\omega)\frac{(e^{j\omega\frac{\sigma-\tau}{2}} + e^{-j\omega\frac{\sigma-\tau}{2}})}{2}e^{-j\omega\frac{\sigma+\tau}{2}} \\ &= H(j\omega)\cos\left(\omega\frac{\sigma-\tau}{2}\right)e^{-j\omega\frac{(\sigma+\tau)}{2}} \qquad (4) \\ &= H(j\omega)\cos\left(\omega\frac{\sigma-\tau}{2}\right)e^{-j\omega T}. \qquad (5) \end{aligned}$$

Finding the phase margin from the plot of

$$\begin{aligned} G &= H(j\omega)\cos\left(\omega\frac{\sigma-\tau}{2}\right) \\ &= H(j\omega)\cos(\omega\sqrt{V}) \end{aligned} \tag{6}$$

on the complex plane will determine the permissable effective delay for the new distributed delay system. Although both G and T are functions of σ and τ, they can be isolated in such a way that T can be varied while keeping G constant and vice versa since V and T can be varied independently but the maximum variance is limited by the effective delay, $\sqrt{V} \leq T$.

Notice that the magnitude of the the new complex function is bounded by that of $|H(j\omega)|$ for all frequencies, but for a given frequency ω, the point may be rotated an angle of π due to the sign change of the cosine term.

In the next section we show how, given the same effective delay, the single delay feedback system can be unstable while the distributed feedback system is stable.

3. CHANGE IN PHASE MARGIN WITH TWO FINITE DELAYS

For the given example, the phase margin of the new system depends on the variance of the distribution. Depending on the distribution, the phase margin can increase or decrease in comparison to the original system. We will show an example where a single delayed feedback will cause instability but if we distribute the delay the new system will have an increased phase margin, stabilizing the system.

Let us investigate the system in Figure 1(a) with

$$H(s) = \frac{.2}{(s+1)(s+.1)}$$

and a fixed delay in the feedback. For the associated distributed feedback, σ and τ are functions of the delay, T, and the variance, V; namely

$$\sigma = T + \sqrt{V} \quad \text{and} \quad \tau = T - \sqrt{V}.$$

For a given effective delay T we can always find σ and τ such that we achieve any desired variance satisfying $\sqrt{V} \in [0, T]$. We investigate the effect of the variance

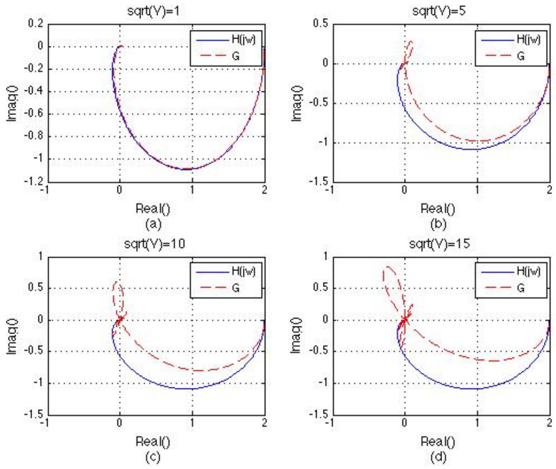

Fig. 2. Comparison of Nyquist plots showing the effects of changing the variance before adding the rotational effects of T.

on the phase margin of the new distributed system which pertains to permissable effective delays in the new system.

Figure 2 shows how the plot of G on the complex plane changes with the variance V. As the variance approaches zero, the plot approaches the Nyquist plot of the original single feedback system as it should. One can see that an increase in the variance improves the phase margin but if we continue to increase the variance we will soon begin to decrease the phase margin due to the sign change in G. The plot of G intersects the origin infinitely times because of the periodicity of the cosine term. Each time the cosine term goes through the origin, it changes sign and we have a rotation of π. So the function G remains bounded in magnitude but has π jumps in phase due to the intersection of the origin.

Figure 3 shows how the maximum permissible time delay changes as a function of the variance for the distributed system. As we recall, the $\sqrt{V}$ cannot exceed T. We mark this boundary by the red dashed line. The curve must lie above this boundary for the system to make physical sense. One can see that the system nears optimality near $\sqrt{V} = 17.5$. After this point, the system is rendered unstable due to the effects of the sign change of G.

