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A Frequency Domain Condition for Stability of Interconnected MIMO Systems

Ather Gattami[1]
Department of Automatic Control
Lund Institute of Technology
Box 118, Lund, SE-22100, Sweden

Richard M. Murray
Control and Dynamical Systems
California Institute of Technology
Pasadena, CA 91125, USA

Abstract

In this paper analysis of interconnected dynamical systems is considered. A framework for the analysis of the stability of interconnection is given. The results from Fax and Murray[1] that studies the SISO-case for a constant interconnection matrix are genralized to the MIMO-case where arbitrary interconnection is allowed. The analysis show existness of a separation principle that is very useful in the sense of the simplicity for stability analysis. Stability could be checked graphically using a Nyquist-like criterion. The problem with time-delays and interconnection variation and robustness appear to be natural special cases of the general framework, and hence, simple stability criteria are derived easly.

1 Introduction

1.1 Motivation

In recent years there has been a large amount of interest in analysis of interconnected systems and networks, where the relation between the interconnection and stability of the resulting systems are related. In particular, there has been an attempt to focus on distributed systems where the controller is decentralized, i.e each plant of the interconnected system makes a decision based on limited information that might be available to it.

Interconnection can be found in our everyday life. There are many examples of such systems, and here we give only a sample of different problems that have the issue of interconnection in common. The internet is a very large network where stability issues are of great interest. The information flow transported along different links could, e.g., be delayed which makes it hard to stabilize the entire network if the delays are not taken into account.

Economy markets are another example of rather complicated pricing system where we do have a lot of manual control, and in the same time a lot of interconnection between different pricing dynamics.

[1] Corresponding author: `ather@control.lth.se`

The power network is probabely one of the most complex networks. We can find stability problems not only when trying to robustly stabilize the physical power network(which is hard enough), but also stabilize the econimcs marked that is embedded into it. Consumers, distributers, and power generators try to optimize their profit. Therefore, we have to take into account the economics network that also could be unstable, where the pricing plays a large role. An example is the California power crisis of 2000.

In later years, even stability of vehicle formations has been of great interest, e.g. formation of unmanned air vehicles (UAV), robots, and sattelites are only few examples.

1.2 Previous Work

There has been a lot of research on interconnected systems where some focused on particular "real-world" problems and some on trying to find a more general approach to analyze interconnected systems and give a constructive way for designing the decetralized controller.

In Fax and Murray [1] a Nyquist-like criterion is derived for stability check under a constant feedback matrix for SISO systems. Also a sufficient condition is given for interconnected MIMO systems. In Olfati-Saber and Murray [2] the average-consensus problem was considered for the case of single integrators. Briefly, the average-consensus problem is about trying to make a group of plants agree on the average of their states or outputs under some interconnection between different plants. Also, [2] touches the idea of introducing an interconnection matrix that is frequency dependent and examine its eigenvalues to derive stability conditions. The work by V. Gupta *et al*[3] derives stability conditions for stochastically varying interconnection. In A. Jadababaie *et al*[4] the problem with switched interconnection is also considered for the case where the switching rule is restricted to certain properties. M. Rotkowitz *et al* [6] introduces the notion of quadratic invariance and how it could be used in constructing a descentralized control law by minimizing the closed-loop norm of feedback system subject to constraints on the controller structure. G. Vinnicombe [5] considers the effect of time-delays in the stability of end-to-end congestion control for the internet.

1.3 Contributions of the Paper

Initially, the problem of how time-delays affected stability of vehicle formations was considered, building on the work done in [1]. Trying to find an approach to solve the problem, the framework given in [1] needed to be extended, starting with translating the problem formulation from the time-domain to the frequency domain. The new formulation of the problem was one of the vital parts of this paper. Interesting properties showed up and proved to be very useful for other kind of problems.

Hence the main goal of the paper is to introduce a general framework for interconnected systems where we try to include all problems discussed in the previous section and state them in a simple and classical form, which hopefully reveals many properties that give us an easy way of stability analysis and system design. Here we try to show how the problem with time-varying connections and delays could be easly modelled using the general framework. Also we show a Nyquist-like criterion inspired by the one in [1], that could ease the analysis of the interconnection.

2 Preliminaries

2.1 Notation

We denote a set of elements $\{a_1, a_2, ..., a_n\}$ by $\{a_i\}$. $A \otimes B$ defines the *Kronecker* product between the matrices A and B. We let I_k be the $k \times k$ identity matrix.

