Nonholonomic Mechanical Systems with Symmetry

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Spring, 1992, this version: June 28, 1995
CDS Technical Report 94-013, California Institute of Technology
URL http://avalon.caltech.edu/cds/reports/cds94-013.ps

Abstract

This work develops the geometry and dynamics of mechanical systems with nonholonomic constraints and symmetry from the perspective of Lagrangian mechanics and with a view to control theoretical applications. The basic methodology is that of geometric mechanics applied to the formulation of Lagrange d'Alembert, generalizing the use of connections and momentum maps associated with a given symmetry group to this case. We begin by formulating the mechanics of nonholonomic systems using an Ehresmann connection to model the constraints, and show how the curvature of this connection enters into Lagrange's equations. Unlike the situation with standard configuration space constraints, the presence of symmetries in the nonholonomic case may or may not lead to conservation laws. However, the momentum map determined by the symmetry group still satisfies a useful differential equation that decouples from the group variables. This momentum equation, which plays an important role in control problems, involves parallel transport operators and is computed explicitly in coordinates. An alternative description using a "body reference frame" relates part of the momentum equation to the components of the Euler-Poincaré equations along those symmetry directions consistent with the constraints. One of the purposes of this paper is to derive this evolution equation for the momentum and to distinguish geometrically and mechanically the cases where it is conserved and those where it is not. An example of the former is a ball or vertical disk rolling on a flat plane and an example of the latter is the snakeboard, a modified version of the skateboard which uses momentum coupling for locomotion generation. We construct a synthesis of the mechanical connection and the Ehresmann connection defining the constraints, obtaining an important new object we call the nonholonomic connection. When the nonholonomic connection is a principal connection for the given symmetry group, we show how to perform Lagrangian reduction in the presence of nonholonomic constraints, generalizing previous results which only held in special cases. Several detailed examples are given to illustrate the theory.

*On leave from Department of Mathematics, Ohio State University, Columbus, OH 43210. Research partially supported by the National Science Foundation PYI grant DMS–91–57556, AFOSR grant F49620-93-1-0037 and MSRI.

†Research partially supported by the AFOSR University Research Initiative Program under grants AFOSR-87-0073 and AFOSR-90-0105 and by the National Science Foundation's Engineering Research Centers Program NSFD CDR 8803012.

‡Research partially supported by the NSF, DOE, a Fairchild Fellowship at Caltech, and the Fields Institute for Research in the Mathematical Sciences.

§Research partially supported by NSF, NASA and the Powell Foundation.

Contents

1 Introduction

Problems of nonholonomic mechanics, including many problems in robotics, wheeled vehicular dynamics and motion generation, have attracted considerable attention. These problems are intimately connected with important engineering issues such as path planning, dynamic stability, and control. Thus, the investigation of many basic issues, and in particular, the role of symmetry in such problems, remains an important subject today.

Despite the long history of nonholonomic mechanics, the establishment of productive links with corresponding problems in the geometric mechanics of systems with configuration space constraints (*i.e.,* holonomic systems) still requires much development. The purpose of this work is to bring these topics closer together with a focus on nonholonomic systems with symmetry. Many of our results are motivated by recent techniques in nonlinear control theory. For example, problems in both mobile robot path planning and satellite reorientation involve geometric phases, and the context of this paper allows one to exploit the commonalities and to understand the differences. To realize these goals we make use of connections, both in the sense of Ehresmann and in the sense of principal connections, to establish a general geometric context for systems with nonholonomic constraints.

A broad overview of the paper is as follows. We begin by recalling the the Lagrange d'Alembert equations of motion for a nonholonomic system. We realize the constraints as the horizontal space of an Ehresmann connection and show how the equations can be written in terms of the usual Euler-Lagrange operator with a "forcing" term depending on the curvature of the connection. Following this, we add the hypothesis of symmetry and develop an evolution equation for the momentum that generalizes the usual conservation laws associated to a symmetry group. The final part of the paper is devoted to extending the Lagrangian reduction theory of Marsden and Scheurle [1993a,b] to the context of nonholonomic systems. In doing so, we must modify the Ehresmann connection associated with the contraints to a new connection that also takes into account the symmetries; this new connection, which is a principal connection, is called the *nonholonomic connection*.

The context developed in this paper should enable one to further develop even further the powerful machinery of geometric mechanics for systems with holonomic constraints; for example, ideas such as the energy-momentum method for stability and results on Hamiltonian bifurcation theory require further general development, although of course many specific problems have been successfully tackled.

