Stability Analysis of Interconnected Nonlinear Systems Under Matrix Feedback

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Abstract

This paper explores the stability analysis problem for nonlinear systems which have general linear feedback interconnections. Systems are often modeled in this manner in the study of decentralized control because many communication topologies can be modeled and analyzed using connections to graph theory. We present necessary conditions for stability of a classification of interconnected systems, and we give some examples to provide insight into this problem. These conditions are related to positive definiteness of matrices associated with the feedback interconnection, and specialize to the common case where the Laplacian matrix of a graph represents the communication topology of the system.

1 Introduction

Stability of interconnected systems is an important area of controls research because of our ambition to be able to understand and control increasingly complex systems. Technological advances in computing and communication over the past few years complement this ambition, and enable us to develop the techniques of distributed and decentralized control for more general systems. We find that fast embedded controllers ("agents") are able to be made cheaper and smaller, and we have ever increasing computing resources to analyze collections ("systems") of these agents on a large scale.

For many reasons, decentralization is a natural extension for control strategies with respect to these systems. Desire for autonomy of a fleet of unmanned aerial vehicles (UAVs), for example, may preclude the use of a fixed, ground based centralized control system. This, in turn, may stem from limitations in bandwidth, communication delay, or range that are imposed by the situation. Centralized control performed by one of the UAVs is also an option, but in general this strategy suffers from limitations in bandwidth, and is completely non-robust to failure of that particular UAV.

Of course, there are many other examples for which decentralized control is an apt framework for analysis of large-scale systems. Satellite formation flying, planetary exploration, rescue operations, automated highway systems, and distributed sensing problems can all benefit from fundamental understanding of decentralized control and of interconnected systems in general. The same can be said for biological and chemical systems where cells, proteins, or molecules can be considered as individual dynamical systems which are part of a rich and complex interconnection. Having tools to analyze these interconnected systems is an important step toward fundamental understanding of them, and toward learning to design practical, high-performance and robust control strategies for such systems.

Regardless of the application, the desire for our collection of dynamical agents to perform a coordinated task necessitates the modeling or design of some type of communication or sensing strategy. This strategy is often represented as a network topology, and graph theoretic tools can be useful in interpreting concepts related to such topologies.

There have been many results achieved in stability analysis of interconnected systems. These have invariably been limited to particular classes of dynamical systems and/or particular classes of interconnections between them. String stability [15] is concerned with a simple form of linear interconnection, and concepts in this area have been extended to a more general form of grid-like interconnection in the study of mesh stability [13]. More general interconnections have also been considered. In particular, for interconnections of linear systems whose interconnection can be modeled as an binary directed graph, there exist necessary and sufficient conditions for stability [6] based on the graph Laplacian matrix. A formal notion of formation stability in the context of graph rigidity is explored in [12], and this notion has been applied to small classes of systems with simple double integrator dynamics [11]. Input-to-state stability for formations is introduced in [16] and sufficient results, in terms of ISS of the individual agents, are given for stability of interconnections which are represented by tree-like graph structures.

There are a number of experimental testbeds that have recently been developed to test new ideas in control of multiple vehicle systems. These include testbeds that are ground-based, hovercraft-like [3, 14], airborne [5, 7], and ground-based, kinematic [4, 8]. Each of these classes of testbeds has unique advantages in exploring the various aspects of decentralized control and the intersection of control theory and communication theory. These testbeds provide a nice motivation to see the tools we develop in decentralized control applied in the face of uncertainty of a physical experiment. In particular, we plan to eventually implement algorithms based on this work on the testbed introduced in [3], which is described in more detail in [2].

In this paper we present a general mathematical framework for the study of interconnected nonlinear systems in order to provide a context for previous analytical work in this field. We explore a class of interconnected systems, and our main result is to provide sufficient conditions for stability of this class in terms of the stability properties of the individual systems and the general properties of the interconnection. We prove our main result with a Lyapunov function approach and along the way exploit some of the properties of Kronecker product algebra in order to prove some preliminary results on positive definiteness of matrices.

Models of systems that we will consider in this paper are presented in Section 2 and followed by mathematical and graph theoretical preliminaries in Sections 3 and 4. Some useful results on positive definiteness are provided in Section 5. The main result of our paper is the theorem and discussion found in Section 6. Example applications of our main result are presented in Section 7 We summarize the paper and discuss directions of future work in Section 8.

2 Problem Statement

Though we will consider only a special case of the following model, we will state the problem we are attempting to solve in its most general form. Given the dynamical description of a set of N agents indexed by the set $\mathcal{I} = \{1, \ldots, N\}$, we wish to determine the most general classification of interconnections between these agents that will cause the interconnected system to be stable, in some respect. A version of this problem can be expressed mathematically in the following manner.