2.2 Matrix Algebra

For a set of N matrices $\{M_1, ..., M_N\}$ of size $r \times s$, we define the *direct sum* as the $Nr \times Ns$ blockdiagonal matrix $\widehat{M}$ whose $r \times s$ diagonal blocks are the matrices $M_1, ..., M_n$ (in this order), and the other entries are zero, which we write as

$$\widehat{M} = \oplus \sum_{i=1}^{N} M_i$$

For a given $N \times N$ matrix Q, define an $Nk \times Nk$ matrix $Q_{(k)}$ by the equation

$$Q_{(k)} = Q \otimes I_k$$

Finally, we state the *Geršgorin disc theorem*(for a proof consult e.g. [12]):

Proposition 1 *Let $A = [A_{ij}]$ be an $n \times n$ matrix, and let*

$$C_j(A) = \sum_{i=1, i \neq j}^{n} |A_{ij}|.$$

Then all eigenvalues of A are located in the union of n discs

$$\bigcup_{j=1}^{n} \{z \in \mathbb{C} : |z - A_{jj}| \leq C_j\}.$$

2.3 Algebraic Graph Theory

A (simple) graph $\mathcal{G}$ is a mathematical structure that consists of finite set of elements $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ called *vertices*, or *nodes*, with a prescribed set $\mathcal{E}$ of *unordered* pairs of *distinct* vertices of $\mathcal{V}$. Every element $e \in \mathcal{E}$ can be written as $e = (v_i, v_j)$, $v_i, v_j \in \mathcal{V}$, and e is called an *edge*, or *arc*, of the graph $\mathcal{G}$. We call v_i and v_j the endpoints of e. We say that v_i and v_j are connected if $(v_i, v_j) \in \mathcal{E}$. We infer also the notion of a *directed* edge $e_{ij} = (v_i, v_j)$, which could be considered geometrically as an arrow from the node v_i to v_j. A graph with directed edges is called a *directed* graph. Consider a matrix A such that that the element a_{ij} is equal to one if $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ and zero otherwise. This matrix is called the *adjacency* matrix of $\mathcal{G}$. The *outdegree* of a vertex is the number of incident edges with the vertex that *point out* from the edge. The set of vertices that point out from vertex v_i is denoted by $\mathcal{J}_i$. Hence, the outdegree of v_i is simply $|\mathcal{J}_i|$.

There is a special matrix that has been used frequently in connection with modelling of networks. Let D be a diagonal matrix with D_{ii} equal to the out-degree of vertex i. The *Laplacian* of a graph is defined as

$$L = D - A.$$

though we will make a slight modification and define the Laplacian as

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{|\mathcal{J}_i|} & \forall j \in \mathcal{J}_i \\ 0 & \text{otherwise} \end{cases}$$

or more algebraically

$$L = D^{-1}(D - A)$$

assuming that $D_{ii} \neq 0$ for all i(note that the first definition does not require the later condition). The question of which definition to be used depends on the application.

The Laplacian is useful since many studies has been focused on its properties, and especially the spectral properties.

3 Main Results

3.1 Stability of Interconnection Represented by Feedback Transfer Matrix

We start by considering a set of N identical plants and its controllers given by the matrix functions $P(s)$ and $K(s)$ of size $n \times m$ and $m \times n$ respectively. Let

$$\widehat{P}(s) = \oplus \sum_{i=1}^{N} P(s)$$

and

$$\widehat{K}(s) = \oplus \sum_{i=1}^{N} K(s)$$

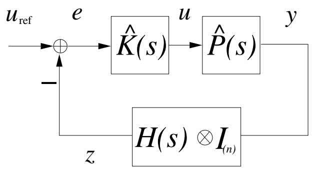

Figure 1: The interconnected system.

