CDC-INV4104

Optimal Control of Affine Connection Control Systems: A Variational Approach

J. Alexander Fax   Richard M. Murray

Engineering and Applied Science
California Institute of Technology
Pasadena, CA 91125
{fax,murray}@cds.caltech.edu

February 20, 2000
Submitted to IEEE CDC 2000

Abstract

In this paper we investigate the optimal control of affine connection control systems. The formalism of the affine connection can be used to describe geometrically the dynamics of mechanical systems, including those with nonholonomic constraints. In the standard variational approach to such problems, one converts an n-dimensional second-order system into a $2n$-dimensional first-order system, and uses these equations as constraints on the optimization.

An alternative approach, which we develop in this paper, is to include the system dynamics as second-order constraints of the optimization, and optimize relative to variations in the configuration space. Using the affine connection, its associated tensors, and the notion of covariant differentiation, we show how variations in the configuration space induce variations in the tangent space. In this setting, we derive second-order equations have a geometric formulation parallel to that of the system dynamics. They also specialize to results found in the literature, (e.g.[4, 11]).

1 Introduction

In this paper we investigate the optimal control of affine connection control systems. The formalism of the affine connection has been shown to be a useful geometric framework for analyzing the dynamics of mechanical systems, including those with nonholonomic constraints (see [1, 8]). From the perspective of control theory, the affine connection has been used to derive controllability tests for mechanical control systems [10] and to design motion control algorithms [3]. Application areas include locomotion of systems with nonholonomic constraints,, control of underwater vehicles, and satellite reorientation [2, 6, 3].

While optimal controls techniques are generally developed for first-order systems, affine connection control systems are inherently second order. The standard approach to including a second order system is to recast an n-dimensional second-order system as a $2n$-dimensional first-order system and applying the machinery of optimal control to this problem instead. While this approach is certainly valid, it abandons aspects of the geometry of the system which may provide insight into the nature of the solution.

The approach we use in this paper is to consider the optimal control problem as a constrained optimization problem and use the technique of Lagrange multipliers to derive the equations of motion. Rather than recast the system as a first-order system, the equations of motion are included as a second-order constraint of the optimization. In optimizing, we consider variations of position and consider how they induce variations in the velocity, rather than considering indpendent variations of the velocity. Central to the development will be the notions from differential geometry of an affine connection and covariant differentiation. The resulting equations of motion of the adjoint variables are second-order differential equations defined using the covariant derivative. This approach has a potential computational advantage in that the number of adjoint variables has been halved. Additionally, recently developed integration schemes based on variational principles [12] can potentially be used in the calculation of optimal trajectories, given a proper understanding of the equations derive from a variational principle.

The variational approach is considered in [4, 11] in the context of finding force-minimizing arcs on Riemannian manifolds. In that case, however, the underlying system was fully actuated, meaning that the constraints could be directly substituted into the cost function, thus eliminating the need for adjoint variables. The results in this paper are more general, in that they apply to underactuated systems, systems with drift, and a broader class of cost functions. The equations we derive match those derived in [9], which considers the same problem from a different perspective. In that paper, the first-order equations given by the Pontryagin Maximum Principle are recast as second-order equations using splittings derived from the affine connection. This paper complements those results, in that it shows how to interpret the resulting equations from the perspective of variational calculus.

2 Theory of Affine Connections

In this section, we introduce terminology which we will use throughout the paper. For a more thorough introduction to the topic, see [5, 7].

2.1 Connections and associated Tensors

Let Q be a manifold, and let $\mathcal{X}(Q)$ be the set of all smooth vector fields on Q. Let $\nabla : \mathcal{X}(Q) \times \mathcal{X}(Q) \to \mathcal{X}(Q)$ be an affine connection defined on Q. The operation of ∇ on vector fields X, Y is denoted $\nabla_X Y$. (for the defining properties of an affine connection, see [5]). The connection in question need not be the Levi-Civita connection associated with a Riemannian metric. Indeed, no metric is used in the results which follow.

