CDC02-REG0610

To be submitted to the 2002 IEEE Conference on Decision and Control

Revised: March 8, 2002

Model predictive control of coordinated multi-vehicle formations*

William B. Dunbar†, Richard M. Murray

Control and Dynamical Systems 107-81
California Institute of Technology
Pasadena, CA 91125
†Corresponding Author: dunbar@cds.caltech.edu

Abstract

A generalized model predictive control (MPC) formulation is derived that extends the existing theory to a multi-vehicle formation stabilization problem. The vehicles are individually governed by nonlinear and constrained dynamics. The extension considers formation stabilization to a set of permissible equilibria, rather than a unique equilibrium. Simulations for three vehicle formations with input constrained dynamics on configuration space $SE(2)$ are performed using a nonlinear trajectory generation (NTG) software package developed at Caltech. Preliminary results and an outline of future work for scaling/decentralizing the MPC approach and applying it to an emerging experimental testbed are given.

1 Introduction

Interest in stabilizing and maneuvering a formation of multiple vehicles has grown in recent years. Application areas include grid searching by coordinating robots, surveillance using multiple unmanned air or ground vehicles, and synthetic aperture imaging with clusters of micro-satellites. Typically, individual vehicle dynamics are decoupled but become coupled by virtue of objectives that involve desired formations. The engineering complexity and challenge of these problems in terms of controls, communications and networking, exceeds that of the traditional controls problem formulated for a system treated as a single entity. To make headway, simplifications must be made, usually by considering simplified vehicle dynamics and/or assuming unlimited communications and networking capability.

The existing literature contains methods of pre-computing control laws to achieve coordinated objectives. Methods utilizing potential functions for coordinating formations include [7] and [12], where graph theoretic tools are also effectively used in the latter reference. In both cases, individual vehicle dynamics correspond to fully actuated second order point masses. Leonard and Fiorelli [7] permit a set of permissible locations for the vehicles in formation and their locations may be

*Partial support for this work was provided by AFOSR under grant F33615-98-C-3613.

interchanged. In contrast, Olfati-Saber and Murray [12] incorporate a formation graph definition that mandates precise relative locations for all vehicles in the graph, which in certain application domains is desirable. Both approaches are free from a strict leader-follower architecture, adding robustness of the group to individual vehicle failures. The same individual vehicle dynamics are considered by Young et al [15], where a leader-follower architecture is experimentally validated on wheeled robots. A group feedback is added from the followers to the leader in an attempt to compensate for the lack of robustness. Stability and controllability by distributed local feedbacks is examined by Yamaguchi et al [13] for formations of kinematic robots. A contribution of this paper is that the individual vehicles may be governed by nonlinear and constrained dynamics.

A generalized multi-vehicle formation stabilization problem, free from a leader-follower architecture, is defined in this paper. The problem is similar, in the formation definition and use of a virtual leader, to that given by Egerstedt and Hu [4] who consider velocity control of kinematic robots. The difference is that the desired formation here is not necessarily a unique state for each vehicle in the formation. Moreover, we provide the necessary definitions and appropriate proofs for stability to a set of equilibria. The results in this paper are new in that an optimization based approach, namely model predictive control (MPC), is applied to the formation stabilization problem that we define.

MPC is the most natural and in some cases the only methodology for control of systems that are governed by constrained dynamics. The current control action is determined by solving on-line, at each sampling instant, a finite horizon open-loop optimal control problem. Each optimization yields an optimal control and the first portion is applied until the next sampling instant. A recent thorough survey of nonlinear MPC stability theory is given by Mayne et al in [9]. The generalized formulation and conditions for stability stated in [9, 8] are used as a guide for the formulation here. A review of proofs of various MPC formulations and new applications is given by Dunbar [3].

There are several appealing aspects of MPC for the coordination of multiple vehicles to stabilize a formation. The main advantage is that optimization based methods in general permit reconfigurability [11]. Specifically, the cost functions and constraints in the optimal control problem can potentially be changed on-the-fly to accommodate new formations and limitations, such as inter-vehicle and obstacle collision avoidance. However, what we buy in reconfigurability, we potentially lose in reliability of the control law itself. Some of the risks of a real-time optimization based method such as MPC for vehicle stabilization are highlighted in [3]. Controller design for a multi-vehicle experimental testbed being developed at Caltech is also a key motivation for this paper [2].

The organization of the paper is as follows. Section 2 details the MPC formulations in a generalized multi-vehicle settings and Section 3 focuses on multi-vehicle simulation examples. The first example considers the constrained dynamics of the testbed vehicles described in [2], while considering simple formation requirements for different MPC parameterizations. The next example considers simplified dynamics but examines distributed, synchronized MPC computations with the effects of model error between vehicles. The software used in the simulations is the Nonlinear Trajectory Generation (NTG) software package developed at Caltech by Milam et al [10]. Conclusive remarks and extensions in the multi-vehicle framework for future research are given in Section 4.

2 MPC for vehicle formation stabilization

In this section, a multi-vehicle problem is posed and a general MPC formulation is stated as a solution. The formation problem definition is motivated by an objective of stability to a set of equilibria and by the requirement that all vehicles have equivalent roles relative to the formation, i.e.

there is no leader/follower architecture. MPC stability results given in [9] are then extended to this new objective. The extension is much like the discrete-time robust MPC (to bounded disturbances) formulated in [8], where stability is guaranteed to a control invariant set. For completeness, a lemma that generalizes Lyapunov functions in continuous time to achieve stability to a desired invariant set is given.

2.1 MPC Problem Statement

The model predictive formulation given in this section is intended to introduce the MPC methodology as it applies to a formation stabilization problem which will be defined. The formulation is generalized in the incorporation of constraints and/or costs to achieve nominal asymptotic stability. In the simulations we apply a formulation that is cost based and the conclusions outline the current constraint based formulations that are being investigated. The cost based formulations here have less guarantees, e.g. collision may occur, but are investigated for future comparisons in numerical feasibility and computation time with constraint approaches.

In this formulations and in the simulations, the common state space for each vehicle $\mathbb{X}$ is either $T(SE(2))$ or $\mathbb{R}^4$. More generally, $\mathbb{X}$ may be a constraint space that is a convex and closed subset of $\mathbb{R}^n$. Consider the k vehicles with system models

$$\dot{x}_i = f_i(x_i, u_i), \quad x_i(t_0) = x_{i0}, \quad t \in [t_0, \infty), \quad i = 1, ..., k,$$

and f_i is a vector field on $\mathbb{X}$ for all $i = 1, ..., k$. For notational ease later, introduce the vector notation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x} \in \mathbb{X}^k, \quad \boldsymbol{u} \in \mathbb{U}^k, \quad \text{where} \tag{1}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}, \ \boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix}, \quad \begin{array}{l} \mathbb{X}^k \triangleq \mathbb{X} \times \cdots \times \mathbb{X}, \\ \\ \mathbb{U}^k \triangleq \mathbb{U} \times \cdots \times \mathbb{U}. \end{array}$$

The vehicles are dynamically decoupled, which permits the statement that $\boldsymbol{x}$ lives in the Cartesian product space $\mathbb{X}^k$.