Now consider the *interconnected* MIMO system given as in Figure 1, where $H(s)$ is the interconnection matrix function with proper dimensions. We can state a simple stability theorem for the system above, but first we need we need a very useful relation that was first shown in [1]:

Lemma 1 *Let Q be a $N \times N$ matrix, M be $r \times s$ matrix with $\widehat{M}$ of size $Nr \times Ns$ such that $\widehat{M} = I_N \otimes M = \mathrm{diag}(M, ..., M)$ and let $Q_{(k)} = Q \otimes I_k$ where $\otimes$ denotes the Kronecker product, and I_k is the $k \times k$ identity matrix. Then*

$$\widehat{M} Q_{(s)} = Q_{(r)} \widehat{M}. \tag{1}$$

Theorem 2 *Let $U(s)$ be a vector of size mN, $Y(s)$ and $Z(s)$ be vectors of size nN, $H(s)$ a matrix of size $N \times N$. Also set*

$$\widehat{P}(s) = \oplus \sum_{i=1}^{N} P(s)$$

and

$$\widehat{K}(s) = \oplus \sum_{i=1}^{N} K(s),$$

where $P(s)$ and $K(s)$ are matrices of size $n \times m$ and $m \times n$ respectively. Let $T(s) = S(s)^* H(s) S(s)$ where $S(s)$ is the unitary Schur-transformation matrix such that $T(s)$ is upper triangular with the eigenvalues $\{\lambda_i(s)\}$ of $H(s)$ on its diagonal. Let p be the number of unstable poles of $T_{(n)}\widehat{P}(s)\widehat{K}(s)$. Then the control law $U(s) = \widehat{K}(s)(U_{\mathrm{ref}} - Z(s))$ stabilizes the system

$$\left\{ \begin{array}{rcl} Y(s) & = & \widehat{P}(s) U(s) \\ Z(s) & = & H_{(n)}(s) Y(s) \end{array} \right. \tag{2}$$

iff the Nyquist plot of

$$\prod_{i=1}^{n} \det[I_N + \lambda_i(s) P(s) K(s)]$$

makes p anti-clockwise encirclements of the origin.

Proof: The closed-loop dynamics are given by

$$\begin{array}{rcl} Z(s) & = & H_{(n)}(s) Y(s) \\ & = & H_{(n)}(s)\widehat{P}(s) U(s) \\ & = & H_{(n)}(s)\widehat{P}(s)\widehat{K}(s)(U_{\mathrm{ref}} - Z(s)). \end{array}$$

Thus, the transfer matrix between $Z(s)$ and U_{ref} is given by

$$(I_{nN} + H_{(n)}(s)\widehat{P}(s)\widehat{K}(s))^{-1} H_{(n)}(s)\widehat{P}(s)\widehat{K}(s) U_{\mathrm{ref}}(s)$$

Using the generalized Nyquist Theorem, we see that the closed-loop system is stable iff the Nyquist plot of

$$\det[I_{nN} + H_{(n)}(s)\widehat{P}(s)\widehat{K}(s)]$$

makes p anti-clockwise encirclements of the origin. But by applying Lemma 1 and the fact that $\det(S_{(n)}) = \det(S^*_{(n)}) = 1$, we get

$$\begin{array}{r} \det[I_{nN} + H_{(n)}(s)\widehat{P}(s)\widehat{K}(s)] = \\ \det[I_{nN} + S_{(n)}(s) T_{(n)} S^*_{(n)}(s)\widehat{P}(s)\widehat{K}(s)] = \\ \det[S_{(n)}(I_{nN} + T_{(n)}(s)\widehat{P}(s)\widehat{K}(s)) S^*_{(n)}] = \\ \det[I_{nN} + T_{(n)}(s)\widehat{P}(s)\widehat{K}(s)]. \end{array} \tag{3}$$

Since $T_{(n)}(s)$ is block upper triangular and both $\widehat{P}(s)$ and $\widehat{K}(s)$ are block diagonal, we get

$$\begin{array}{r} \det[I_{nN} + T_{(n)}(s)\widehat{P}(s)\widehat{K}(s)] = \\ \prod_{i=1}^{N} \det[I_n + \lambda_i(s) P(s) K(s)]. \end{array} \tag{4}$$

so the number of anti-clockwise encirclements of the origin made by the Nyquist plot of

$$\det[I_{nN} + T_{(n)}(s)\widehat{P}(s)\widehat{K}(s)]$$

is the same as the number of encirclements of the origin made by the Nyquist plot of

$$\prod_{i=1}^{N} \det[I_n + \lambda_i(s) P(s) K(s)],$$

■

3.2 Robustness

In the previous section we developed a method for analysis when homogeneous linear systems are interconnected. In the real world, the plants could be effected by nonlinearities and model errors. In this section we will propose a simple method to analyze this problem based on the framework that was presented in the previous section.

There are many perturbations models, but we will focus on two general structures that cover the low and high frequency parameter errors.