The connection allows us to introduce the notion of covariant differentiation, which is the differentiation of vectors (or arbitrary tensors, as we shall see) along a path $c(t)$ in Q. We let $\frac{DX}{dt} \in T_{c(t)}Q$ denote the covariant derivative of a vector field X defined along $c(t)$. If $X(t)$ can be thought of as an element of $\mathcal{X}(Q)$, i.e. $X(t) = Y(c(t))$, then the covariant derivative and the connection coincide: $\frac{DX}{dt} = \nabla_{c'(t)}Y$. For the covariant derivative of X to be well-defined, we need know only $X(c(t))$, $c'(t)$, and the rate of change of X along $c(t)$.

While the connection ∇ is not a tensor, it has two tensors associated with it which will used later in the paper. The first is the curvature form, which is a (1,3) tensor defined to be

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1}$$

and the second is the torsion form, a (1,2) tensor defined to be

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{2}$$

We can write these tensors in coordinates using the Christoffel symbols Γ_{ij}^k associated with ∇:

$$T_{ij}^s = \Gamma_{ij}^s - \Gamma_{ji}^s \tag{3}$$

$$R_{ijk}^s = \frac{\partial \Gamma_{jk}^s}{\partial q^i} - \frac{\partial \Gamma_{ik}^s}{\partial q^j} + \Gamma_{jk}^l \Gamma_{il}^s - \Gamma_{ik}^l \Gamma_{jl}^s \tag{4}$$

where the index notation is given by:

$$T(X,Y) = T_{ij}^s X^i Y^j \frac{\partial}{\partial q^s} \tag{5}$$

$$R(X,Y)Z = R_{ijk}^s X^i Y^j Z^k \frac{\partial}{\partial q^s} \tag{6}$$

Note that although the definitions in Equations (1),(2) use the affine connection, the tensors themselves are functions of the vectors at a given point, since the elements of right-hand side of Equations (1),(2) which involve the local variation of the operands are internally canceled.

We conclude this section with some additional terminology, borrowed from [9], which will be useful. Given an $(1,r)$ tensor A, we associate with it an $(2,r-1)$ tensor A^* which satisfies the identity

$$\langle A(X_1,\ldots,X_r),\alpha\rangle = \langle X_1, A^*(\alpha, X_2,\ldots,X_r)\rangle \tag{7}$$

where $\langle,\rangle$ denotes the natural pairing of vectors and covectors.

We will also associate with any $(0,r)$ tensor A an $(1,r-1)$ tensor $\hat{A}$ such that

$$A(X_1,\ldots,X_{r-1},Y) = \langle Y, \hat{A}(X_1,\ldots,X_{r-1})\rangle \tag{8}$$

Finally, will associate with any $(r,2)$ tensor A another $(r,2)$ tensor $\tilde{A}$ which is the same with the indices for the vector inputs reversed, that is:

$$A(X,Y) = \tilde{A}(Y,X) \tag{9}$$

2.2 Differentiation of Tensors

In this section, we present the formulas for covariant differentiation of an arbitrary covariant tensor. Let A be a tensor of order $(0,r)$. Then its covariant differential ∇A is a tensor of order $(0,r+1)$, defined by (do Carmo, p. 102)

$$(\nabla A)(X_1,\ldots,X_r,Z) = Z[A(X_1,\ldots,X_r)] - A(\nabla_Z X_1,\ldots,X_r) - \cdots - A(X_1,\ldots,\nabla_Z X_r) \tag{10}$$

and the covariant derivative of A in the direction of Z is an $(0,r)$ tensor $\nabla_Z A$ defined by

$$(\nabla_Z A)(X_1,\ldots,X_r) = (\nabla A)(X_1,\ldots,X_r,Z) \tag{11}$$

In particular, if we wish to differentiate a one-form α, we see that

$$\langle X, \nabla_Z \alpha\rangle = Z(\langle X,\alpha\rangle) - \langle \nabla_Z X, \alpha\rangle \tag{12}$$

If Z is the tangent vector to a path $c(t)$, we can rewrite this identity as

$$\frac{d}{dt}\langle X,\alpha\rangle = \langle \frac{DX}{dt},\alpha\rangle + \langle X, \frac{D\alpha}{dt}\rangle \tag{13}$$

We see that our definition for covariant differentiation of tensors leads to a product rule as one would expect. This identity is true for any connection, even if it is not a special one such as the Levi-Civita connection. This equation will be significant for two reasons. Firstly, we will use it to integrate by parts expressions involving covariant derivatives. Secondly, if we replace t with ϵ, we see that this expression will be useful in understanding how variational principles enter equations with covariant derivatives.