Remark 1. Collision avoidance can be accounted for by incorporating constraints, resulting in a state space that is a restriction of $\mathbb{X}^k$. Stabilizing MPC formulations that accommodate this is a subject of future work.

Given an initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ and a control trajectory $\boldsymbol{u}(\cdot) \in \mathbb{U}^k$, the state trajectory $\boldsymbol{x}^u(\cdot; \boldsymbol{x}_0)$ is the curve satisfying

$$\boldsymbol{x}^u(t; \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{x}^u(\tau; \boldsymbol{x}_0), \boldsymbol{u}(\tau)) \, \mathrm{d}\tau, \quad \forall t \geq t_0, \tag{2}$$

when constraints on the state are not active. At any current time t and for current state $\boldsymbol{x}(t) = \boldsymbol{x}$, the general optimal control problem is

$$\inf_{\boldsymbol{u}(\cdot)} J(\boldsymbol{x}, \boldsymbol{u}(\cdot), T), \quad \text{where} \tag{3}$$

$$J(\boldsymbol{x}, \boldsymbol{u}(\cdot), T) = \int_{t}^{t+T} q(\boldsymbol{x}^u(\tau; \boldsymbol{x}), \boldsymbol{u}(\tau)) \, \mathrm{d}\tau + V(\boldsymbol{x}^u(t+T; \boldsymbol{x})) \tag{4}$$

subject to equation (1) and terminal constraint

$$\boldsymbol{x}^u(t+T;\boldsymbol{x}) \in X_f,$$

where X_f is assumed to be compact. In equation (4) we can set $t=0$, i.e. we can regard the current initial time as zero since $f(\cdot)$, $V(\cdot)$ and $q(\cdot)$ are time-invariant. The optimal control problem is denoted $\mathbb{P}_T^{mv}(\boldsymbol{x})$. The optimal cost function and control and state trajectories (setting current time t to 0) are denoted $J^*(\boldsymbol{x},T)$, $\boldsymbol{u}^*(\tau;\boldsymbol{x},T)$ and $\boldsymbol{x}^*(\tau;\boldsymbol{x},T)$, where $\tau \in [0,T]$.

Definition 1. The Model Predictive Control or *MPC Problem* is

1. solve $\mathbb{P}_T^{mv}(\cdot)$ from state $\boldsymbol{x}$ at current time t,
2. implement the optimal input trajectory $\boldsymbol{\kappa}(\tau;\boldsymbol{x},T) \triangleq \boldsymbol{u}^*(\tau;\boldsymbol{x},T)$ for $\tau \in [0,\delta]$, where $0 \le \delta < T$,
3. repeat step 1. from state $\boldsymbol{x} \leftarrow \boldsymbol{x}^*(\delta;\boldsymbol{x},T)$ at current time $t \leftarrow t+\delta$ until $\boldsymbol{x} \in X_f$.

Henceforth we assume the following

- A1 The minimum of the value function $J^*(\cdot,T)$, $T \ge 0$, is attained.
- A2 Perfect knowledge of each vehicles dynamics governed by equation (1), and initial condition when necessary, is available to all other vehicles.
- A3 Computation times are negligible.

Assumption A1 does not imply uniqueness of the optimal solution. Assumption A2 is typical and A3 almost universal in the MPC literature. By ignoring uncertainty (absence of disturbances included) we can proceed by incorporating all vehicles in one (centralized) optimization over each horizon. Having more than one copy of such an optimization, say one per vehicle, would be redundant since they would all produce the same result. Assumption A3 permits $\delta = 0$, in which case $\mathbb{P}_T^{mv}(\cdot)$ is continuously resolved. The MPC controller in this case is denoted $\boldsymbol{\kappa}(\boldsymbol{x},T)$ and the closed-loop system becomes

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\kappa}(\boldsymbol{x},T)), \ \forall \boldsymbol{x} \in X_T - X_f \\ \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\kappa_f}(\boldsymbol{x})), \ \forall \boldsymbol{x} \in X_f \end{cases} \tag{5}$$

The theoretical results that follow apply to equation (5). Extension of the results for practical MPC controllers ($\delta > 0$), as applied the simulations, could be performed by following the procedure in [1]. An important paper that requires none of these assumptions and will be used as a basis for future work on constraint based approaches is [14]. In the next sections we define a multi-vehicle formation and give generalized conditions on (q,V,X_f) for proving stability.

2.2 Multi-vehicle formation objective

The control objective is to steer the set of states $\mathbb{R} \ni t \mapsto \{x_1,\ldots,x_k\} \in \mathbb{X}^k$ to an *equilibrium formation*, which will be defined. The general formation set includes non unique permissible states at any given t, rather than precise locations for each vehicle at any t. A formation where a precise but variable location for each vehicle is also defined as a subset. Naturally, the equilibrium formation satisfies equilibrium conditions for all of the vehicles in the formation. Although modelled separately, the vehicle (closed-loop) dynamics become coupled by virtue of the formation objective.

Suppose that the formation is specified by the integrated cost in the MPC problem posed for each vehicle; each cost would then incorporate some or all of the states and inputs of the other vehicles. For any such vehicle, we refer to the other vehicles that are referenced in it's formation specification as *neighbors*. In this sense the vehicles are *cooperative*, i.e. the closed-loop dynamics of every vehicle is directly affected by it's neighbors in the formation.

Partitioning the state vector in terms of position and velocity subvectors will be useful for notational reasons. When $\mathbb{X} = T(SE(2))$ or $\mathbb{X} \subseteq \mathbb{R}^4$, denote, respectively

$$x_i = \begin{bmatrix} z_i \\ \theta_i \\ \dot{z}_i \\ \dot{\theta}_i \end{bmatrix} \quad \text{or} \quad x_i = \begin{bmatrix} z_i \\ \dot{z}_i \end{bmatrix}, \quad \forall i = 1, ..., k,$$

where z_i and $\dot{z}_i$ live in $\mathbb{R}^2$ and $(\theta_i, \dot{\theta}_i) \in TS^1$.

A precise definition of a formation of vehicles is now given. A $\mathbb{U}^k$-controlled invariant set M of equation (1) defines a subset of $\mathbb{X}^k$ for which

$$\forall \boldsymbol{x}(t_0) \in M, \text{ there exits } \boldsymbol{u}(t) \in \mathbb{U}^k, \forall t \in \mathbb{R} \text{ such that } \boldsymbol{x}(t) \in M, \forall t \in \mathbb{R}.$$

Definition 2. Given a $\mathbb{U}^k$-controlled invariant set $M \subset \mathbb{X}^k$ of equation (1) and a formation reference $\mathcal{X}_r(t) \in \mathbb{X}$, $\forall t \geq t_0$, a k-vehicle formation associated with equation (1) is denoted $\mathcal{F}(M, k, \mathcal{X}_r(t))$ and defined as

$$\mathcal{F}(M, k, \mathcal{X}_r(t)) = \left\{ \boldsymbol{x} \in \mathbb{X}^k \mid (\boldsymbol{x}(t) - \boldsymbol{\mathcal{X}_r}(t)) \in M, \ \forall t \geq t_0 \right\}, \tag{6}$$

where $\boldsymbol{\mathcal{X}_r}$ is a column vector with k copies of $\mathcal{X}_r$ for each component.