3.2.1 The Multiplicative Uncertainty Structure:

Let plant i be perturbed by the multiplicative uncertainty $(I + W_2^i(s)\Delta_i(s)W_1^i(s))$ such that the real plant dynamics is given by

$$P_\Delta(s) = (I + W_2^i(s)\Delta_i(s)W_1^i(s))P(s)$$

where $W_1^i(s)$ and $W_2^i(s)$ are transfer matrices that charecterize the spatial and frequency structure of the uncertainty $\Delta_i(s)$, where $||\Delta_i(s)||_\infty \leq 1$. Now let $\Delta(s) = \mathrm{diag}\{\Delta_1(s), ..., \Delta_N(s)\}$, $W(s) = \mathrm{diag}\{W_1^1(s), ..., W_1^N(s)\}$ and $W_2(s) = \mathrm{diag}\{W_2^1(s), ..., W_2^N(s)\}$. Then, the total system is given by the feedback-loop shown in Figure 2.

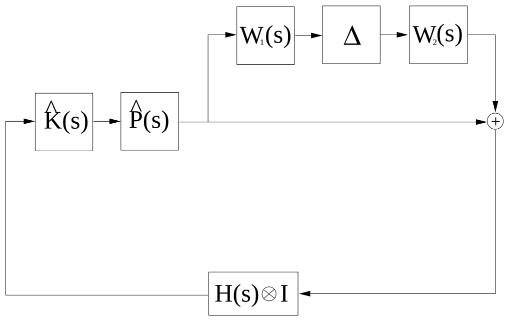

Figure 2: The interconnected system with additive unceratainty.

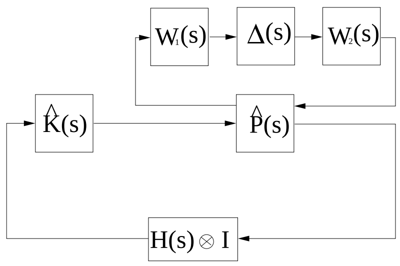

Figure 3: The interconnected system with uncertainty of the M-Δ structure.

3.2.2 The M-Δ Loop Structure: Here we consider plant i to be interconnected with a stable uncertainty matrix Δ_i where $||\Delta_i||_\infty \leq 1$ and weighted by the stable matrices $W_1^i(s)$ and $W_2^i(s)$. Hence, the real plant dynamics are given by

$$P_\Delta(s) = (I_n + P(s)W_2^i(s)\Delta_i(s)W_1^i(s))^{-1}P(s)$$

Constructing $\Delta(s)$, $W_1(S)$, and $W_2(s)$ as we did in the previous perturbation model, we obtain the following equation for the transfer function of the whole system

$$(I_{nN} + H_{(n)}(s)\tilde{\Delta}(s)\widehat{P}(s)\widehat{K}(s))^{-1}H_{(n)}(s)\tilde{\Delta}(s)\widehat{P}(s)\widehat{K}(s)$$

which is similar to the previous model except that

$$\tilde{\Delta}(s) = (I_{nN} + \widehat{P}(s)W_2(s)\Delta(s)W_1(s))^{-1}.$$

The feedback loop is shown in Figure 3. Both perturbation models introduced a new uncertainty $\widetilde{\Delta}$ that could be introduced as a separate block, i.e. $\widehat{P}_\Delta(s) = \widetilde{\Delta}\widehat{P}$ and the new loop becomes like in Figure 4.

Consider the interconnected system in Figure 4. Assume that $\widehat{P}(s)$, $H(s)$, and $\widetilde{\Delta}(s)$ are stable. Let

$$||\widetilde{\Delta}(s)||_\infty \leq \beta. \tag{5}$$

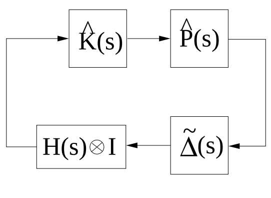

Figure 4: The interconnected system with unceratainty $\widetilde{\Delta}$.