3 Calculus of Variations

In this section, we recall how variations are defined, and we derive properties of the variations which will be used later on. We consider a family of trajectories $q(t,\epsilon)$ defined on some interval $[t_0, t_f] \times [-\epsilon_0, \epsilon_0]$, and we denote $q(t,0)$ as $q(t)$. We can also freeze t and consider a path dependent on ϵ. We now define the variations of $q(t)$ in the standard way:

$$\delta q(t) = \left.\frac{\partial q}{\partial \epsilon}\right|_{\epsilon=0} \in T_{q(t)}Q \tag{14}$$

We also note that the velocity vector field is given by

$$V(t,\epsilon) = \frac{\partial q}{\partial t} \in T_{q(t,\epsilon)}Q \tag{15}$$

When $\epsilon = 0$, we denote the velocity as $V(t)$. We see that $\delta q(t)$ can be thought of as a vector field defined along a path $q(t)$, and if we hold t fixed, $V(t,\epsilon)$ can be thought of as a vector field defined along a path $q(t,\epsilon)$ parametrized by ϵ. With that in mind, we see that $\frac{DV}{d\epsilon}$ is well-defined at $\epsilon = 0$ and $\frac{D\delta q}{dt}$ is also well defined for all t. We now prove three propositions using the coordinate definition of covariant differentiation.

Proposition 1: $\frac{\partial V^k}{\partial \epsilon} = \frac{\partial \delta q^k}{\partial t}$

Proof: This follows immediately from the definitions and the equality of mixed partials.

Proposition 2: $T(\delta q(t), V(t)) = \frac{DV}{d\epsilon} - \frac{D\delta q}{dt}$.

Proof: The definition of covariant differentiation in coordinates is

$$\frac{DV}{d\epsilon} = \left(\frac{\partial V^k}{\partial \epsilon} + V^j \delta q^i \Gamma^k_{ij}\right) \frac{\partial}{\partial q^k} \tag{16}$$

and

$$\frac{D\delta q}{dt} = \left(\frac{\partial \delta q^k}{\partial t} + \delta q^j V^i \Gamma^k_{ij}\right) \frac{\partial}{\partial q^k} \tag{17}$$

Subtracting and applying Proposition 1 yields

$$\frac{DV}{d\epsilon} - \frac{D\delta q}{dt} = V^j \delta q^i \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right) \frac{\partial}{\partial q^k} \tag{18}$$

which is the coordinate definition of $T(\delta q, V)$ given in Equation (3).

Proposition 3: $R(\delta q(t), V(t))Z = \left(\frac{D}{d\epsilon}\frac{D}{dt} - \frac{D}{dt}\frac{D}{d\epsilon}\right) Z$.

Proof: The proof follows in exactly the same fashion as Proposition 2. If one expands the right-hand side in coordinates and cancels like terms via Proposition 1, one is left with the coordinate definition of $R(\delta q(t), V(t))Z$ as given in Equation (4).

We now discuss how covariant derivatives enter the picture. If we fix t, we can consider $V(t,\epsilon)$ as a vector field varying along a trajectory $q(t,\epsilon)$. Suppose we had a one-form $\alpha(t,\epsilon)$ similarly defined, and we consider the variation of

$$\int \langle V, \alpha \rangle dt \tag{19}$$

Using Equation (13), we see that

$$\delta \int \langle V, \alpha \rangle dt = \int \frac{d}{d\epsilon} \langle V, \alpha \rangle dt \tag{20}$$

$$= \int \langle \frac{DV}{d\epsilon}, \alpha \rangle + \langle V, \frac{D\alpha}{d\epsilon} \rangle dt \tag{21}$$

Indeed, the expression $\frac{DV}{d\epsilon}$ is precisely the variation of V as ϵ is varied. It is important to note that variations of V arise solely from the family of trajectories $x(t,\epsilon)$.