The formation reference can be considered a virtual leader that is free from the possibility of fault [4, 3]. In sub-vector notation, $\mathcal{X}_r$ is denoted $(\mathcal{Z}_r, \theta_r, \dot{\mathcal{Z}}_r, \dot{\theta}_r)$ or $(\mathcal{Z}_r, \dot{\mathcal{Z}}_r)$ depending on $\mathbb{X}$. Example $\mathbb{U}^k$-controlled invariant sets that will be used in simulation studies are given below.

Example 1. For $\mathbb{X} = T(SE(2))$ or $\mathbb{X} \subseteq \mathbb{R}^4$,

$$M_k^1 = \Big\{ \boldsymbol{x} \in \mathbb{X}^k \ \Big| \ ||z_i|| = \rho_i, \ ||z_i - z_p|| = d_{ip}, \ \dot{z}_i = 0, \\ z_i \neq z_j, \ \forall i, j = 1, 2, ..., k, i \neq j, \text{ and for some } p \in \{1, 2, ..., k\} \Big\}.$$

Example 2. For $\mathbb{X} = T(SE(2))$,

$$M_k^2 = \Big\{ \boldsymbol{x} \in M_k^1 \ \Big| \ \dot{\theta}_i = 0, \ \theta_i = \theta', \ \forall i = 1, 2, ..., k \Big\}.$$

where the value of θ' is given.

Example 3.

$$M_k^3 = \{\hat{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\xi})\}, \text{ for some } \hat{\boldsymbol{x}} \in M_k^2,$$

where $(\boldsymbol{\alpha}, \boldsymbol{\xi})$ are scheduling parameters based on the locations of the vehicles at the end of each optimization horizon.

The norms in this example are the standard norm on $\mathbb{R}^2$ and $\rho_i, d_{ij} \in \mathbb{R}^+$ (positive, real) for all $i, j = 1, 2, ..., k$, $i \neq j$. The choices for d_{ip} need to be compatible with the radius of the circle(s). The neighbors of a vehicle with state x_i are the vehicles with state x_j for which d_{ij} is given. The number of neighbors affects the number of possible permutations of vehicle configurations also. Even if a given set M is a $\mathbb{U}^k$-controlled invariant set, the feasibility of $\mathcal{F}$ for the k vehicles depends greatly on $\mathcal{X}_r(t)$. Example 1 and Example 2 illustrate the non-uniqueness of permissible states in a desired formation. Example 3 defines a formation with precise locations for each vehicle, but these locations change with each optimization, as will be shown. An appropriate set that includes the restriction of $\mathcal{F}(M, k, \mathcal{X}_r(t))$ to equilibrium conditions is now given.

Definition 3. An *equilibrium formation* S_{eq} associated with $\mathcal{F}(M, k, \mathcal{X}_r(t))$ is the subset of $\mathbb{X}^k \times \mathbb{U}^k$ defined as

$$\begin{aligned} S_{eq} &= S_{eq}^X \times S_{eq}^U \\ &= \Big\{ (\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{X}^k \times \mathbb{U}^k \mid \dot{\boldsymbol{x}} = \boldsymbol{0},\ \boldsymbol{x} \in \mathcal{F}(M, k, \mathcal{X}_r(t)), \\ &\qquad\qquad \text{and } \dot{\mathcal{Z}}_r(t) = \dot{\theta}_r(t) = 0,\ \forall t \geq t_0 \Big\} \end{aligned} \tag{7}$$

By definition, S_{eq}^X is a S_{eq}^U-controlled invariant set with respect to equation (1). Either S_{eq}^X is M or, if the constants $(\mathcal{Z}_r, \theta_r)$ are nonzero, it is a translated and rotated version of M with respect to an inertial frame in $\mathbb{X}^k$.

2.3 Stability

MPC of constrained systems is nonlinear, warranting the use of Lyapunov stability theory. The value function is almost universally employed as a Lyapunov function for stability analysis for nonlinear (constrained or not) and constrained linear systems. The generalized conditions in [9] regarding the terminal cost $V(\cdot)$, terminal constraint set X_f and local controller $\kappa_f(\cdot)$ are here used as a guide. We make the following assumptions

- A4 $\boldsymbol{f}$ is C^2 and $f_i(x_i, u_i)$ linearized around any (x_i, u_i) in the equilibrium set is controllable for any $i = 1, ..., k$.
- A5 For all t of interest, $\boldsymbol{u}(t) \in \mathbb{U}^k$, a convex compact subset of $\mathbb{R}^{km}$ containing S_{eq}^U and $S_{eq}^U \equiv \{\boldsymbol{0}\}$. If $\dot{\boldsymbol{x}} = \boldsymbol{0}$ requires a constant $\boldsymbol{u} \neq \boldsymbol{0}$, asume we can translate the equilibrium input to the origin.
- A6 For all t of interest, $\boldsymbol{x}(t) \in \mathbb{X}^k$, which must contain S_{eq}^X.
- A7 $q(\cdot)$ is C^2 and $\boldsymbol{u} \mapsto q(\boldsymbol{x}, \boldsymbol{u})$ is convex for each $\boldsymbol{x} \in \mathbb{X}^k$.
- A8 $q(\cdot)$ is positive definite in $\boldsymbol{u}$ and semi-definite in $\boldsymbol{x}$, satisfying $q(S_{eq}) = 0$.

From A5 and Definition 3, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0}$ for each $\boldsymbol{x} \in S_{eq}^X$. Inside X_f we assume there exist local stabilizing controllers $\boldsymbol{\kappa_f}^T = [\kappa_f^1, ..., \kappa_f^k]$. Design of such controllers raise some interesting issues, as stabilization is desired to a set and not a particular point. To generalize these terminal controllers we define their domain and range spaces as $\kappa_f^i : X_f \to \mathbb{U}$, $i = 1, ..., k$. Since controllability is assumed, linear control techniques may be used for local X_f.

Remark 2. The incorporation of $\mathcal{F}(M, k, \mathcal{X}_r)$ in $\mathbb{P}_T^{mv}$ may be done by enforcing constraints over the entire horizon time, by design of the integrated cost or a combination of both. From an implementation perspective, accuracy of the formation improves by incorporating it in the form of constraints but this is generally at the price of reduced computational feasibility. In the examples in Section 3, only the case of designing the integrated cost q to accommodate the desired formation is considered.