Consider the set of N subsystems described by

$$\begin{aligned} \dot{x}_i &= f_i(x_i, u_i) \\ y_i &= h_i(x_i, u_i), \end{aligned} \qquad i \in \mathcal{I} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^{p_i}$, and the interconnection described by

$$u_i = k_i(y) \qquad i \in \mathcal{I}, \tag{2}$$

where y is the "stacked" version of y_i, i.e. $y = [y_1^T \ \cdots \ y_N^T]^T$ (similarly for the variables u and x when the subscript is dropped). We wish to determine the equilibria and stability properties of the interconnected system.

The special case of this problem addressed in this paper is that for which $n_i = n_j$, $m_i = m_j$ and $p_i = p_j$, $\forall i, j \in \mathcal{I}$, where the dependence of y_i on u_i is absent, and for which the individual agents are affine and linear in the control, with linear output equations and identical corresponding B and C matrices. That is, our system equations reduce to

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + Bu_i \\ y_i &= Cx_i \end{aligned} \qquad i \in \mathcal{I} \tag{3}$$

We also consider only linear feedback interconnections, i.e. $u = -Ky$, where $K \in \mathbb{R}^{Nm \times Nm}$. In many cases, the linear interconnection is modeled as a graph, and K can be related to the Laplacian matrix of the graph [6]. Note that the dynamics of the agents represented in (3) are not necessarily identical due to the subscript i of f.

3 Mathematical Preliminaries

We will make use of the Kronecker product in describing and analyzing the interconnected systems described in Section 2. For that reason we present the definition of the Kronecker product and some relevant properties here. We also present Gershgorin's theorem, which will be used later in the paper.

Definition 3.1. Given the matrices $A \in \mathbb{R}^{n \times m}$ (with elements $A = [a_{i,j}]$) and $B \in \mathbb{R}^{p \times q}$, the *Kronecker product* (also called the matrix direct product) of A and B, denoted $A \otimes B$, is the $np \times mq$ matrix

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \dots & a_{1,m}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \dots & a_{n,m}B \end{bmatrix} \in \mathbb{R}^{np \times mq}.$$

The Kronecker product has several useful properties, including the following [17]. We assume here that A, B, C and D are real-valued matrices of the appropriate dimension.

1. The Kronecker product is a bilinear operator. Given $\alpha \in \mathbb{R}$,

$$\begin{aligned} A \otimes (\alpha B) &= \alpha(A \otimes B) \\ (\alpha A) \otimes B &= \alpha(A \otimes B) \end{aligned} \tag{4}$$

2. The Kronecker product distributes over addition:

$$\begin{aligned}(A+B)\otimes C &= (A\otimes C)+(B\otimes C)\\ A\otimes (B+C) &= (A\otimes B)+(A\otimes C)\end{aligned} \tag{5}$$

3. Transpose distributes over the Kronecker product (and does *not* invert order)

$$(A\otimes B)^T = A^T \otimes B^T. \tag{6}$$

4. When dimensions are appropriate,

$$(A\otimes B)(C\otimes D) = (AC\otimes BD). \tag{7}$$

Theorem 3.1 (Gershgorin, [9]). *Let $B = [b_{ij}] \in \mathbb{R}^{N\times N}$, and let*

$$r_i(B) = \sum_{j=1, j\neq i}^{N} |b_{ij}|, \quad 1 \leq i \leq N$$

denote the deleted absolute row sums of B. Then all of the eigenvalues of B are located in the union of n discs

$$G(B) \equiv \bigcup_{i=1}^{N} \{z \in \mathbb{C} : |z - b_{ii}| \leq r_i(B)\}.$$

Gershgorin's theorem will be useful in proving positive definiteness for matrices of interest in Section 5. Finally, we denote the spectrum of a square matrix L as $\Lambda(L) = \{\lambda_i(L)\}$, and (unless otherwise stated) index the eigenvalues from minimum to maximum modulus.

4 Graph Theory Preliminaries

We make use of some standard results from graph theory; a thorough exposition of the field can be found in, for example, [1]. The graphs considered here are pairs (V, E) where V is the set of vertices (also called nodes) of the graph, and E is the set of *weighted* and directed edges on the graph. An element of E assigns a weight $w_{ij} \in \mathbb{R}, w_{ij} \geq 0$ to an ordered pair of vertices $(v_i, v_j) \in V \times V$. In this paper an edge from node i to node j indicates that node i has access in some way to information from node j, so the flow of information is opposite the direction of arrows in the graph. In addition, the term "graph" here is intended to always indicate a directed graph; symmetric graphs are to be considered special cases.

Remark 4.1. The reader should note that *all* graphs can be considered weighted and directed. In this paper, we will refer to graphs whose weights take values in the set $\{0, 1\}$ as *binary* and those graphs whose adjacency matrices are symmetric as *symmetric*. We will abstain from using the terms "unweighted" and "undirected" to indicate binary and symmetric graphs, respectively, because the terminology seems to imply that these classes are mutually exclusive to the general class of (weighted and directed) graphs. The Venn diagram of Figure 1 solidifies these concepts.