4 Applications to Optimal Control

4.1 Affine Connection Control Systems

We define an affine connection control system to be a manifold Q, a connection ∇ on Q, a (1,1) tensor $F : TQ \to TQ$, a vector field P, and a set of vector fields $Y_i \in TQ$, $i = 1 \ldots m$, for which $q \in Q$ evolves according to the second-order equation on Q:

$$\frac{DV}{dt} = Y_i(q)u^i + F(V) + P(q) \tag{22}$$

where $u^i \in \mathbb{R}$ are the controls. The vector fields Y_i map the controls to TQ. The vector field P represents the drift vector field (in a second-order sense). In a mechanical setting, we would write this as the gradient of a potential function, but we need not make this restriction here. The term $F(V)$ represents dissipation. The latter two terms are not generally included in discussions of affine connection control systems, but here they have been included, since they do not complicate the theory in this case.

4.2 Cost Functions

We now need to define a cost function to complete the optimal control problem. To do so, we borrow the framework of [9]. Let $i \in \{1, \ldots, s\}$, r_i be a nonnegative integer, and A_i be a symmetric $\mathbb{R}^m$-dependent $(0, r_i)$ tensor field on Q. Our integral cost function J is therefore

$$J(q,u,V) = \sum_{i=1}^{s} A_i(q,u)(V, \ldots, V) \tag{23}$$

where V is repeated r_i times as an operand for each A_i. Thus, when $r_i = 0$, the tensor represents a cost term associated with position and control effort alone. If $r_i = 2$, then the cost is quadratic in velocity and possibly dependent on q and u.

Also, let $\phi_i(q)$ be a symmetric $(0, b_i)$ tensor field on Q. We define a terminal cost function J_f as

$$J_f(q,V) = \sum_{i=1}^{w} \phi_i(q)(V, \ldots, V) \tag{24}$$

4.3 Problem Statement

We can now state two optimal control problems:

P1 (Fixed Final State) Given an initial condition $(q_0, V_0) \in TQ$ and a final condition $(q_f, V_f) \in TQ$, what are trajectories which minimize

$$\int_{t_0}^{t_f} J(q,u,V)dt \tag{25}$$

subject to the constraint of Equation (22).

P2 (Free Final State) Given an initial condition $(q_0, V_0) \in TQ$, what are trajectories which minimize

$$\int_{t_0}^{t_f} J(q,u,V)dt + J_f(q(t_f), V(t_f)) \tag{26}$$

subject to the constraint of Equation (22).

4.4 Problem Solution: Lagrange Multipliers

To solve this problem we use the technique of Lagrange multipliers and calculus of variations. For the standard, first-order case, this technique yields the Euler-Lagrange equations predicted by the Pontryagin Maximum Principle. We follow the same technique here, only cast as a second-order problem. Rather than artifically doubling the constraints to conform to the standard structure, we consider only the given constraint. The unconstrained cost function takes the form:

$$\int_{t_0}^{t_f} J(q,u,V) + \langle \lambda, -\frac{DV}{dt} + F(V) + P + Y_i u^i \rangle dt + J_f(q(t_f), V(t_f)) \tag{27}$$

We see that our constraint lives in TQ, meaning the Lagrange multiplier is a one-form field on Q. When we take the variation, we consider the variation of u and q. Unlike the first-order case, we do not consider the variation of V independently of that of q. Instead, we look at how variations of q affect the velocity vector field, and we do so using the covariant derivative, as discussed above.

Our governing equation is now

$$\delta \left(\int_{t_0}^{t_f} J(q,u,V) + \langle \lambda, -\frac{DV}{dt} + F(V) + P + Y_i u^i \rangle dt + J_f(q(t_f), V(t_f)) \right) = 0 \tag{28}$$

or

$$\int_{t_0}^{t_f} \frac{d}{d\epsilon} \left(J(q,u,V) + \langle \lambda, -\frac{DV}{dt} + F(V) + P + Y_i u^i \rangle \right) dt + \frac{d}{d\epsilon} J_f(q(t_f), V(t_f)) = 0 \tag{29}$$

We now consider the variation of each term in this expression with respect to q. In the following section, the term V^{r_i} indicates V repeated r_i times as an operand.