The general conditions here require a Lyapunov function with stabilizing properties toward a set. An appropriate lemma, combining Lyapunov's stability theorem and LaSalle's theorem, is now given for the system in equation (1) with closed loop state-feedback control

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x})). \tag{8}$$

Lemma 1. *Let M and Ω be a positively invariant sets for equation (8) with $M \subset \Omega \subset \mathbb{X}^k$ and Ω is compact. Let $V : \Omega \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(\boldsymbol{x}) = 0 \text{ in } M \text{ and } V(\boldsymbol{x}) > 0 \; \forall x \in \Omega - M \tag{9}$$

$$\dot{V}(\boldsymbol{x}) = 0 \text{ in } M \text{ and } \dot{V}(\boldsymbol{x}) < 0 \; \forall x \in \Omega - M \tag{10}$$

Then, M is an asymptotically stable invariant set.

The construction of the set Ω does not have to be associated with the construction of V. However, an appropriate choice for Ω is the largest bounded level of V contained in $\mathbb{X}^k$, containing M and satisfying equation (10). We now state a theorem based upon conditions B1-B4 in [9] that incorporates a generalized MPC implementation for stabilization of multiple vehicles to an equilibrium formation.

Theorem 1. *Let X_T denote the set of states $\boldsymbol{x}$ that can be steered to X_f by an admissible control in time T. Assume that $J^*(\cdot)$ is C^1 and that X_T is compact. If $(V(\cdot), X_f, \kappa_f)$ satisfy*

1. *$X_f \subset \mathbb{X}^k$, X_f compact, $S_{eq}^X \subseteq X_f$.*
2. *$\kappa_f^i(X_f) \subset \mathbb{U}$, $\forall i = 1, ..., k$.*
3. *X_f is positively invariant for $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\kappa_f}(\boldsymbol{x}))$.*
4. *$V : X_f \to \mathbb{R}$ is C^1, satisfies equation (9) with $M = S_{eq}^X$ and $\left[\dot{V} + q\right](\boldsymbol{x}, \boldsymbol{\kappa_f}(\boldsymbol{x})) < 0$, $\forall \boldsymbol{x} \in X_f - S_{eq}^X$,*

then S_{eq}^X is an asymptotically stable invariant set of equation (5) with region of attraction X_T.

The proof of this theorem is given in Appendix A. The generalized conditions 1-4 in Theorem 1 contain all of the following variants of MPC (see [3] for other variants):

- V1 *Terminal equality constraint.* Set $X_f = S_{eq}^X$, $V(x) = 0$ and $\boldsymbol{\kappa_f} \in S_{eq}^U$ and the conditions are trivially satisfied. Note that computational feasibility is likely to improve over the (unique) terminal state case that this type of constraint usually incorporates.

- V2 *Terminal cost.* In this case $X_f = \mathbb{R}^{kn}$, which naturally enlarges the domain of the terminal cost V. For nonlinear unconstrained problems, the work of Jadbabaie [5] can be applied here, where V is a modified CLF in that it satisfies condition 4. In this formulation, the horizon is chosen long enough such that a level set of V (which can be thought of as X_f although it is not enforced) is reached at the horizon time. This variant is examined in the simulation examples.

V3 *No terminal cost or constraints.* For unconstrained nonlinear systems, it has been shown [5] that there always exists a horizon time (long enough) such that stability to the origin can be attained. Some simulation examples also employ this variant.

Remark 3. Constraints may invalidate the assumption that J^* is C^1; there are also proofs that do not require this assumption [1]. Also, if J^* is radially unbounded and X_T is taken as a large level set of J^* (subset of X_T defined in theorem), then X_T is compact and we can delete this assumption.

3 MPC Coordinated Multi-Vehicle Simulations

This section details simulation examples of multi-vehicle coordinated control problems solved using model predictive control. The first example considers vehicle dynamics on configuration space $SE(2)$ with constrained inputs. A simple formation reference and the effects of adding a local terminal cost to an integrated cost for stabilization of the formation are investigated. The terminal cost satisfies the conditions of Theorem 1. The second example considers vehicles with linear, 2-D second order dynamics with only an integrated cost and no constraints. This example investigates what happens when assumption A2 is no longer true, i.e. when the vehicle models are no longer perfect and the MPC computations are distributed on each vehicle. The simulations are done using the NTG software package developed at Caltech [10].

3.1 Desired Formation and Reference

From the examples, the following invariant sets will be referenced $k = 3$ vehicles

$$\begin{aligned} M_3^1 &= \left\{ \boldsymbol{x} \in \mathbb{X}^3 \;\middle|\; ||z_i|| = 1,\ ||z_i - z_p|| = \sqrt{3},\ \dot{z}_i = 0,\ \forall i,j = 1,2,3 \right\}. \\ M_3^2 &= \left\{ \boldsymbol{x} \in M_3^1 \;\middle|\; \dot{\theta}_i = 0,\ \theta_i = 0,\ \forall i = 1,2,3 \right\}. \\ M_3^3 &= \{ \hat{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\xi}) \},\ \text{for some } \hat{\boldsymbol{x}} \in M_3^2, \end{aligned}$$

where $(\boldsymbol{\alpha}, \boldsymbol{\xi})$ will be defined in Section 3.2. The state space $\mathbb{X}$ for each vehicle in the simulations is $T(SE(2))$ for Example 1 and $\mathbb{R}^4$ for Example 2.

3.2 Example 1

The dynamics of the individual vehicles are taken from the multi-vehicle wireless testbed [2] (schematic and pictures given). Denoting the configuration $(x, y, \theta) \in SE(2)$ and assuming viscous friction, the equations of motion of a vehicle are:

$$\begin{aligned} m\ddot{x} &= -\eta\dot{x} + (F_s + F_p)\cos\theta \\ m\ddot{y} &= -\eta\dot{y} + (F_s + F_p)\sin\theta \\ J\ddot{\theta} &= -\psi\dot{\theta} + (F_s - F_p)r. \end{aligned} \tag{11}$$

The starboard and port fan forces are denoted F_s and F_p, respectively, and r denotes the (common) moment arm of the forces. To match the previous notation, $z_i^T = [x_i, y_i]$. An equilibrium point for the dynamics in equation (11) is any constant position and orientation (x_c, y_c, θ_c) with zero velocity. However, the linearized dynamics are not controllable around any equilibrium (uncontrollable subspace has rank 2). To achieve controllability, we can look at the error dynamics around tracking a constant velocity $\dot{x}_{nom}$ and heading θ_{nom} reference

$$\mathcal{X}_r(t) = [x_r(t_0) + t\dot{x}_{nom},\ y_r(t_0) + t\dot{y}_{nom},\ \theta_{nom},\ \dot{x}_{nom},\ \dot{y}_{nom},\ 0],\ \forall t \geq t_0 \tag{12}$$

where $\dot{y}_{nom} = \dot{x}_{nom}\tan(\theta_{nom})$. The error state and inputs are $(x_{ei}, F_{sei}, F_{pei}) = (x_i - \mathcal{X}_r, F_{si} - F_{nom}, F_{pi} - F_{nom})$, $i = 1, 2, 3$, and the error dynamics are