Then the system is stable iff

$$||H_{(n)}(s)\widehat{P}(s)\widehat{K}(s)||_\infty < \frac{1}{\beta} \tag{6}$$

We note that equation 7 is equivalent to

$$||\lambda_i(s)P(s)K(s)||_\infty < \frac{1}{\beta}, \quad \forall i. \tag{7}$$

since

$$\begin{aligned} ||H_{(n)}(s)\widehat{P}(s)\widehat{K}(s)||_\infty &= ||H_{(n)}(s)\widehat{P}(s)\widehat{K}(s)||_\infty \\ &= ||S_{(n)}T_{(n)}(s)\widehat{P}(s)\widehat{K}(s)S^*_{(n)}||_\infty \\ &= ||T_{(n)}(s)\widehat{P}(s)\widehat{K}(s)||_\infty \\ &= ||\lambda_i(s)P(s)K(s)||_\infty \end{aligned} \tag{8}$$

Also, since $\widetilde{\Delta}(s)$ is blockdiagonal, we have that $\beta = \max_i\{||\widetilde{\Delta}_i||_\infty\}$. Now we give another stability result for the interconnected system where we use Theorem 2:

Corollary 1 *The interconnected system in Figure 4 is stable iff*

$$\prod_{i=1}^{N} \det[I_n + \lambda_i(s)P(s)K(s)]$$

makes p *anti-clockwise encirclements of the origin, where* p *is the number of unstable poles of* $H_{(n)}\widetilde{\Delta}(s)\widehat{P}(s)\widehat{K}(s)$ *and* $\lambda_i(s)$ *is the* i*th eigenvalue of* $H(s)\widetilde{\Delta}(s)$.

Of course, we don't know the eigenvalues of $H(s)\widetilde{\Delta}(s)$ since $\widetilde{\Delta}(s)$ is an uncertainty. On the other hand, it could be considered as a perturbation of the interconnection, e.i., a perturbation of the eigenvalues of $H(s)$. Tools from robust control theory might be useful to give upperbounds for these perturbations.

4 Applications

Theorem 2 is stated in such a way to give as general framework as possible for the interconnection of systems with homogeneous dynamics. There are many interesting special cases that are far from being trivial when trying to use traditional techniques. But using the results of Theorem 2 enable us to analyse complicated interconnections easily.

4.1 The Consensus Problem

It is of great interest to make a group of plants, e.g. aerial vehicles, to reach agreement, or *consensus* which as it is called in Olfati-Saber and Murray[2]. We would like, for instance, to make these plants to agree on some common state or output. We want to explore how the information topology and dynamics affects the stability of the interconnected system. We will consider the problem based on the framework presented in Fax and Murray [1], but with a frequency-domain approach.

Consider a system of N plants $\mathcal{P} = \{P_i\}^N$ such that each plant has m inputs and n outputs. Note that the assuming that the plants have the same dimensions does not imply any loss of generality. We assume that the dynamics for each plant are decoupled from the other $N-1$ plants in the system. Then we can write the system for plant i in the frequency domain as

$$Y_i(s) \quad = \quad P_i(s)U_i(s) \tag{9}$$

for all $i \in \{1,...,N\}$. The output $Y_i(s)$ is considered as a sensed information which represents the *internal* state measurement for plant i. The *external* state measurements $Z_{ij}(s)$ for V_i *relative* to other plants is given by

$$Z_{ij}(s) = Y_i(s) - Y_j(s), \forall j \in \mathcal{J}_i \tag{10}$$

where $\mathcal{J}_i \subset \{1,...,N\}\backslash\{i\}$ represents the set of plants that P_i can sense. For simplicity, we assume that $|\mathcal{J}_i| \geq 1, \forall i \in \{1,...,N\}$. This condition implies that each plant can sense at least one other plant. Notice that a single plant cannot drive all the terms $Z_{ij}(s)$ to zero simultaneously. Therefore, all errors must be synthesized into one signal. We introduce the new error measurement $Z_i(s)$ by building a weighted sum over the relative state measurements. For simplicity, we assume that the terms $Z_{ij}(s)$ are equally weighted, hence

$$Z_i(s) = \frac{1}{|\mathcal{J}_i|}\sum_{j\in\mathcal{J}_i} Z_{ij}(s). \tag{11}$$

Note that this assumption does *not* give us a weaker result. Let $K_i(s)$ denote the decentralized control law for plant i. Introduce $U(s) = (U_1(s),...,U_N(s))$, $Y(s) = (Y_1(s),...,Y_N(s))$ and $Z(s) = (Z_1(s),...,Z_N(s))$, where So letting $L^i_{(n)}$ denote the ith row of $L_{(n)}$, we see that

$$Z_i(s) = L^i_{(n)}Y(s)$$

Hence, the equation for the *total* system is given by

$$\left\{ \begin{array}{lcl} Y(s) & = & \widehat{P}(s)U(s) \\ Z(s) & = & L_{(n)}Y(s) \end{array} \right. \tag{12}$$

where $\widehat{P}(s)$ is the direct sum for the set of plants $\mathcal{P} = \{P_1(s),\ldots,P_N(s)\}$.