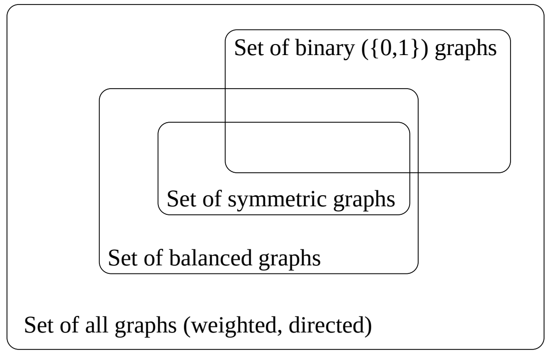


Figure 1: A Venn diagram to visualize the classifications of graphs. Note that all graphs can be considered weighted and directed; binary and symmetric graphs are special cases. Note also that all symmetric graphs are balanced.

We will make use of a few key matrices associated with graphs, which are defined here along with some special classifications of graphs. Throughout, we consider only simple graphs (i.e. those with no self-loops or multiple edges), but unless otherwise noted we are considering arbitrarily *weighted* graphs. In this way we can accommodate multiple edges by combining their respective weights into a single edge, but we can not distinguish between multiple edges. We denote the order of the graph as N, which is also indexed by the set $\mathcal{I}$.

Definition 4.1. The *weighted adjacency matrix* of a graph, $A \in \mathbb{R}^{N\times N}$ is a positive matrix whose ijth element represents the weight of the edge from node i to node j. Because we consider only simple graphs (which contain no self-loops), the diagonal elements of this graph are all zero.

Definition 4.2. A graph is *symmetric* if the weighted adjacency matrix of the graph is symmetric, $A = A^T$. Sometimes the term *undirected* is used, but we abstain from using it here.

Definition 4.3. The *(weighted) outdegree matrix* of a graph, $\Delta \in \mathbb{R}^{N\times N}$ is the positive diagonal matrix whose diagonal elements are the row sums of the (weighted) adjacency matrix, $\Delta_{ii} = \sum_{j=1}^{N} a_{ij}$.

Definition 4.4. The *(weighted) Laplacian matrix* of a graph, $L \in \mathbb{R}^{N\times N}$ is defined in terms of the (weighted) adjacency matrix and (weighted) outdegree matrix, $L = \Delta - A$. The Laplacian matrix is sometimes defined to be normalized by its outdegree matrix, i.e. $L = \Delta^{-1}(\Delta - A)$. We refer to this as the *normalized* Laplacian matrix.

The term Laplacian matrix in this paper indicates the non-normalized Laplacian matrix. There are some notable features of this Laplacian matrix. Since the row sums are all zero, $\lambda_1 = 0$ is an eigenvalue of L with associated eigenvector $v = \vec{\mathbf{1}}$. Also, Gershgorin's theorem tells us that the eigenvalues of L are restricted to a circle centered at $\Delta_{max} \triangleq \max_i(\Delta_{ii})$ and of radius Δ_{max}.

Definition 4.5. The *indegree matrix* of a graph, $\Delta^{in} \in \mathbb{R}^{N\times N}$ is the positive diagonal matrix whose diagonal elements are the column sums of the adjacency matrix, $\Delta^{in}_{ii} = \sum_{j=1}^{N} a_{ji}$.

The outdegree matrix and indegree matrix are useful for classifying the notion of a *balanced* graph, which is presented in [10].

Definition 4.6. Node i of a graph is *balanced* if its outdegree equals its indegree, i.e. $\Delta_{ii} - \Delta^{in}_{ii} = 0$. It is *overbalanced* if $\Delta_{ii} - \Delta^{in}_{ii} > 0$ and *underbalanced* if $\Delta_{ii} - \Delta^{in}_{ii} < 0$.

Definition 4.7. A simple graph is *balanced* if its outdegree matrix equals its indegree matrix, $\Delta = \Delta^{in}$.

It is clear by definition that a graph is balanced if and only if all of the nodes of the graph are balanced. Clearly, every unbalanced graph has at least one overbalanced node and one underbalanced node. It can be shown that symmetric graphs are balanced, and that balanced graphs consist only of superimposed cycles.

5 Results on Positive Definiteness

We present a set of useful lemmas regarding positive definiteness of matrices. Here $\text{sym}(L) \triangleq \frac{1}{2}(L + L^T)$ and $\text{skew}(L) \triangleq \frac{1}{2}(L - L^T)$. We use the following definition of a positive (semi)definite matrix:

Definition 5.1. The square matrix $L \in \mathbb{R}^{N \times N}$ is *positive semidefinite* ("$L \geq 0$") if $x^T L x \geq 0$, $\forall x \in \mathbb{R}^N$. It is *positive definite* ("$L > 0$") if the inequality is strict for $x \neq 0$.