$$\begin{aligned} m\ddot{x}_{ei} &= -\eta(\dot{x}_{ei} + \dot{x}_{nom}) + (F_{sei} + F_{pei})\cos(\theta_{ei} + \theta_{nom}) \\ m\ddot{y}_{ei} &= -\eta(\dot{y}_{ei} + \dot{y}_{nom}) + (F_{sei} + F_{pei})\sin(\theta_{ei} + \theta_{nom}) \\ J\ddot{\theta}_{ei} &= -\psi\dot{\theta}_{ei} + (F_{sei} - F_{pei})r, \end{aligned} \tag{13}$$

and $F_{nom} = (\eta\dot{x}_{nom})/(2\cos\theta_{nom})$. No state constraints are enforced, so $\mathbb{X} = T(SE(2))$. The inputs (F_s, F_p) live in the constraint space $\mathbb{U} = [0, 6] \times [0, 6] \subset \mathbb{R}^2$. The reachable space of the inputs is used to determine that the controllable equilibrium of equation (13) is now any constant position (x_c, y_c) with θ and velocity equal to zero. For the desired formation with three vehicles, the integrated cost function is

$$\begin{aligned} q(\boldsymbol{x}, \boldsymbol{u}) = &\sum_{i=1}^{3} \left\{ W_i \left[\sqrt{x_{ei}^2 + y_{ei}^2} - 1 \right]^2 + V_i \left[\dot{x}_{ei}^2 + \dot{y}_{ei}^2 \right] + U_i \left[F_{sei}^2 + F_{pei}^2 \right] \right\} \\ &+ \frac{1}{2} \sum_{i,j=1, i\neq j}^{3} W_{ij} \left[\sqrt{(x_{ei} - x_{ej})^2 + (y_{ei} - y_{ej})^2} - \sqrt{3} \right]^2, \end{aligned} \tag{14}$$

In the simulations, $W_i = V_i = 1.0$, $U_i = 0.05$, and $W_{ij} = W_{ji}$ is 1.0 or 2.0 for all i, j. With respect to assumption A7, q is not C^1 or C^2 at the origin. Replacing the distance error-squared penalties above with distance squared error squared penalties (4th-order), e.g. $(x^2 - \rho^2)^2$, restores continuity as is done in [4]. Instead, equation (14) was implemented with a small constant (0.0001) under every radical to satisfy assumption A7. It was noted that since the formation is not near the origin, problems were seldom encountered without the small constant term and that performance was superior to the cost with 4th order distance penalty. The cost q is positive in $\mathbb{X}^3 - M_3^1$ and zero on M_3^1. Any element of S_{eq} must be in the form $((x_c, y_c, 0, 0, 0, 0), (0, 0))$ so an appropriate terminal cost function must be designed for stability.

3.2.1 A Formation Terminal Cost Function

A terminal cost that satisfies Theorem 1 with $(\Omega, M) = (X_f, M_3^3)$ is now given. For an LQR problem associated with the linearization of equation (13) around any equilibrium, denote the corresponding positive-definite Riccati matrix as P. The terminal cost designed for this formation is a scheduled, quadratic penalty on an error state $\boldsymbol{e}_i$, for each vehicle $i = 1, 2, 3$, with P as a weighting matrix. Specifically

$$V(\boldsymbol{x}) = \gamma \left(\boldsymbol{e}_1^T P \boldsymbol{e}_1 + \boldsymbol{e}_1^T P \boldsymbol{e}_1 + \boldsymbol{e}_3^T P \boldsymbol{e}_3 \right), \tag{15}$$

where γ is a positive, scalar weighting and the error state for vehicle $i = 1, 2, 3$ is

$$\boldsymbol{e}_i = \left[\begin{array}{cccccc} x_{ei} - g_{i1}(x_i, y_i), & y_{ei} - g_{i2}(x_i, y_i), & \theta_{ei}, & \dot{x}_{ei}, & \dot{y}_{ei}, & \dot{\theta}_{ei}, \end{array} \right].$$

The functions g_{i1}, g_{i2} are defined as

$$\begin{aligned} &g_{i1}(x_i, y_i) = \cos\left(\bar{\xi} + \alpha_i\right), \quad g_{i2}(x_i, y_i) = \sin\left(\bar{\xi} + \alpha_i\right), \quad \text{where} \\ &\bar{\xi} = (\xi_1 + \xi_2 + \xi_3)/3, \quad \xi_i = \arctan\left((y_i - y_r)/(x_i - x_r)\right) = \arctan\left(y_{ei}/x_{ei}\right), \end{aligned}$$

for each $i = 1, 2, 3$ and the scheduling variable α_i is defined as

$$\alpha_i = \begin{cases} 0, & \text{when } |\xi_i - \bar{\xi}| < |\xi_j - \bar{\xi}| \ \forall j \neq i \\ 2\pi/3, & \text{when } |\xi_i - (\bar{\xi} + 2\pi/3)| < |\xi_j - (\bar{\xi} + 2\pi/3)| \ \forall j \neq i \\ -2\pi/3, & \text{when } |\xi_i - (\bar{\xi} - 2\pi/3)| < |\xi_j - (\bar{\xi} - 2\pi/3)| \ \forall j \neq i. \end{cases} \quad (16)$$

If $\xi_i = \xi_j$ for some $i \neq j$, then $\alpha_k, k \notin \{i, j\}$ is identified according to equation (16) and α_i and α_j are arbitrarily chosen to be (distinctly) what is left of $\{0, 2\pi/3, -2\pi/3\}$. If $\xi_1 = \xi_2 = \xi_3$, then each α_i is arbitrarily chosen to be one of $\{0, 2\pi/3, -2\pi/3\}$, not equal to any other α_j.

The idea behind this terminal cost is as follows: at the end of each optimization horizon, the vehicles are in some location relative to each other and the formation set. Calculating the angle ξ_i for each vehicle gives its angular location in the relative formation set frame. Taking the average of these locations ($\bar{\xi}$), one desired vehicle state is on the circle with angular location $\bar{\xi}$ and all other state variables matching the reference. This is the desired state for the vehicle with angular location closest to $\bar{\xi}$. The desired states for the other vehicles, equilaterally spaced on the set, are chosen also according to equation (16). The terminal cost penalizes the weighted 2-norm of the error between each vehicle's state and it's desired state. No terminal constraint is enforced in this example and the results with and without a terminal cost are reported. Given the LQR-based design of the terminal cost above, there exists a corresponding local set X_f in which κ_f can be taken to be the LQR controller. Instead of estimating X_f and enforcing it as a terminal constraint set, the effects of the terminal cost alone are investigated. The cost V is C^1 as long as $(x_{ei}, y_{ei}) \neq (0, 0)$ for all $i = 1, 2, 3$, which can be enforced by putting a constraint such that no vehicle can come within a small distance of the reference.