Now we will pick a value of T that destabilizes the original system but when we distribute it, the system becomes stable. We set $T = 14$ and $\sqrt{V} = 10$ which gives $\sigma = 24$ and $\tau = 4$. Figure 4 shows the plots of G and H on the complex plane before and after applying the delay. For $T = 14$ the original system crosses the critical point marked by a cross which makes the system unstable. The distributed delay system remains stable. Figure 5 shows the step responses for the two systems with $T = 14$.

Now that we have shown the benefits of delay distribution we apply these principles to an example in the next section.

4. A DELAYED CONTROLLER FOR THE REPRESSILATOR

Many genetic regulatory systems have multiple feedback loops that serve the same function. The Gal network

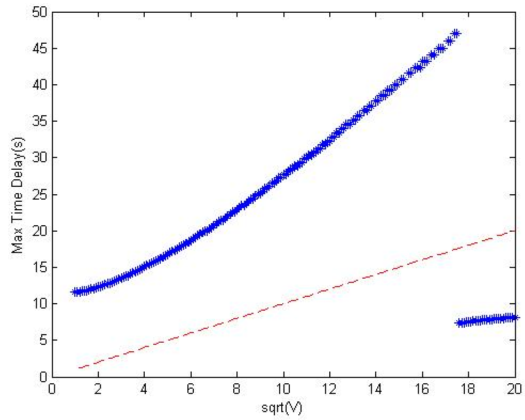

Fig. 3. Maximum permissible time delay as a function of variance.

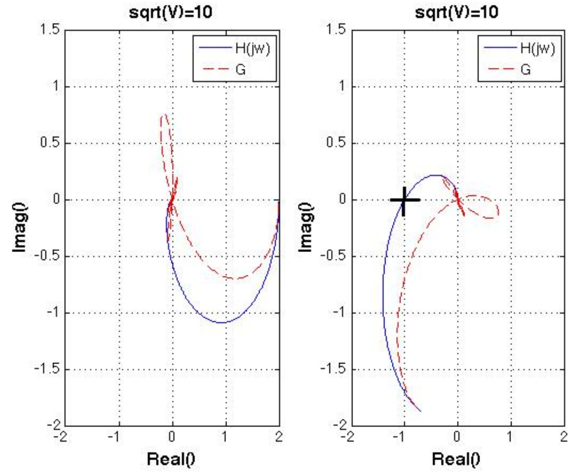

Fig. 4. Comparison of Nyquist plots for G and H at $T = 0$ and $T = 14$.

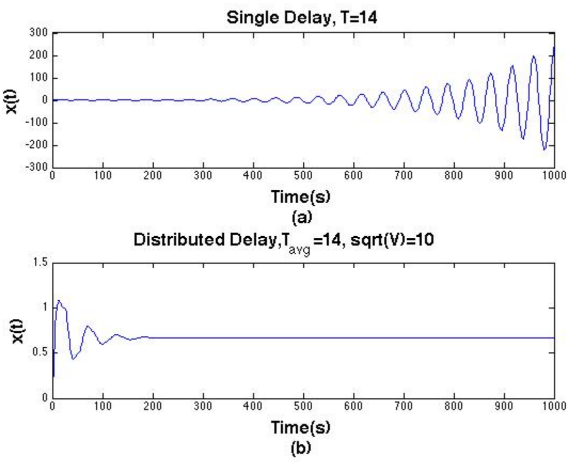

Fig. 5. (a)Step response for a single finite delayed feedback. (b) Step response for the distributed delayed feedback.

[Bennett et al. (2008)] has two transcription factor proteins that regulate the expression of the same gene. Their functions are almost identical but they have different translation rates. This seemingly redundant feature may prove to be functional in making the system robustly stable aside from the obvious advantage if one of the feedback loops gets knocked out.

Although biological systems have inherent finite delays like the time required for transcription and translation, the system remains stable under highly stochastic events. A potential reason may be the distribution of these delays by way of multiple feedbacks or by the stochastic nature of the delays themselves.

In Orosz et al. (2010) the authors show how the repressilator Elowitz and Leibler (2000) can be stabilized with delays. The repressilator is a synthetic oscillatory genetic regulatory network. A regulating gene with delayed expression of mRNAs and proteins is added to the repressilator as a controller. The system can be stabilized by tuning parameters that pertain to the delay and weight of the controller. A potential reason that this delayed controller is effective, with the right parameters, is the creation of a second feedback path.

We will describe the delayed controller in Orosz et al. (2010), and show how it falls into framework posed in section 2. The linearized system for the modified system with the controller is massaged into the form of Figure 1(c).