We will explore the stability of the interconnection with plants of *equal* dynamics, i.e. $P_i(s) = P(s)$ for all $i \in \{1,...,N\}$.

4.2 Stable SISO Plants and Interconnection

Let us consider the case when $P(s)$ and $K(s)$ are SISO-stable, and the interconnection matrix $H(s)$ is stable, that is $H(s)$ has no poles in the RHP. Then the criterion for stability of the interconnected system (1) is that the Nyquist plot of

$$\prod_{i=1}^{n}(1+\lambda_i(s)P(s)K(s)),$$

makes zero encirclements around the origin, or equivalently that the Nyquist plot of

$$\lambda_i(s)P(s)K(s)$$

makes no encirclements around $-1+0j$, for $i = 1,...,n$.

Now let the interconnection matrix be the Laplacian matrix, that is $H(s) = L$. Then we see that the system is stable iff the Nyquist plot of $\lambda_i P(s)K(s)$ makes no encirclements around $-1+0j$, which is equivalent to that the Nyquist plot of $P(s)K(s)$ does not encircle $-\frac{1}{\lambda_i}$ for $i = 1,...,n$.

4.3 Interconnection with Fixed Time-delays

A common problem with interconnected systems is the presence of time-delays. In this section, we will find necessary and sufficient conditions, using the techniques discussed earlier.

Consider the interesting case where $H(s) = L$, that is the interconnection is given by the Laplacian matrix. Suppose that there is a fixed time-delay τ_{ij} for plant i to get the sensed measurement from plant j that it is connected with. Then we can write the interconnection matrix as

$$[H(s)]_{ij} = L_{ij}e^{-\tau_{ij}s}.$$

For instance, if the plants are SISO and stable, necessary and sufficient conditions for stability of the interconnected system is that the Nyquist plot of

$$\prod_{i=1}^{N}(1+\lambda_i(s)P(s)K(s))$$

makes zero encirclements around the origin, where λ_i is the ith eigenvalue of $H(s)$. This is equivalent to that the Nyquist plot of

$$\lambda_i(s)P(s)K(s)$$

makes zero encirclements around -1+0j, for all i. So we can see that stability of the interconnected system depends on the structure of the interconnection given by the matrix $H(s)$, which is spanned by the topology of the interconnection(the Laplacian), and the structure of the time-delays.

A similar argument is easy to obtain for MIMO plants $P(s)$ which are not necessarily stable, using the results in section 3.

4.4 Interconnection with Random Delays

In this section we will assume that the plants are stable, and plant i recieves the sensed measurements from plant j after τ_{ij} time units. The delays τ_{ij} could be time-varying. Using the same arguments as in the fixed-delays section, we see that the interconnections matrix $H(s)$ this time has eigenvalues that are uncertain, because of the time-variation of the delays. One way is to use robustness analysis, where we consider the eigenvalues $\{\lambda_i(s)\}$ of $H(s)$ as perturbed, and the perturbation is given by regions that are depndent on the characteristics of variation of $\tau_i j$. Another way to do the analysis, which is much more conservative, is to use the Small Gain Theorem. Since we know that the entries of $H(s) = [L_{ij}e^{-\tau_{ij}s}]$ are characterised by the delays and Laplacian for the interconnection graph, we could easily calculate

$$||H(s)||_\infty \leq 2.$$

The inequality above is obtained using the Geršgorin disc theorem. Thus, a sufficient condition is to have

$$||\widehat{P}(s)\widehat{K}(s)||_\infty = ||P(s)K(s)||_\infty < \frac{1}{2}.$$

We illustrate the method above. Assume that plant i gets *information* from plant j telling the size of the delay. So when making the external state measurements, we can set

$$\tau_i = \max\{\tau_{ij}, j \in \mathcal{J}_i\},$$

and $Z_{ij} = [Y_i(s) - Y_j(s)]e^{-\tau_i s}$. Thus $Z(s) = \Delta LY(s)$, where $\Delta = \text{diag}\{e^{-\tau_1 s}, ..., e^{-\tau_n s}\}$. Let $\mathbf{\Delta}$ be the set of all possible time-delay connections Δ, such that the closed-loop system in Figure 5 is stable. The transfer function from $U_{\text{ref}}(s)$ to $Z(s)$ is given by $(I + \Delta L\widehat{P}\widehat{K})^{-1}\Delta L\widehat{P}\widehat{K}$. This is equivalent to $L\widehat{P}\widehat{K}$ being the transfer function from $U(s) = U_{\text{ref}}(s) - \Delta Z(s)$ to $Z(s)$, that is $L\widehat{P}\widehat{K}$ is the open-loop system with the feedback system given by Δ.