Lemma 5.1. $L \geq 0 \iff sym(L) \triangleq \frac{1}{2}(L + L^T) \geq 0$ *and* $L > 0 \iff sym(L) > 0$.

Proof. $x^T L x \geq 0 \iff x^T(\text{sym}(L) + \text{skew}(L))x \geq 0 \iff x^T \text{sym}(L) x \geq 0$, since $x^T \text{skew}(L) x = 0$ for any L. The inequality can trivially be made strict. □

Lemma 5.2. *If λ_i are the n eigenvalues of $A \in \mathbb{R}^{n \times n}$ and μ_j are the m eigenvalues of $B \in \mathbb{R}^{m \times m}$, then the mn eigenvalues of $A \otimes B$ are given by $\lambda_i \mu_j$, $i = \{1, \ldots, n\}$, $j = \{1, \ldots, m\}$.*

Proof. Note that if $Av_i = \lambda_i v_i$, $Bw_j = \mu_j w_j$, then

$$(A \otimes B)(v_i \otimes w_j) = (Av_i \otimes Bw_i) = \lambda_i \mu_j (v_i \otimes w_j)$$

□

Lemma 5.3. *L represents the Laplacian matrix of a symmetric graph $\implies L \geq 0$.*

Proof. A is symmetric $\implies L = \Delta - A$ is symmetric $\implies$ all of the eigenvalues of L are real. Since L is a Laplacian, Gershgorin's theorem places these (real) eigenvalues on the real line between 0 and $2\Delta_{max} > 0$. This implies that L is positive semidefinite. □

The next lemma is a relaxation of Lemma 5.3 to the more general classification of balanced graphs; it states that all Laplacian matrices of balanced graphs are positive semidefinite.

Lemma 5.4. *L represents the Laplacian matrix of a balanced graph $\implies L \geq 0$.*

Proof. By definition, if $L = \Delta - A$ is balanced, then $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$. Note that since $a_{ii} = 0, \forall i \in \mathcal{I}$ and $a_{ij} \geq 0, \forall i, j \in \mathcal{I}$, then

$$\Delta_{ii} = \sum_{j=1}^N a_{ij} = \sum_{j=1, j \neq i}^N |a_{ij}|,$$
$$\Delta_{ii}^{in} = \sum_{j=1}^N a_{ji} = \sum_{j=1, j \neq i}^N |a_{ji}|.$$

We first prove that $\text{sym}(L) = \frac{1}{2}(L + L^T)$ is positive semidefinite. Consider that

$$B \triangleq \text{sym}(L) = \tfrac{1}{2}(L + L^T) = \tfrac{1}{2}(\Delta - A + \Delta - A^T) = \Delta - \tfrac{1}{2}(A + A^T).$$

Applying Gershgorin's theorem (Theorem 3.1) to this matrix, we find that $b_{ii} = \Delta_{ii}$ and that

$$\begin{aligned} r_i(B) &= \tfrac{1}{2} \sum_{j=1, j\neq i}^{N} |-a_{ij}| + \tfrac{1}{2} \sum_{j=1, j\neq i}^{N} |-a_{ji}| \\ &= \tfrac{1}{2} \sum_{j=1, j\neq i}^{N} |a_{ij}| + \tfrac{1}{2} \sum_{j=1, j\neq i}^{N} |a_{ji}| \\ &= \tfrac{1}{2}\Delta_{ii} + \tfrac{1}{2}\Delta_{ii}^{in} \end{aligned}$$

By definition, if L is balanced, then $\Delta_{ii} = \Delta_{ii}^{in}$ and $r_i(B) = r_i(\text{sym}(L)) = \Delta_{ii}, \forall i \in \mathcal{I}$. In this case the Gershgorin disks that bound the eigenvalues of $\text{sym}(L)$ are centered at Δ_{ii} and are of radius Δ_{ii}, and their union is the disk of radius Δ_{max} centered at Δ_{max}. Since these eigenvalues are real, they are nonnegative and therefore $\text{sym}(L) \geq 0$. By Lemma 5.1, this is equivalent to $L \geq 0$. □

By contrast, note that since every unbalanced graph has at least one underbalanced node i, the Gershgorin disk for $\text{sym}(L)$ associated with that node has a center $b_i = \Delta_{ii}$ and radius $r_i > \Delta_{ii}$. Therefore, we cannot bound the real eigenvalues of $\text{sym}(L)$ to the nonnegative real axis. Although Gershgorin's theorem only tells us that the eigenvalues of $\text{sym}(L)$ lie in the union of these disks, it has been thus far observed that for all unbalanced graphs one of the eigenvalues of $\text{sym}(L)$ lies on the negative real axis within a disk corresponding to an underbalanced node.

The next lemma is an interesting result which is useful in application to the main result of our paper. In it the properties of the Kronecker product presented in Section 3 are used, in particular the distributive property of Equation (5).