1. $\frac{d}{d\epsilon} J(q,u,V)$

   Suppose J consists of a single tensor A_i. We will calculate the variation of this term, and sum over the resulting expression for the case where J consists of multiple tensors. We assume $r_i \neq 0$, apply Equation (10) and exploit the symmetry of A_i to arrive at

$$\frac{dJ}{d\epsilon} = \nabla A_i (V^{r_i}, \delta q) + r_i A_i (\frac{DV}{d\epsilon}, V^{r_i - 1}) \tag{30}$$

which, using Proposition 2, becomes

$$\frac{dJ}{d\epsilon} = \nabla A_i(V^{r_i}, \delta q) + r_i A_i(\frac{D\delta q}{dt} + T(\delta q, V), V^{r_i - 1}) \tag{31}$$

Using the notation in Equation (8), we rewrite this as

$$\frac{dJ}{d\epsilon} = \langle \widehat{\nabla A}_i(V^{r_i}), \delta q\rangle + r_i \langle \frac{D\delta q}{dt} + T(\delta q, V), \widehat{A}_i(V^{r_i - 1})\rangle \tag{32}$$

We split the latter term into two and, using the terminology of Equation (7), we write this as

$$\frac{dJ}{d\epsilon} = \langle \widehat{\nabla A}_i(V^{r_i}), \delta q\rangle + r_i \langle \frac{D\delta q}{dt}, \widehat{A}_i(V^{r_i - 1})\rangle + r_i \langle T^*(\widehat{A}_i(V^{r_i-1}), V), \delta q\rangle \tag{33}$$

The second term of this equation is integrated by parts following Equation (13):

$$\begin{aligned} \tfrac{dJ}{d\epsilon} = \quad & \langle \widehat{\nabla A}_i(V^{r_i}), \delta q\rangle - r_i \langle \delta q, \nabla_V \widehat{A}_i(V^{r_i-1})\rangle + r_i\langle T^*(\widehat{A}_i(V^{r_i-1}), V), \delta q\rangle \\ & + \tfrac{d}{dt} r_i \langle \delta q, \widehat{A}_i(V^{r_i-1})\rangle \end{aligned} \tag{34}$$

The last term can be removed from the integrand, and will be considered later. Finally, we differentiate the second term on the right hand side and collect terms using the symmetry of A_i, s we did before, to arrive at

$$\begin{aligned} \tfrac{dJ}{d\epsilon} = \quad & \langle \widehat{\nabla A}_i(V^{r_i}), \delta q\rangle - r_i \langle \delta q, \left(\nabla_V \widehat{A}_i\right)(V^{r_i-1})\rangle \\ & + r_i \langle T^*(\widehat{A}_i(V^{r_i-1}), V), \delta q\rangle - r_i(r_i - 1)\langle \delta q, \widehat{A}_i(\tfrac{DV}{dt}, V^{r_i-2})\rangle \end{aligned} \tag{35}$$

If we collect terms, we arrive at

$$\begin{aligned} \tfrac{dJ}{d\epsilon} = \quad & \langle \left(\widehat{\nabla A}_i - r_i \nabla_V \widehat{A}_i\right)(V^{r_i}) \\ & - r_i(r_i-1)\widehat{A}_i(\tfrac{DV}{dt}, V^{r_i-2}) + r_i T^*(\widehat{A}_i(V^{r_i-1}), V), \delta q\rangle \end{aligned} \tag{36}$$

Of course, we can substitute in for $\frac{DV}{dt}$ using Equation (22). Note that if $r_i = 0$, this reduces to ∇A_i, which is simply the gradient of the function A_i, so the notation is consistent even for this case.

2. $\frac{d}{d\epsilon}\langle \lambda, \frac{DV}{dt}\rangle$

   As before, we write

$$\frac{d}{d\epsilon}\langle \lambda, \frac{DV}{dt}\rangle = \langle \lambda, \frac{D}{d\epsilon}\frac{DV}{dt}\rangle \tag{37}$$

   Applying Proposition 3, we rewrite this as

$$\frac{d}{d\epsilon}\langle \lambda, \frac{DV}{dt}\rangle = \langle \lambda, \frac{D}{dt}\frac{DV}{d\epsilon} + R(\delta q, V)V\rangle \tag{38}$$