3.2.2 Simulation Cases and Results

In the simulations, the horizon and update times are 5.0 and 1.0. In addition to matching the appropriate initial condition for the state at each update, the 3 accelerations for each vehicle are also initially constrained to ensure continuity of the input forces. Most initial conditions examined with only an integrated cost were observed to be stabilizing without collisions of the vehicles. Figure 1 shows an initial condition that resulted in stability but two vehicles passed through each other (an unacceptable scenario resulting in collision for real vehicles). The (black) vehicle in the center

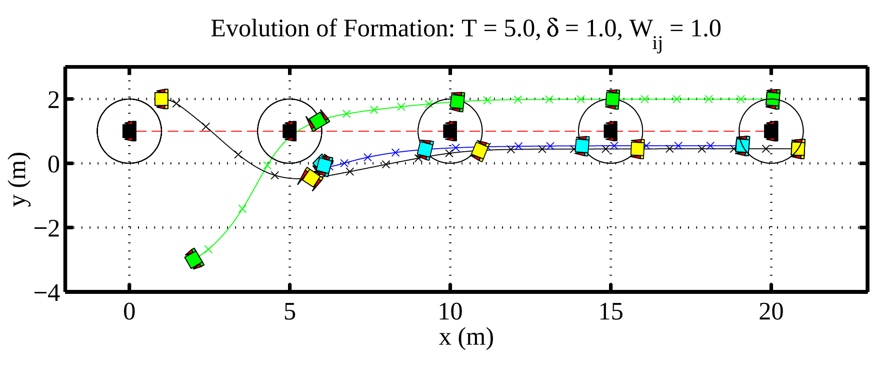

Figure 1: MPC formation with integrated cost only: collision occurs

of the circle represents the formation reference and the x's along the trajectories of the formation vehicles represent the updates in the MPC controller. In an attempt to avoid collision, the relative distance cost weighting W_{ij} was increased from 1.0 to 2.0 and the result is shown in Figure 2. Collision is avoided but at the sacrifice of performance. The formation eventually stabilizes but

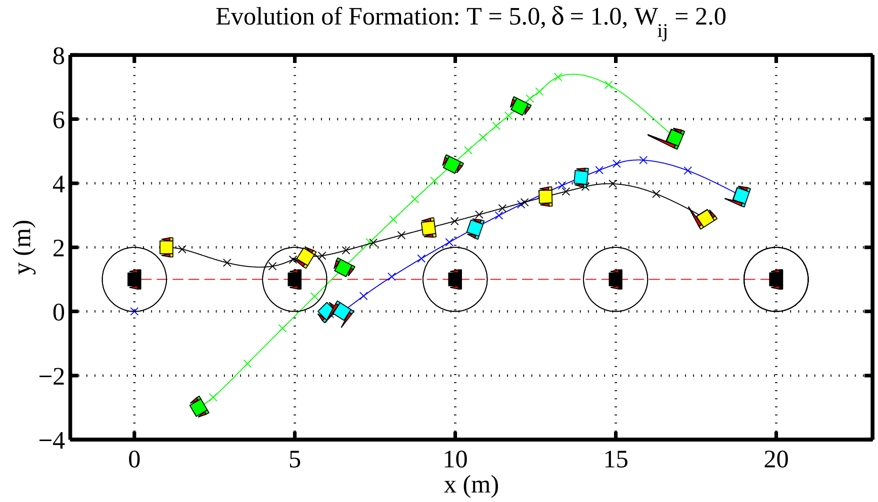

Figure 2: Larger relative distance weighting: collision avoided but performance decreases

not within the 25 second window shown in the figure. The affect of adding the terminal cost in equation (15), keeping the relative weight at 1.0, is shown in Figure 3. Other simulations cases for

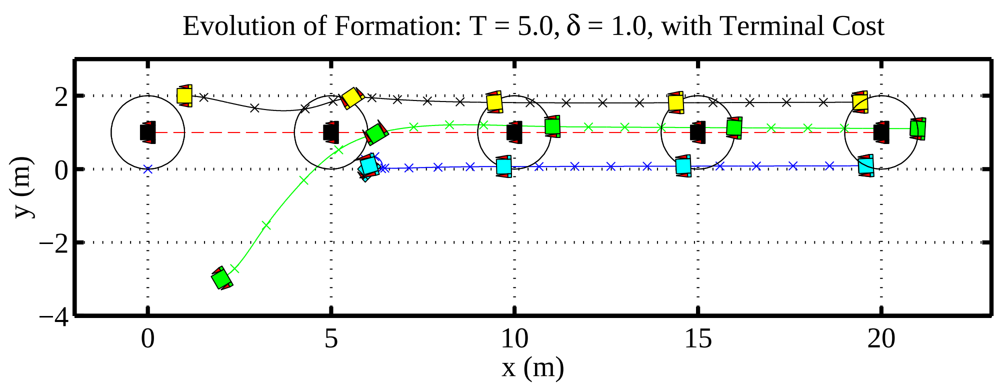

Figure 3: Addition of terminal cost to stabilize formation

various initial conditions and cost weight values are detailed in [3].

3.3 Example 2

In this example, individual vehicle dynamics are simplified to observe how a stabilizing global MPC policy is affected when it becomes distributed and there is model uncertainty between the vehicles in the formation. The dynamics for each (point mass) vehicle are linear, double-integrators on state space $\mathbb{X} = \mathbb{R}^4$ and input space $\mathbb{R}^2$. The same cost function in equation (14) is used. In the global version, one (centralized) optimization is performed to compute the MPC law for every vehicle at every update time δ. In the local version, there are 3 separate MPC optimizations at each update time, one for each vehicle. Each local optimization incorporates the correct model for the host vehicle and assumes that the other vehicles go in straight lines according to the shared initial conditions. In both cases, a nontrivial reference is implemented (see [3] for details).

The horizon length is $T = 2.0$ seconds and update time $\delta = 0.5$ seconds. In the plots, the vehicles are represented as triangles, where each triangle points in the direction of it's velocity vector, and the reference is represented as a red square. As before, the reference trajectory is marked by a dashed red line. The global (*full info*) MPC solution vehicles are represented by the three triangles in black, with colored squares at the center of each triangle. The *local info* vehicles are represented by triangles in full color, with matching colors corresponding to matching initial conditions for the first optimization. Each triangle's trajectory is marked by a line and each figure shows the formation at points in time along the entire time history. A global and local model predictive result is shown in Figure 4 for a velocity error weighting of $V_i = 1.0$, $i = 1, 2, 3$. The top plot corresponds (roughly) to the first half of the time history and the bottom plot shows the remaining portion. The global formation is stable throughout the entire time history, while the