Clearly, $||\Delta||_\infty = 1$. Applying the Small Gain Theorem yields that the interconnection with random time-delays is stable *iff*

$$||L_{(n)}\widehat{P}(s)\widehat{K}(s)||_\infty = ||\lambda_i P(s)K(s)||_\infty < 1$$

for all i, e.i.

$$||P(s)K(s)||_\infty < \frac{1}{|\lambda_i|}$$

for all i, where $\{\lambda_i\}$ are the eigenvalues of L.

4.5 Time-varying Interconnection

It is very interesting to explore the robustness of an interconnected system, where links between different plants could be broken or intentionally changed to achieve performance. Consider a system interconnected by the Laplacian matrix L. Let $\mathcal{L}$ be a set of Laplacian matrices such that the closed loop system with respect to every Laplacian $\tilde{L} \in \mathcal{L}$ is stable.

4.5.1 Stable Plants and Multiplicative Uncertainty:

Consider time-varying interconnection in the case where the plants in the system are stable. Let Δ denote the uncerttain Laplacian matrix for the system. A diagram for the closed-loop system is given by Figure 7.

It is known that the eigenvalues for the Laplacian lie inside the unit disc centered at $1 + 0j$(an is proof could be derived using the Geršgorin disc theorem presented in the second section). Thus $||L||_\infty = \bar{\sigma} \leq 2$.

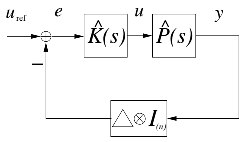

Figure 5: The interconnected system with interconnection uncertainty.

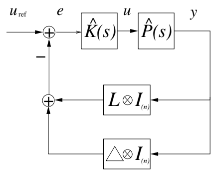

Figure 6: The interconnected system with interconnection uncertainty.

Then by the Small Gain Theorem, the interconnected system is stable *iff*

$$\gamma = ||\widehat{P}(s)\widehat{K}(s)||_\infty = ||P(s)K(s)||_\infty < \frac{1}{2}.$$

4.5.2 Unstable Plants and Additive Uncertainty:

Set

$$\Delta = \{\tilde{L} - L | \tilde{L} \in \mathcal{L}\}$$

It is not hard to find that $||\tilde{L} - L||_\infty \leq 2$ using Geršgorin disc theorem.

Now consider the closed-loop system shown in Figure 8:

By the Small Gain Theorem, the system is stable *iff*

$$\begin{aligned}\gamma &= ||L\widehat{P}(s)\widehat{K}(s)(I + L\widehat{P}(s)\widehat{K}(s))^{-1}||_\infty \\ &= \max_i ||\lambda_i P(s)K(s)(I + \lambda_i P(s)K(s))^{-1}||_\infty < \frac{1}{2}.\end{aligned} \tag{13}$$

There are many ways of stabilizing the system. One straight-forward way is to change the feedback gain, which is simply multiplying the Laplacian matrix L with some proper real constant. Another way is to use $\mathcal{H}_\infty$ control to minimize γ.

4.6 A Numerical Example

Consider a system of 6 stable plants $\{P_i\}_{i=1}^6$ with equal dynamics $P(s) = \frac{b_1 s + b_0}{s+a}$ associated with identical stable controllers $K(s) = \frac{c_1 s + c_0}{s+d}$. Suppose that plant P_i can sense plant P_{i+1} and P_{i-1} for $i = 1, ..., 6$, $P_7 = P_1$ and $P_{-1} = P_6$. The graph representing the interconnection is
Let τ_i be the time-delay for P_i to recieve the sensed signal of P_{i+1} and P_{i-1}. Then building the relative measurement $Z_{ij} = [Y_i(s) - Y_{i+1}(s)]e^{-\tau_i s}$ gives us the following interconnection matrix for the system(compare with the problem

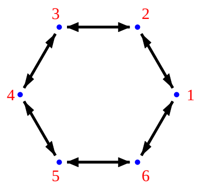

Figure 7: The graph representing the interconnection between the plants.