The following are the dynamics of the repressilator with the added controller gene expressed by p_4 Orosz et al. (2010):

$$\begin{aligned}
\dot{m}_i(t) &= -m_i(t) + \alpha f(p_{i+1}(t)) \quad \text{for } i = 1, 2,\\
\dot{p}_i(t) &= -\beta(p_i(t) - m_i(t)),\\
\dot{m}_3(t) &= -m_3(t) + \alpha f((1-\eta)p_1(t) + \eta p_4(t)),\\
\dot{p}_3(t) &= -\beta(p_3(t) - m_3(t)),\\
\dot{m}_4(t) &= -m_4(t) + \alpha f(p_2(t-\sigma)),\\
\dot{p}_4(t) &= -\beta(p_4(t) - m_4(t-\tau)),\\
f(p) &= \frac{1}{1+p^n} + f_0.
\end{aligned}$$

The dynamics of the repressilator can be recovered by taking $\eta = 0$ and discarding the dynamics of p_4 and m_4 which are then disconnected. The system above corresponds to Figure 6.

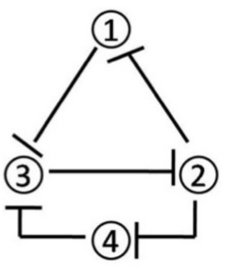

Fig. 6. Network diagram for repressilator with regulating gene.

The linearized system is derived in Orosz et al. (2010):

$$\frac{d}{dt}x(t) = Ax(t) + A_1 x(t-\sigma) + A_2 x(t-\tau),$$

where

$$x(t) = \begin{bmatrix} \Delta m(t) \\ \Delta p(t) \end{bmatrix}.$$

$\Delta m(t) = m(t) - m^*$ and $\Delta p(t) = p(t) - p^*$. The asterisk indicates equilibrium point. The dynamics matrices are

$$A = \begin{bmatrix} -I & \alpha\kappa A_0 \\ \beta B_0 & -\beta I \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \alpha\kappa A_\sigma \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ \beta B_\tau & 0 \end{bmatrix},$$

with $\kappa = f'(p^*) < 0$ and

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-\eta & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_\tau = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A_0, B_0, A_σ and B_τ reflect the topological structure of the system. The full nonlinear system can be represented by the block diagram shown in Figure 7. For this system

$$L = A_0 + A_\sigma,$$

$$P = \begin{bmatrix} P(s) & 0 & 0 & 0 \\ 0 & P(s) & 0 & 0 \\ 0 & 0 & P(s) & 0 \\ 0 & 0 & 0 & P(s)e^{-s\tau} \end{bmatrix},$$

$$P(s) = \frac{\alpha\beta}{(1+\beta)(1+s)},$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-s\sigma} \end{bmatrix},$$

and matrix N is a diagonal with the nonlinear function $f(p)$ in the nonzero entries.

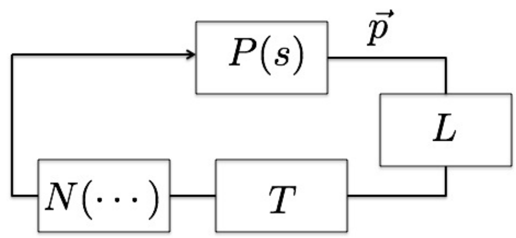

Fig. 7. Framework for the GRN network

From here on we apply all analysis to the linearized system to determine stability. We consider the nonzero "symmetric" equilibrium point, $p_i^* = m_i^* = p_{i+1}^* = m_{i+1}^* \neq 0$ for $i = 1, 2$ and 3. The matrix N is now a diagonal matrix with κ in the nonzero entries. We would like to focus our attention on stabilizing the output concentration of the third protein. The structure in Figure 7 is used to find the transfer function for the third protein. In a linear system the transfer functions commute, so it works out that the total delay in the regulating gene is the sum of the two delays corresponding to the production of mRNAs and proteins respectively.

We can now define in Figure 1(c) what $H(s)$ is and what the general distribution function $g(t)$ is for a SISO system with the output being the concentration of the third protein in the represillator with controller proposed in Orosz et al. (2010). We find

$$H(s) = -\kappa^3 \left(\frac{\alpha\beta}{(s+1)(s+\beta)} \right)^3 \quad (7)$$

and

$$g(t) = \eta\delta(t) + (1-\eta)\delta(t-(\sigma+\tau)). \quad (8)$$

The distribution function is like the example given in the beginning, two finite delayed feedbacks.