Lemma 5.5. *Assuming* $A = A^T$ *or* $B = B^T$, *then* $A \geq 0$, $B \geq 0 \implies A \otimes B \geq 0$.

Proof. The Lemma is proved as follows.

$$\begin{aligned} A, B \geq 0 &\iff \text{sym}(A), \text{sym}(B) \geq 0 \\ &\implies \text{(by Lemma 5.2) } (\text{sym}(A) \otimes \text{sym}(B)) \geq 0 \\ &\iff (\tfrac{1}{2}(A + A^T)) \otimes (\tfrac{1}{2}(B + B^T)) \geq 0 \\ &\iff A \otimes B + A \otimes B^T + A^T \otimes B + A^T \otimes B^T \geq 0 \\ &\iff \text{sym}(A \otimes B) + \text{sym}(A \otimes B^T) \geq 0 \\ &\iff \text{sym}(A \otimes B) + \text{sym}(A^T \otimes B) \geq 0 \\ &\iff (A = A^T \text{or } B = B^T) \ \text{sym}(A \otimes B) \geq 0 \\ &\iff (A \otimes B) \geq 0 \end{aligned}$$

□

Note that the necessity of this condition is only broken in the second step, which uses Lemma 5.2. The assumption that one of A or B is symmetric (or other assumptions) are necessary for Lemma 5.5 to hold. Necessary and sufficient conditions for the positive definiteness of the Kronecker product of two matrices are given in Lemma 5.6.

Lemma 5.6. *sym*$(A) \otimes$ *sym*$(B) \geq 0 \iff$ *sym*$(A) \geq 0,$ *sym*$(B) \geq 0$ ***OR*** *sym*$(A) \leq 0,$ *sym*$(B) \leq 0$

Proof. This lemma can be proved by application of Lemma 5.2. □

6 Main Result

We now present the main theorem of the paper, which establishes sufficient conditions for stability of a class of interconnected systems that are based on the properties of the individual systems and the properties of the interconnection. We will compare the result of this theorem with other work, discuss its implications and describe possible extensions to it. In the theorem, $f(x) = (f_1(x_1), \cdots, f_N(x_N))$.

Theorem 6.1. *Consider the interconnection of N (not necessarily identical) linear agents described by*

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + Bu_i \\ y_i &= Cx_i \end{aligned} \tag{8}$$

where $x_i \in \mathbb{R}^n$, $u_i, y_i \in \mathbb{R}^m$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and the interconnection is given by $u = -(L \otimes I_m)y$, where ($L \in \mathbb{R}^{N\times N}$, $u, y \in \mathbb{R}^{Nm}$). If the following two conditions hold

1. $x_i^T f_i(x_i) < 0 \ \forall x_i \neq 0, \forall i \in \mathcal{I}$, *and*
2. $L \otimes BC \geq 0$,

then the origin of the interconnected system is asymptotically stable.

Proof. With the given feedback interconnection, the closed loop interconnected system is given by

$$\begin{aligned} \dot{x} &= f(x) + (I_N \otimes B)u \\ &= f(x) - (I_N \otimes B)(L \otimes I_m)y \\ &= f(x) - (I_N \otimes B)(L \otimes I_m)(I_N \otimes C)x \\ &= f(x) - (L \otimes BC)x, \end{aligned} \tag{9}$$

where in the last step we use the Kronecker product property of Equation (7). Using the Lyapunov function candidate $V(x) = \frac{1}{2}x^T x$, we find that

$$\dot{V} = x^T f(x) - x^T(L \otimes BC)x \tag{10}$$

Assumption 1 implies that the first term is negative definite and Assumption 2 implies that the second term is negative semidefinite, therefore $\dot{V}$ is negative definite and is hence a Lyapunov function for the closed loop system. Therefore, the origin of the interconnected system is asymptotically stable. □

Remark 6.1. Assumption 1 of Theorem 6.1 implies that the individual systems are globally asymptotically stable. There is an obvious extension of this theorem to local asymptotic stability. Also, the relatively simple proof of the theorem relies on each of the agents having a common Lyapunov function which take a particularly simple form. Clearly, it is of interest to extend this theorem to the case where each of the systems has an arbitrary Lyapunov function that shows asymptotic stability of the individual systems.

Remark 6.2. There is a natural dual to this theorem based on Lyapunov's instability theorem. If the individual systems are asymptotically unstable, then if $L \otimes BC$ is not negative definite, then the interconnected system is unstable. This is the case when L is a Laplacian matrix because the zero eigenvalue of L and the eigenvector $\vec{\mathbf{1}}$ correspond to a neutrally stable mode for which the interconnection cannot recover stability.

Remark 6.3. Notions of formation stability cannot be directly applied here, as the assumption of the Theorem 6.1 relies on each of the agents being stable to a common equilibrium point. Extensions can be made in the case where we consider neutrally stable individual systems, or systems whose stable equilibria are offset by desired amounts.