   We integrate the first term by parts, and rewrite the resulting expression using Equation (7) and Proposition 2:

$$\frac{d}{d\epsilon}\langle \lambda, \frac{DV}{dt}\rangle = -\langle \frac{D\lambda}{dt}, \frac{D\delta q}{dt} + T(\delta q, V)\rangle + \langle \delta q, R^*(\lambda, V)V\rangle + \frac{d}{dt}\langle \lambda, \frac{DV}{d\epsilon}\rangle \tag{39}$$

We integrate by parts again and use Equation (7) to write this term as

$$\begin{aligned} \tfrac{d}{d\epsilon}\langle\lambda, \tfrac{DV}{dt}\rangle = \; & \langle \tfrac{D^2\lambda}{dt^2} - T^*(\tfrac{D\lambda}{dt}, V) + R^*(\lambda, V)V, \delta q\rangle \\ & + \tfrac{d}{dt}\left(\langle\lambda, \tfrac{DV}{d\epsilon}\rangle - \langle\tfrac{D\lambda}{dt}, \delta q\rangle\right) \end{aligned} \tag{40}$$

Again, the final term can be integrated directly, and will be addressed later. The remaining term in the integrand is

$$\frac{d}{d\epsilon}\langle\lambda, \frac{DV}{dt}\rangle = \langle\frac{D^2\lambda}{dt^2} - T^*(\frac{D\lambda}{dt}, V) + R^*(\lambda, V)V, \delta q\rangle \tag{41}$$

3. $\frac{d}{d\epsilon}\langle\lambda, F(V)\rangle$

   Following the steps used earlier, we have

$$\begin{aligned} \frac{d}{d\epsilon}\langle\lambda, F(V)\rangle &= \langle\lambda, \nabla F(V, \delta q) + F(\frac{DV}{d\epsilon})\rangle \\ &= \langle\lambda, \widetilde{\nabla F}(\delta q, V) + F\left(\frac{D\delta q}{dt} + T(\delta q, V)\right)\rangle \\ &= \langle\widetilde{\nabla F}^*(\lambda, V), \delta q\rangle + \langle F^*(\lambda), \frac{D\delta q}{dt} + T(\delta q, V)\rangle \\ &= \begin{array}{l} \langle\widetilde{\nabla F}^*(\lambda, V) + T^*(F^*(\lambda), V) - \nabla F^*(\lambda, V) - F^*(\frac{D\lambda}{dt}), \delta q\rangle \\ + \frac{d}{dt}\langle F^*(\lambda), \delta q\rangle \end{array} \end{aligned} \tag{42}$$

   As before, the final term is integrated directly, and the remaining term in the integrand is

$$\frac{d}{d\epsilon}\langle\lambda, F(V)\rangle = \langle\widetilde{\nabla F}^*(\lambda, V) + T^*(F^*(\lambda), V) - \nabla F^*(\lambda, V) - F^*(\frac{D\lambda}{dt}), \delta q\rangle \tag{43}$$

4. $\frac{d}{d\epsilon}\langle\lambda, P(q)\rangle$

   This term evaluates to

$$\begin{aligned} \frac{d}{d\epsilon}\langle\lambda, P(q)\rangle &= \langle\lambda, \nabla P(\delta q)\rangle \\ &= \langle\delta q, \nabla P^*(\lambda)\rangle \end{aligned} \tag{44}$$

5. $\frac{d}{d\epsilon}\langle\lambda, Y_i(q)u^i\rangle$

   This term evaluates similarly to the previous one to be

$$\frac{d}{d\epsilon}\langle\lambda, Y_i(q)u^i\rangle = \langle\delta q, \nabla Y_i^* u^i(\lambda)\rangle \tag{45}$$

We now have expressed the variation of each term in the form $\langle\cdot, \delta q\rangle$. If we collect the expressions in Equations (36),(41),(43),(44), and (45) and set the integrand to zero by setting $\delta q = 0$, we arrive at the following equations of motion for λ:

$$\begin{aligned} & \tfrac{D^2\lambda}{dt^2} - T^*(\tfrac{D\lambda}{dt}, V) + R^*(\lambda, V)V = \textstyle\sum_i \Big[\Big(\widehat{\nabla A}_i - r_i \nabla_V \widehat{A}_i\Big)(V^{r_i}) \\ & -r_i(r_i - 1)\widehat{A}_i(F(q, V) + P(q) + Y_i(q)u^i, V^{r_i-2}) + r_i T^*(\widehat{A}_i(V^{r_i-1}), V)\Big] \\ & + \widetilde{\nabla F}^*(\lambda, V) + T^*(F^*(\lambda), V) - \nabla F^*(\lambda, V) - F^*(\tfrac{D\lambda}{dt}) + \nabla P^*(\lambda) \\ & + \textstyle\sum_{i=1}^m \left(\nabla Y_i^* u^i(\lambda)\right) \end{aligned} \tag{46}$$

Note that the equations of motion are a second order differential equation on T^*Q, paralleling Equation (22).