Looking at the Nyquist plot of $H(s)$ in Figure 8 we see that the single feedback system is unstable. The added delayed feedback path stabilizes the unstable system.

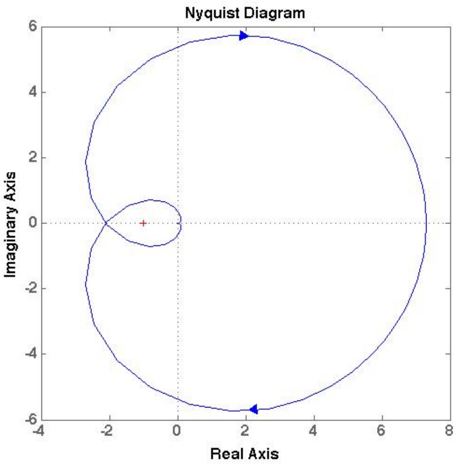

Fig. 8. Nyquist plots for $H(s)$ in Equation 7 with parameters $\alpha = 215.52$, $\beta = .2069$ and $\kappa = -.009$.

It is this distribution of delays that helps to stabilize the system for appropriate values of η and $\sigma + \tau$. The distribution creates a weighted average over the output at the current time and some past time.

5. STABILIZATION USING PARALLEL DELAYS

Now we revisit the previous example and pose a different problem. We previously explored the effects of distributing a delay about a given mean. We would like to take a different approach on our analysis in order for our results to be useful in application. If the delays are inherent in a system then there is a minimum delay that can be achieved. With this in mind, if a second feedback is to be added there would be a naturally imposed constraint on the size of the added delay. We will show that in some cases it may be possible to stabilize the system by adding a longer delay.

Now, we are given the constraint

$$T_{min} = \sigma < \tau$$

where T_{min} is the known minimum achievable delay in the feedback. We find the range of the second delay τ that stabilizes the system, assuming the original system with the single delayed feedback is unstable.

A utility function is defined as

$$J_V = \frac{\theta_\beta}{\omega_\beta} - T, \quad (9)$$

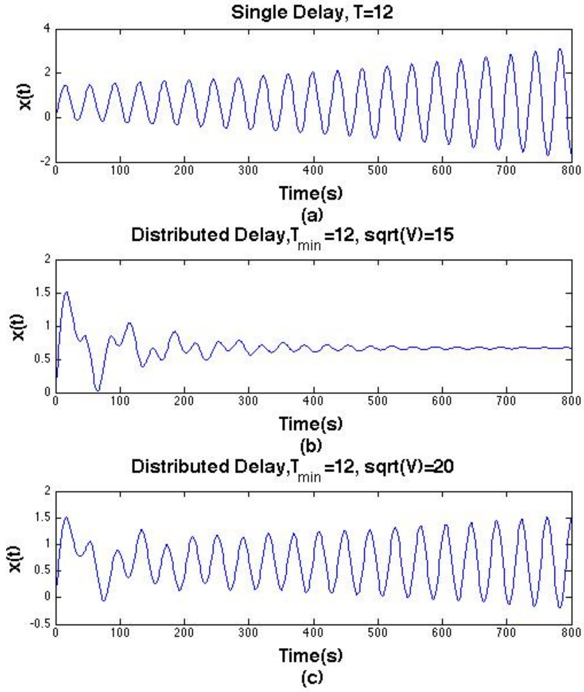

Fig. 9. (a) Simulation for the single delayed feedback system with $T = 12s$. (b) Simulation plot for and added delay of $\tau = 42s$. (c) Simulation plot for and added delay of $\tau = 52s$.

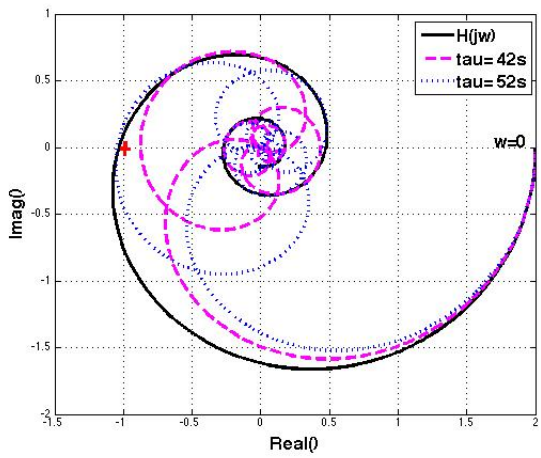

Fig. 10. Nyquist plot for the single delayed feedback system and the two distributed delayed systems simulated in Figure 10.

where θ_β and ω_β are the phase margin and gain-crossover frequency of the newly defined open loop function G for the double feedback system as defined in Equation (6). T is the expected or average value of the two delays. J_V gives the difference between the permissible delay,

$$T_{PM} = \frac{\theta_\beta}{\omega_\beta}, \quad (10)$$

due to the new increased phase margin and the new effective delay,

$$T = \sqrt{V} + \sigma, \tag{11}$$

as a result of the second feedback. For the system to be stable we require $J_V > 0$.