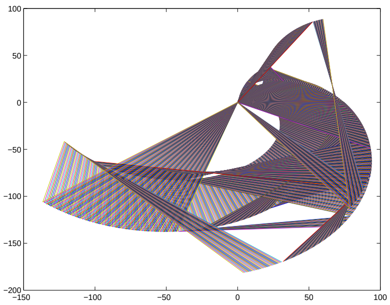

Figure 8: The Nyquist-like plot of the interconnected system .

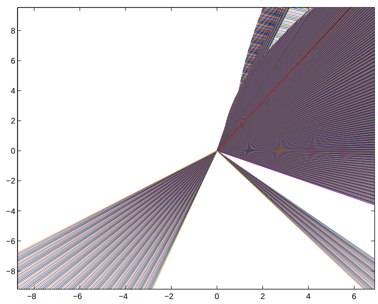

Figure 9: The Nyquist-like plot of the interconnected system zoomed around -1+ 0j.

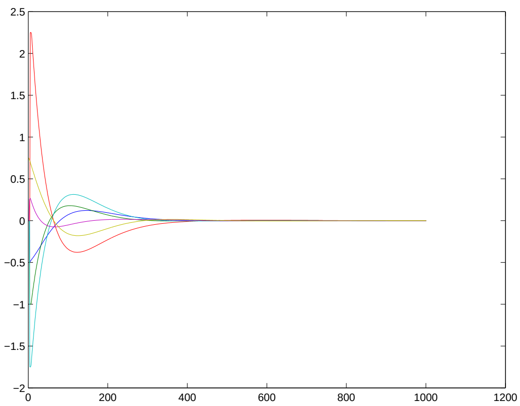

Figure 10: Simulation results of the interconnected system with time-delays.

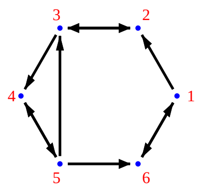

Figure 11: The graph to switch to in the example.

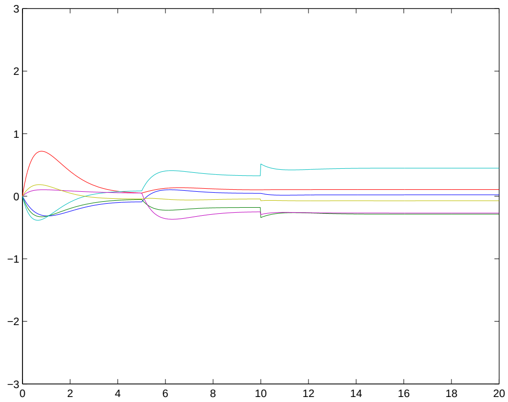

Figure 12: Switching between two topologies of the interconnection

setup):

$$H(s) = \begin{pmatrix} e^{-\tau_1 s} & -\frac{1}{2}e^{\tau_1 s} & 0 & \cdots & -\frac{1}{2}e^{\tau_1 s} \\ -\frac{1}{2}e^{\tau_2 s} & e^{-\tau_2 s} & -\frac{1}{2}e^{\tau_2 s} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}e^{\tau_6 s} & 0 & -\frac{1}{2}e^{\tau_6 s} & \cdots & e^{-\tau_6 s} \end{pmatrix}$$

Checking the Nyquist-like plot we see that the net encirclement of the origin is zero, hence the system must be stable.

Simulation of the system is shown in Figure 10. Now consider the case when we switch from the Laplacian above to another Laplacian for the graph in Figure 11. Choosing a controller such that

$$||L\widehat{K}(s)\widehat{P}(s)||_\infty = ||\lambda_i K(s)P(s)||_\infty < 1$$

guarentees stability, as seen in the simulation result shown in Figure 12.

5 Conlclusions and Future Work

In this paper a framework for interconnected systems was extended from the one introduced in Fax and Murray [1].

An efficient Nyquist-like method for stability check is developed for homogeneous and almost homogeneous interconnected systems with arbitrary connection. We have seen that many problems that arise in the context of system networks could be modelled in a way that fits our general framework, e.g. time-varying interconnection and the problem with time-delays.

There is still a lot to explore. The problem where the plants are heterogeneous is still important to analyze. It is very interesting to find out whether there is a similar separation principle when the plant dynamics are different. Another important issue is the problem with *bounded* time-varying delays. It is also of great interest to explore the role of robust control theory to obtain less conservative results and improve on the framework.

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