4.5 Variations of u: stationarity condition

We now turn to the requirement that cost function be a critical point with respect to variations of u. Since $u \in \mathbb{R}^m$, this variation is far simpler to calculate. The variation with respect to u^i yields the equation

$$\frac{\partial J}{\partial u^i}\delta u^i + \langle \lambda, Y_i \delta u^i \rangle = 0 \tag{47}$$

Since δu^i is a scalar, we can pull it outside the pairing, collect terms, and arrive at

$$\frac{\partial J}{\partial u^i} + \langle \lambda, Y_i \rangle = 0 \tag{48}$$

We thus have m conditions which define u as a function of q, V, λ. If we assume that the cost function is smooth and convex with respect to u, and that there are no constraints on the system states or controls, then Equations (22),(46) and (48) together represent necessary conditions which the optimal trajectory must satisfy.

As desired, we arrive at a second order equation on T^*Q rather than a first-order equation on TT^*Q. While the notation is cumbersome, it is not particularly complicated. In many cases, the dissipation and/or torsion terms will be zero, which will simplify the expansion significantly.

4.6 Endpoint Conditions

In this section, we state the endpoint conditions needed to solve the differential equations which govern the optimal control system. The optimal control ODE is now two second-order differential equations, one on Q, the other on T^*Q. We therefore need four endpoint conditions to solve for the controller which generates the optimal trajectory. We consider the two optimal control problems separately:

P1: In this case, the initial conditions $q(t_0), V(t_0)$ and the final time conditions $q(t_f), V(t_f)$ are fixed. These conditions completely specify optimal control and trajectory. Because the endpoints are fixed, the variations of the terminal cost, as well as the endpoint terms generated by the integrations by parts, are zero.

P2: In this case, the final time conditions $q(t_f), V(t_f)$ are not supplied. This means that the variation of the trajectory at t_f is nonzero, and therefore we derive the final time condition by considering the variations of the terminal cost and the endpoint terms derived from the integrations by parts. Specifically, using Equation (10), we have

$$\delta J_f = \sum_{i=1}^{w} \langle \widehat{\nabla \phi_i}(V^{b_i}), \delta q \rangle + b_r \langle \widehat{\phi}_i(V^{b_i - 1}), \frac{DV}{D\epsilon} \rangle \tag{49}$$

with all terms evaluated at t_f. For the integral cost term we saw in Equations (13),(40),(42), that certain terms were integrated directly. Those terms are also zero at t_0, but at the final time, we have

$$\delta J|_{t=t_f} = \sum_{i=1}^{s} r_i \langle \widehat{A}_i(V^{r_1 - 1}), \delta q \rangle + \langle \lambda, \frac{DV}{d\epsilon} \rangle - \langle \frac{D\lambda}{dt}, \delta q \rangle + \langle F^*(\lambda), \delta q \rangle \tag{50}$$

Summing these and setting to zero, we have

$$\langle \sum_{i=1}^{w} \widehat{\nabla \phi_i}(V^{b_i}) + \sum_{i=1}^{s} r_i \widehat{A}_i(V^{r_1 - 1}) - \frac{D\lambda}{dt} + F^*(\lambda), \delta q \rangle + \langle \lambda + \sum_{i=1}^{w} b_r \widehat{\phi}_i(V^{b_i - 1}), \frac{DV}{d\epsilon} \rangle = 0 \tag{51}$$

At a given point in time, the term $\frac{DV}{d\epsilon}$ is independent of δq. We thus have two final time conditions:

$$\lambda(t_f) = -\sum_{i=1}^{w} b_r \widehat{\phi}_i(V^{b_i-1}) \tag{52}$$

$$\frac{D\lambda}{dt}(t_f) = \sum_{i=1}^{w} \widehat{\nabla\phi_i}(V^{b_i}) + \sum_{i=1}^{s} r_i \widehat{A}_i(V^{r_1-1}) + F^*(\lambda) \tag{53}$$

We thus recover two final-time conditions for λ, $\frac{D\lambda}{dt}$ which, when paired with $q(t_0), V(t_0)$, provide endpoint conditions to solve for the optimizing controller.