Remark 6.4. Assumption 2 of Theorem 6.1 will hold, for instance, if L is a balanced Laplacian matrix and BC is symmetric positive definite, as can be seen by application of Lemmas 5.4 and 5.5.

7 Application to Examples

We now present some examples of the application of Theorem 6.1, first to a system of damped nonlinear pendulums, and then to a linearization of this system, or equivalently, to a set of mass-spring-dampers.

7.1 Nonlinear Nonidentical Dynamics

Consider a set of pendulum equations of the form of (3) given by

$$\begin{aligned} \ddot{\theta}_i &= -\frac{g_i}{l_i}\sin\theta_i - \frac{\psi_i}{m_i l_i^2}\dot{\theta} + u_i \\ y_i &= \theta_i \end{aligned} \tag{11}$$

where m_i, g_i, l_i and ψ_i are positive constants and each of the pendulum subsystems is given a feedback law of the form

$$u_i = -\sum_{j=1}^{N} a_{ij}(y_i - y_j). \tag{12}$$

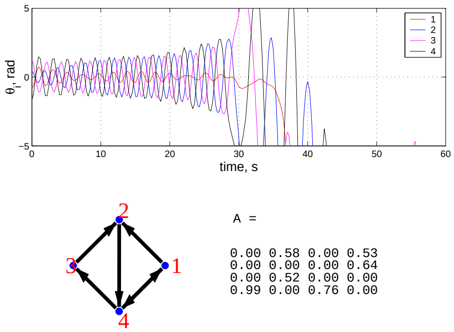
If $A = [a_{ij}]$ corresponds to the adjacency matrix of the interconnection between pendulums, it can be shown that the stacked input vector u is given by

$$u = -(L \otimes I_m)y. \tag{13}$$

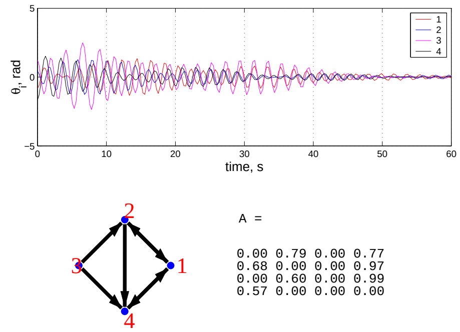
where I_m is the $m \times m$ identity matrix (for this case $m = 1$). The local feedback law of Equation (12) corresponds to each pendulum trying to match outputs with the rest of the pendulums for which $a_{ij} \neq 0$.

This set of equations was simulated for $N = 4$ pendulums with parameters $g_i = 9.8\ \frac{m}{s^2}$, $l_i = 1.0\ m$, $\psi_i = 0.1\ \frac{kg\ m^2}{s}$ and $m_i = 1.0\ kg$, for different sets of interconnections represented by the adjacency matrix A. The initial conditions provided were $\theta_i = \{-0.8,\ 0.4,\ 1.2,\ -1.6\}\ rad$ and $\dot{\theta}_i = \{0,\ 0,\ 0,\ 0\}\ rad/s$. Note that for this system $B = [0\ 1]^T$ and $C = [1\ 0]$, and that BC is an indefinite matrix ($\Lambda(\text{sym}(BC)) = \{-0.5,\ 0.5\}$), so the assumptions of Theorem 6.1 cannot be satisfied. We demonstrate the range of behaviors for this system nonetheless.

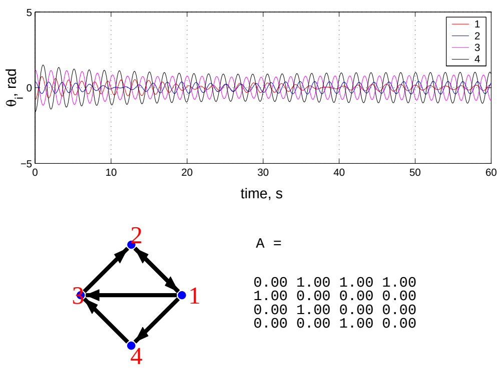
The time traces for the four pendulum system are shown in Figure 2. Note that this simple linear interconnection of asymptotically stable subsystems can display various stability properties, including asymptotic stability in Figure 2(b), limit cycles in Figure 2(c) and instability in Figure 2(a). These three results, however, are cases for which the $L \otimes BC \geq 0$ assumption of Theorem 6.1 is not satisfied. The stable example of Figure 2(b) is immediate evidence that the conditions of Theorem 6.1 are not necessary for stability and are to some degree conservative.



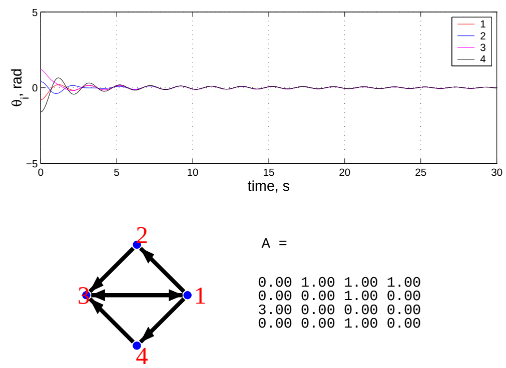
(a) Adjacency matrix A_1, $L \otimes BC \ngeq 0$.