We solve for ω_β in the same usual manner gain-crossover frequencies are calculated. The gain-crossover frequency is the frequency at which the magnitude of the open loop transfer function is equal to 1,

$$|cos(\omega_\beta \sqrt{V})H(j\omega_\beta)| = |cos(\omega_\beta \sqrt{V})||H(j\omega_\beta)| = 1 \tag{12}$$

$$|cos(\omega_\beta \sqrt{V})| = \frac{1}{|H(j\omega_\beta)|}. \tag{13}$$

The first equality in Equation (12) holds since one can factor out the cosine term in calculating the magnitude. This gives the constraint $|H(j\omega_\beta)| \geq 1$

Solutions to Equation 13 are given by

$$\sqrt{V} = \frac{1}{\omega_\beta} \cos^{-1}\left(\pm \frac{1}{|H(j\omega_\beta)|}\right). \tag{14}$$

The phase margin θ_{PM} is the smallest CW rotation in radians required for the point $G|_{\omega_\beta}$ to cross the negative real axis.

We use T^+_{PM} and T^-_{PM} to refer to the solutions corresponding to the plus and minus term in Equation (14) respectively. Figure 11 shows the solution curves to Equations (10) and (11) as functions of the variance for the system with

$$H(s) = \frac{.2}{(s+1)(s+.1)}. \tag{15}$$

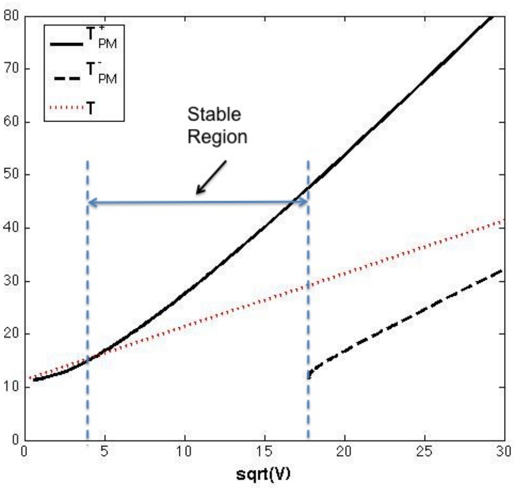

Fig. 11. Phase margin for open loop system and average delay as a function of variance with $\sigma = 11.39s$.

The minimum delay σ was calculated from the phase margin of the original system, therefore, the unmodified system is unstable. Positive values of J_V correspond to the region where the T_{PM} curve lies above the T curve. The plot shows a minimum variance of 4.4^2 is required to stabilize the system. This corresponds to a second delay value of $\tau = 20.19s$.

Maximizing the utility function is not necessarily the best choice when designing the added feedback. This occurs at approximately $\sqrt{V} = 17.7s$ and it is obvious that the system is not robust to uncertainty or variance in the delay. It can easily perturb into the unstable region.

Figure 10 shows simulations for step inputs into the single feedback system with delay $T_{min} = 12s$ and the double feedback system with two different values for the second delay.

We see that the system can indeed be stabilized by adding a longer delay to the feedback path.

6. SUMMARY

We have shown how the shape of the distribution function of the feedback influences the stability of the closed loop system. The results in this paper suggest that multiple feedbacks can serve to stabilize a system if their distribution function can achieve high variance with little cost to increasing the effective delay. We looked at a posed delayed controller for the repressilator and showed that the resulting stabilizing effects can be attributed to the distribute of delay. Last, we demonstrated how adding delays can have surprising results.

ACKNOWLEDGEMENTS

This work was supported by NSF funding source # CNS-0911041 (Division of Computer and Network Systems). Ideas were partly inspired through conversations with colleague Ophelia Venturelli about the galactose regulatory network in yeast. The authors would like to thank Gábor Orosz for valuable feedback.

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