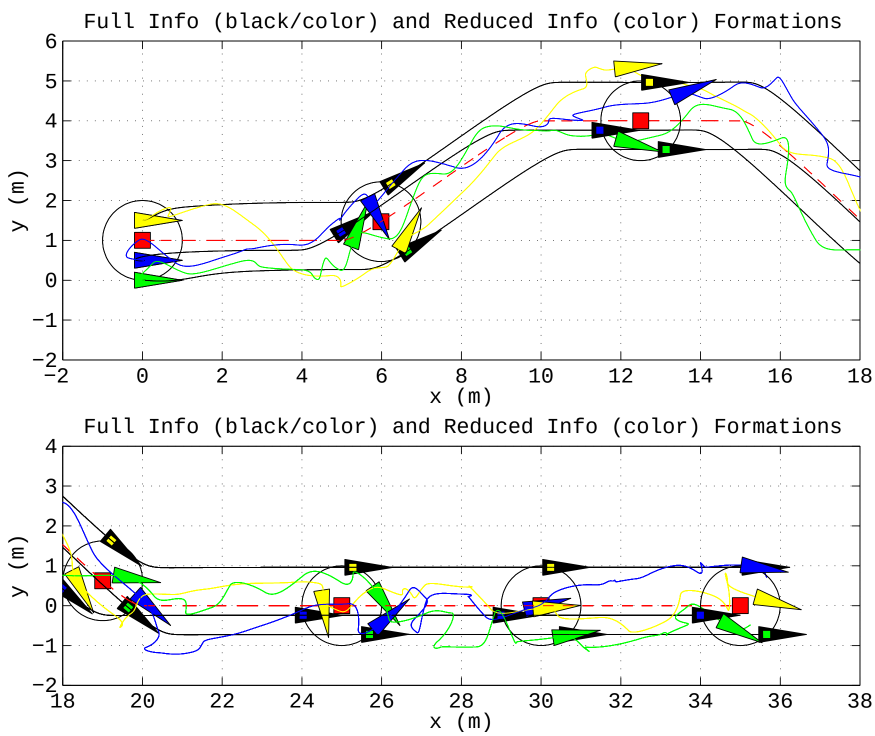

Figure 4: Global and local MPC simulations ($V_i = 1.0$, $i = 1, 2, 3$)

local formation is very unstable. In fact, some of the local vehicles are observed to intersect each other, resulting in collision in a real implementation.

The response from the same set of initial conditions but with increased weight on the velocity error penalty (20.0) is shown in Figure 5. For the particular choice of local models there is a large degree of sensitivity to such weighting changes, as is evident by the local formation responses in Figure 5. The global formation appears on the other hand to be insensitive to the weighting change. The degree of sensitivity of the local formation could likely be reduced by making a more "educated" model for the reduced order vehicles. A good choice might be to assume that the other vehicles travel at constant acceleration, perhaps equal to the known initial value of the reference

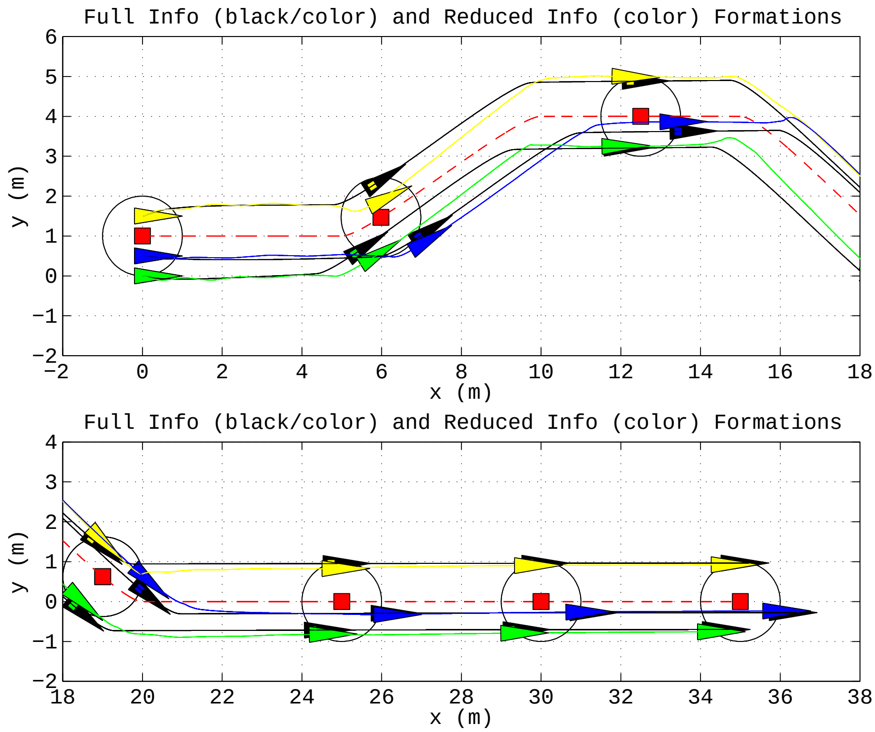

Figure 5: Global and local MPC simulations ($V_i = 20.0$, $i = 1, 2, 3$)

acceleration. Development of the theory of distributed MPC would likely guide the model classes from which distributed models could be chosen, given the full (nominal) models of actual vehicles.

4 Conclusions and Extensions

A generalized constrained and nonlinear MPC formulation with guaranteed stability has been detailed in this chapter for asymptotic stabilization of multiple vehicles to an equilibrium formation. Real-time implementation issues such as uncertainty and computational delay time are not addressed. A multi-vehicle coordination problem that admits a generalized formation objective was then posed. The objective allows that vehicles are stabilized to a set of permissible equilibria, rather than a precise location for each vehicle in the formation. There is also no particular role assignment in this formulation, although a formation reference is defined and could be considered a virtual leader. The theory of MPC is continuing to branch out to address uncertain environments and recent results have investigated real-time issues associated with this methodology [3]. It is realistic to assume that computational tools for MPC will only improve with time. The extension of MPC to a distributed problem adds new elements of complexity to the theory, from which many new interesting problems can be examined. Of particular interest is computational uncertainty (distributed and local) and reduced order model effects of the environment. Environment uncertainty could mean the model of the other vehicles in the formation to any one vehicle could be approximate. Section 3 explored simulations for this type of problem.

The unification of these topics is related to a multi-vehicle experimental testbed being developed at Caltech [2]. The individual vehicle dynamics and inputs are subject to constraints and the objectives include real-time formation maneuvers while avoiding collision and terrain. The MPC framework outlined in this paper is thus a natural choice to meet these objectives. The testbed will be subject to the issues that arise from applying MPC real-time, e.g. model-uncertainty and non-trivial computation times. These topics as well as exploring other variants of MPC and the theoretic implications of distributing the computations over networks will explored in future work.

A Proof of Theorem 1

We first state two lemmas that will be utilized in the proof.