(b) Adjacency matrix A_2, $L \otimes BC \ngeq 0$.



(c) Adjacency matrix A_3, $L \otimes BC \ngeq 0$.

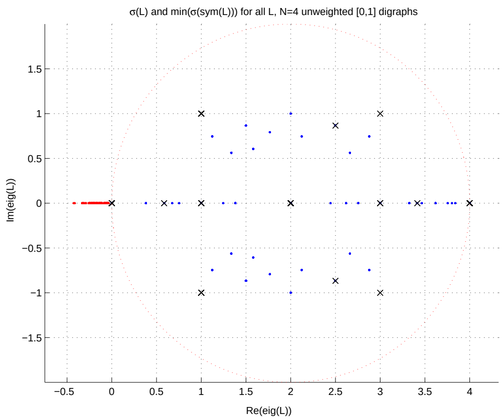


(d) Adjacency matrix A_4, $L \otimes BC \geq 0$, corresponds to balanced graph.

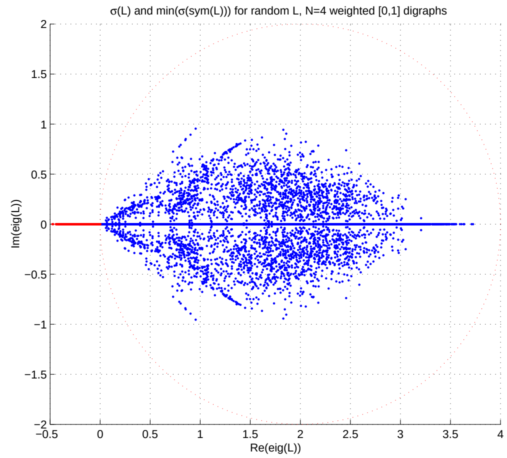
Figure 2: Time traces and graph depictions of the four pendulums example. The stability conditions of Theorem 6.1 are locally satisfied only for (d). Systems (a), (b) and (c) are described exactly by Equation (11), but for system (d) the output equation is modified to be $y_i = \dot{\theta}_i$ so that BC is positive semidefinite.

The conditions of Theorem 6.1 are satisfied for the simulation of Figure 2(d), for which the output equation is modified to be $y_i = \dot{\theta}_i$. In this case, BC is positive semidefinite. Also, for this case L corresponds to the Laplacian matrix of a balanced graph, and is therefore positive semidefinite according to Lemma 5.4. Since neither L nor BC is symmetric, we cannot apply Lemma 5.5 to prove positive semidefiniteness of their Kronecker product. This condition was verified manually.

It has been observed that Laplacian matrices L which are not balanced tend to be indefinite, so when considering only Laplacian matrices we restrict ourselves significantly regarding the classes of interconnections for which we can apply Theorem 6.1. The spectra of all binary and a large sample of random graphs of order four are shown in Figure 3. These are plotted along with the Gershgorin disk inside which all of the eigenvalues lie, and the minimum value of $\Lambda(\text{sym}(L))$. The latter value for each graph will indicate to us whether L is positive semidefinite, which is only true if $\min \Lambda(\text{sym}(L)) = 0$. The scatter of eigenvalues of $\text{sym}(L)$ on the negative real axis is an



(a) Spectra of binary graphs of order 4.



(b) Spectra of random (weighted) graphs of order 4.

Figure 3: Eigenvalue locations for Laplacian matrices of graphs of order 4. The spectra of all binary ($a_{ij} \in \{0,1\}$) graph Laplacians are shown on the left, with those corresponding to balanced graphs shown with '×'s. On the nonpositive real axis are shown the minimum eigenvalues of sym(L) for each graph plotted. The spectra of random weighted Laplacians are shown in on the right.

indication that Laplacian matrices of unbalanced graphs are not positive semidefinite.