5 Example: Splines on Manifolds

One example to which this theory can be applied is the problem of calculating force-minimizing curves which link two points in the tangent space of some Riemannian manifold Q. This problem is considered in [4, 11], where it is noted that the problem can be phrased either as an optimal control problem. In that case, the system is fully actuated, so the problem can be stated as an unconstrained higher-order variational problem, and there is no need to use adjoint variables. Nonetheless, the theory developed here covers this case.

Let $\langle\langle\ ,\ \rangle\rangle$ be a Riemannian metric on Q, and let $g : TQ \to T^*Q$ also denote the map with the property $\langle\langle X, Y\rangle\rangle = \langle g(X), Y\rangle$. The problem statement is to minimize

$$J = \frac{1}{2}\int_{t_0}^{t_f} \langle\langle u, u\rangle\rangle \, dt \tag{54}$$

subject to

$$\frac{DV}{dt} = u \tag{55}$$

Substituting these into Equations (46),(48) yields the conditions:

$$\frac{D^2\lambda}{dt^2} + R^*(\lambda, V)V = 0 \tag{56}$$

(recall that the Riemannian metric is torsion-free and has the property $\nabla g = 0$), and

$$\lambda = g(u) \tag{57}$$

Subtituting in the equation for λ yields

$$\frac{D^2}{dt^2}\left(g\frac{DV}{dt}\right) + R^*(g\frac{DV}{dt}, V)V = 0 \tag{58}$$

Using the cyclic properties of the curvature tensor and the fact that g commutes with covariant differentiation, we can rewrite this equation as

$$g\left(\frac{D^3V}{dt^3}\right) + g\left(R(\frac{DV}{dt}, V)V\right) = 0 \tag{59}$$

which is equivalent to

$$\frac{D^3V}{dt^3} + R(\frac{DV}{dt}, V)V = 0 \tag{60}$$

which matches the results of [4, 11].

6 Acknowledgment

The authors wish to thank Prof. Andrew Lewis for his input in developing this paper.

References

[1] A. Bloch and P. Crouch. Nonholonomic control systems on Riemannian manifolds. *SIAM Journal of Control and Optimization*, 33(1):126–148, 1995.

[2] A. Bloch, P. Krishnaprasad, J. Marsden, and R. Murray. Nonholonomic mechanical systems with symmetry. *Arch. Rat. Mech. Anal.*, 136:21–99, 1996.

[3] F. Bullo, N. Leonard, and A. D. Lewis. Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups. *To appear, IEEE Transactions on Automatic Control, Nov. 97.*

[4] P. Crouch and F. Silva Leite. The dynamic interpolation problem: on Riemannian manifolds, Lie groups, and symmetric spaces. *Journal of Dynamical and Control Systems*, 1(2):177–202, 1995.

[5] M. P. do Carmo. *Riemannian Geometry.* Birkhauser, Boston, 1992.

[6] S. Kelly and R. Murray. Geometric phases and locomotion. *Journal of Robotic Systems*, 12(6):417–431, 1995.

[7] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Vol I.* Interscience Publishers, New York, 1963.

[8] A. D. Lewis. Affine connections and distributions with applications to nonholonomic mechanics. *Reports on Mathemaical Physics*, 42(1/2):135–164, 1998.

[9] A. D. Lewis. The geometry of optimal control for affine connection control systems. *Preprint*, 1999.

[10] A. D. Lewis and R. Murray. Controllability of simple mechanical control systems. *SIAM Review*, 41(3):555–574, 1999.

[11] L. Noakes, G. Heinzinger, and B. Paden. Cubic splines on curved spaces. *IMA J. Math. Control and Inf.*, 6(4):465–473, 1989.

[12] J. Wendlandt and J. Marsden. Mechanical integrators derived from a discrete variational principle. *Physica D*, 106:223–246, 1997.