Lemma 2. *Assuming $J^*(\boldsymbol{x}, T)$ is at least C^1, we have the following*

$$\frac{\partial J^*(\boldsymbol{x}, T)}{\partial T} = q\left(\boldsymbol{x}, \boldsymbol{\kappa}(\boldsymbol{x}, T)\right) + \left[\frac{\partial J^*(\boldsymbol{x}, T)}{\partial \boldsymbol{x}}\right]^T \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{\kappa}(\boldsymbol{x}, T)\right).$$

Lemma 3. *Assuming $J^*(\cdot)$ and $V(\cdot)$ are at least C^1,*

$$\frac{\partial J^*(\boldsymbol{x}, T)}{\partial T} \leq q(\hat{\boldsymbol{x}}, \boldsymbol{\kappa_f}(\hat{\boldsymbol{x}})) + \frac{\partial V}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}, \boldsymbol{\kappa_f}(\hat{\boldsymbol{x}}))\boldsymbol{f}(\hat{\boldsymbol{x}}, \boldsymbol{\kappa_f}(\hat{\boldsymbol{x}})),\ \forall \boldsymbol{x} \in X_T,$$

where $\hat{\boldsymbol{x}} \triangleq \boldsymbol{x}^*(T; \boldsymbol{x}, T) \in X_f$.

Theorem 1. *Let X_T denote the set of states $\boldsymbol{x}$ that can be steered to X_f by an admissible control in time T. Assume that $J^*(\cdot)$ is C^1 and that X_T is compact. If $(V(\cdot), X_f, \kappa_f)$ satisfy*

1. *$X_f \subset \mathbb{X}^k$, X_f compact, $S_{eq}^X \subseteq X_f$.*
2. *$\kappa_f^i(X_f) \subset \mathbb{U}$, $\forall i = 1, ..., k$.*
3. *X_f is positively invariant for $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\kappa_f}(\boldsymbol{x}))$.*
4. *$V : X_f \to \mathbb{R}$ is C^1, satisfies equation (9) with $M = S_{eq}^X$ and*
   $\left[\dot{V} + q\right](\boldsymbol{x}, \boldsymbol{\kappa_f}(\boldsymbol{x})) < 0$, $\forall \boldsymbol{x} \in X_f - S_{eq}^X$,

then S_{eq}^X is an asymptotically stable invariant set of the closed-loop autonomous system with region of attraction X_T.

Proof. The strategy of the proof is to apply Theorem 1 to V and then to J^*. By definition of S_{eq}, $\dot{\boldsymbol{x}} = 0$ and so $\dot{V} = 0$ in S_{eq}^X (technically in S_{eq}). By Condition 4 and assumption A8, $X_f - S_{eq}^X$ is the set of all points in X_f where $\dot{V}(\boldsymbol{x}, \boldsymbol{\kappa_f}(\boldsymbol{x})) < 0$. Since S_{eq}^X is an invariant set by Definition 3, Theorem 1 says that S_{eq}^X is an asymptotically stable invariant set of the closed-loop autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\kappa_f}(\boldsymbol{x}))$ with region of attraction X_f. Condition 4, Lemma 2 and Lemma 3 imply that

$$\left[\dot{J}^*(\boldsymbol{x}, \boldsymbol{\kappa}(\boldsymbol{x}, T), T) + q(\boldsymbol{x}, \boldsymbol{\kappa}(\boldsymbol{x}, T))\right] < 0,\ \forall \boldsymbol{x} \in X_T - S_{eq}^X. \tag{17}$$

To apply Theorem 1, observe that $J^*(\cdot, T) : X_T \to \mathbb{R}$ satisfies equation (9) with $M = S_{eq}^X$ and that X_T contains X_f and is positively invariant with respect to

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\kappa}(\boldsymbol{x}, T)),\ \forall \boldsymbol{x} \in X_T - X_f \\ \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\kappa_f}(\boldsymbol{x})),\ \forall \boldsymbol{x} \in X_f. \end{cases}$$

A proof by contradiction that X_T is positively invariant is as follows: take any trajectory that starts in X_T and leaves X_T at some time t ($0 < t < T$) to come back to X_T and eventually to X_f in time T. Take any point on this trajectory x' not in X_T. An admissible control that gets x' to X_f in time T is the concatenation of the control left for the original trajectory with the controller κ_f inside X_f, so x' must also be in X_T. Thus J^* satisfies Theorem 1 with $(\Omega, M) = (X_T, S_{eq}^X)$ by the same reasoning above that V does with $(\Omega, M) = (X_f, S_{eq}^X)$. ■

References

[1] H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive scheme with guaranteed stability. *Automatica*, 14(10):1205–1217, 1998.

[2] L. Cremean, W.B. Dunbar, D. van Gogh, J. Meltzer, R.M. Murray, E. Klavins, and J. Hickey. The Caltech multi-vehicle wireless testbed. In *Submitted: 2002 Conference on Decision and Control*, Las Vegas, NV, 2002.

[3] William B. Dunbar. Model predictive control: extension to coordinated multi-vehicle formations and real-time implementation. CDS Technical Memo CIT-CDS 01-016, California Institute of Technology, Pasadena, CA 91125, Dec 2001.

[4] M. Egerstedt and X. Hu. Formation constrained multi-agent control. *IEEE Transactions on Robotics and Automation*, 17(6):947–951, 2001.

[5] Ali Jadbabaie. *Receding Horizon Control of Nonlinear Systems: A Control Lyapunov Function Approach.* PhD thesis, Control and Dynamical Systems, California Institute of Technology, Pasadena CA 91125, October 2000.

[6] Hassan K. Khalil. *Nonlinear Systems, Second Edition.* Prentice Hall, 1996.

[7] Naomi Ehrich Leonard and Edward Fiorelli. Virtual leaders, artificial potentials and coordinated control of groups. In *2001 Conference on Decision and Control*, Florida, 2001.

[8] D.Q. Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7(2-3):87–99, 2001.

[9] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Contrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.

[10] Mark B. Milam, Kudah Mushambi, and Richard M. Murray. A new computational approach to real-time trajectory generation for constrained mechanical systems. In *Conference on Decision and Control*, 2000.

[11] R. M. Murray, J. Hauser, A. Jadbabaie, M. B. Milam, N. Petit, W. B. Dunbar, and R. Franz. Online control customization via optimization-based control. In *Software-Enabled Control: Information Technology for Dynamical Systems*, T. Samad, G. Balas, editors, IEEE Press (in preparation), Piscataway, NJ, 2002.

[12] Reza Olfati-Saber and Richard M. Murray. Distributed cooperative control of mulitple vehicle formations using structural potential functions. In *Accepted: 2002 IFAC World Congress*, Barcelona, Spain, 2002.

[13] H. Yamaguchi, T. Arai, and G Beni. A distributed control scheme for multiple robotic vehicles to make group formations. *Robotics and Autonomous Systems*, 36:125–147, 2001.

[14] T. H. Yang and E. Polak. Moving horizon control of nonlinear systems with input saturation, disturbances and plant uncertainty. *International Journal of Control*, 58(4):875–903, 1993.

[15] B. J. Young, R. W. Beard, and J. M. Kelsey. A control scheme for improving multi-vehicle formation maneuvers. In *Proceedings of the American Control Conference*, 2001.