7.2 Linear Nonidentical Dynamics

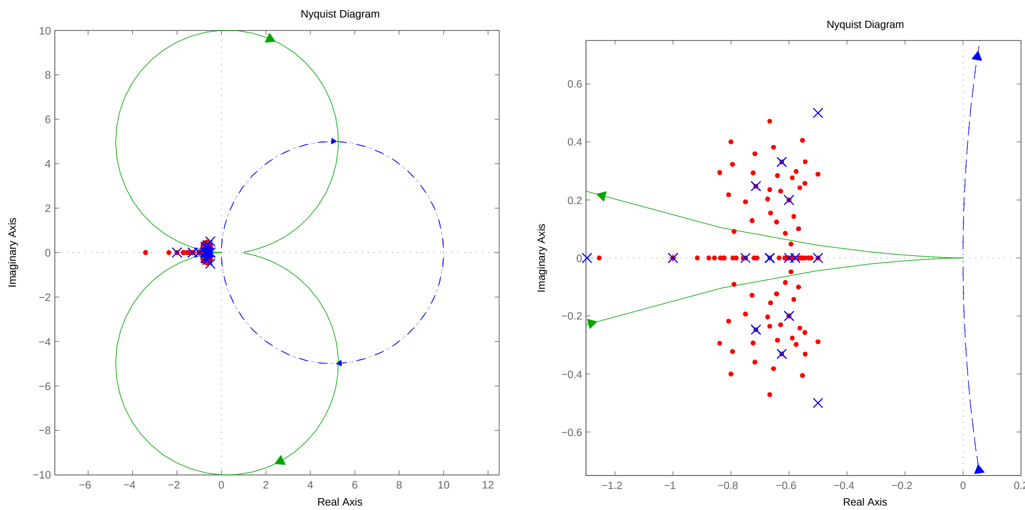
We now discuss the case of the linearization of the system described in Section 7.1, mainly so we can provide comparison with results of previous work on stability of linear interconnected systems. It should be noted that Assumption 1 of Theorem 6.1 is not actually satisfied for the nonlinear pendulums example since the origin of the system is only *almost* globally asymptotically stable. Therefore, in the pendulums example case, we have implicitly appealed to the natural local extension of the result in Theorem 6.1.

The set of systems which we consider in this section are those for which the dynamical equations can be written as

$$\begin{aligned} \ddot{x}_i &= -\alpha_i x_i - \beta_i \dot{x}_i + u_i \\ y_i &= x_i \quad OR \quad y_i = \dot{x}_i. \end{aligned} \tag{14}$$

Simulation for the case when $y_i = \theta_i$, depending on the interconnection, results in both stable and unstable solutions similar to the results of Figure 2. For the case when $y_i = \dot{\theta}_i$, however, the interconnected system is always stable. This result can be proven using the Nyquist-like stability criteria of [6], which states that the interconnected system is stable if and only if the Nyquist plot of our linear system does not encircle any of the points $-1/\lambda_i$, where λ_i are the N eigenvalues of L.

The locations of $-1/\lambda_i$ for all binary graphs of order $N = 4$ are plotted in the complex plane of Figures 4, along with the Nyquist plots of the two systems of (14). For the case where $y_i = x_i$, some of the points are encircled by the (solid) Nyquist plot and their corresponding interconnections are unstable according to the result of [6]. Theorem 6.1 is inconclusive regarding these systems since BC is indefinite.



(a) Some eigenvalues on the negative real axis are off this full scale plot.

(b) Same plot zoomed in to area of interest.

Figure 4: The Nyquist plots for the systems of Equation (14). The plot for $y_i = x_i$ is solid and the one for $y_i = \dot{x}_i$ is dash-dotted. The locations of $-1/\lambda_i(L)$ are plotted as dots for unbalanced graphs and as '$\times$'s for balanced graphs.

For the case where $y_i = \dot{x}_i$, the Nyquist plot is restricted to the right-half plane, and conformal mapping techniques can show that $-1/\lambda_i$ are restricted to a region in the left-half plane as a result of Gershgorin's theorem for Laplacian matrices L. By applying the result in [6], it can be shown that *all* of these interconnections are stable. Theorem 6.1 provides a proof of stability for the balanced graphs, for which $L \geq 0$ by Lemma 5.4. The locations of $-1/\lambda_i$ for these graphs are indicated by '$\times'$s in Figure 4(b).

The comparison of Theorem 6.1 with the result of [6] for linear systems provides a good indication of the conservative nature of the sufficient conditions for stability given in Theorem 6.1. A main contribution of this new result, however, is the extension to a class of nonlinear systems and to allow for collections of systems with nonidentical dynamics. In addition, this new result makes inroads on development and use of tools from nonlinear control theory to prove stability of interconnected systems.

8 Summary and Future Directions

We have provided sufficient conditions for the interconnection of a class of nonlinear, nonidentical dynamical systems, and we have discussed some implications of this result regarding general classes of nonlinear interconnected systems, particularly in regard to the Laplacian matrix of graphs. We presented some applicable results with respect to positive definiteness of matrices and indicated through use of Kronecker products a method by which to analyze the stability of interconnected systems. We also illustrated through examples the nature of interconnected systems when subject to a commonly used linear interconnection strategy related to the Laplacian matrix.

There are many extensions to the work presented here that should lead to stronger results in the field of decentralized control for nonlinear systems. A few of these were noted in the remarks of Section 6. Extension of these results to even more general systems and finding less conservative conditions for stability are both avenues of exploration. In addition, we are pursuing results of this nature for neutrally stable and unstable systems, as well as connections to structural and formation stability with respect to these systems.

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