Previous progress in realizing the goals of this paper has been made by, amongst others, Chaplygin [1897 etc.], Cartan [1928], Neimark & Fufaev [1972], Rosenberg [1977], Weber [1986], Koiller [1992], Bloch & Crouch [1992], Krishnaprasad, Dayawansa & Yang [1992], Yang [1992], Yang, Krishnaprasad & Dayawansa [1993], Bates & Sniatycki [1993], Marle [1994], and van der Schaft & Maschke [1994].

Nonholonomic systems come in two varieties. First of all, there are those with *dynamic nonholonomic constraints*; that is, constraints preserved by the basic Euler-Lagrange or Hamilton equations, such as angular momentum, or more generally momentum maps. Of course, these "constraints" are not externally imposed on the system, but rather are consequences of the equations of motion, and so it is sometimes convenient to treat them as conservation laws rather than constraints *per se*. On the other hand, *kinematic nonholonomic constraints* are those imposed by the kinematics, such as rolling constraints, which are constraints linear in the velocity.

There have, of course, been many classical examples of nonholonomic systems studied (we thank Hans Duistermaat for informing us of much of this history.) For example, Routh [1860] showed that a uniform sphere rolling on a surface of revolution is an integrable system (in the classical sense). Another example is the rolling disk (not necessarily vertical), which was treated in Vierkandt [1892]; this paper shows that the solutions of the equations on what we would call the reduced space (denoted $\mathcal{D}/G$ in the present paper) are all periodic. (For this example from a more modern point of view, see, for example, Hermans [1995], O'Reilly [1994] and Getz & Marsden [1994b]). A related example is the bicycle; the bicycle; see Getz & Marsden [1995]). The work of Chaplygin

[1897a] is a very interesting study of the rolling of a solid of revolution on a horizontal plane. In this case, it is also true that the orbits are periodic on the reduced space (this is proved using a nice technique of Birkhoff utilizing the reversible symmetry in HERMANS [1995]). One should note that a limiting case of this result (when the body of revolution limits to a disk) is that of Vierkandt. CHAPLYGIN [1897b], [1903] also studied the case of a rolling sphere on a horizontal plane but which may have an inhomogeneous mass distribution.

Another classical example is the wobblestone, studied in a variety of papers and books such as WALKER [1896], CRABTREE [1909], BONDI [1956]. See HERMANS [1995] BURDICK, GOODWINE & OSTROWSKI [1994] for additional information and references. In particular, the paper of Walker establishes important stability properties of relative equilibria by a spectral analysis; he shows, under rather general conditions (including the crucial one that the axes of principal curvature do not align with the inertia axes), that rotation in one direction is spectrally stable (and hence linearly and nonlinearly asymptotically stable). By time reversibility, rotation in the other direction is unstable. On the other hand, one can have a relative equilibrium with eigenvalues in both half planes, so that rotations in opposite senses about it can *both* be unstable, as Walker has shown. Presumably this is consistent with the fact that some wobblestones execute multiple reversals. However, the global geometry of this mechanism is still not fully understood analytically.

In this paper we give several examples to illustrate our approach. Some of them are rather simple and are only intended to clarify the theory. For example the vertical rolling disk and the spherical ball rolling on a rotating table are used as examples of systems with *both* dynamic and kinematic nonholonomic constraints. In either case, the angular momentum about the vertical axis is conserved; see BLOCH, REYHANOGLU & MCCLAMROCH [1992], BLOCH & CROUCH [1994], BROCKETT & DAI [1992] and YANG [1992].

A related modern example is the snakeboard (see LEWIS, OSTROWSKI, MURRAY, & BURDICK [1994]), which shares some of the features of these examples but which has a crucial difference as well. This example like many of the others, has the symmetry group $SE(2)$ of Euclidean motions of the plane. However, now, the corresponding momentum is *not* conserved. However, the *equation* satisfied by the momentum associated with the symmetry is useful for understanding the dynamics of the problem and how group motion can be generated. The nonconservation of momentum occurs even with no forces applied (besides the forces of constraint) and is consistent with the conservation of energy for these systems. In fact, nonconservation is crucial to the generation of movement in a control theoretic context.

One of the important tools of geometric mechanics is reduction theory (either Lagrangian or Hamiltonian), which provides a well developed method for dealing with dynamic constraints. In this theory the dynamic constraints and the symmetry group are used to lower the dimension of the system by constructing an associated *reduced system*. We develop the Lagrangian version of this theory for nonholonomic systems in this paper. We have focussed on Lagrangian systems because this is a convenient context for applications to control theory. Reduction theory is important for many reasons, amonst which is that it provides a context for understanding the theory of geometric phases (see KRISHNAPRASAD [1989], MARSDEN, MONTGOMERY & RATIU [1990], BLOCH, KRISHNAPRASAD, MARSDEN & SÁNCHEZ DE ALVAREZ [1992] and references therein) which, as we discuss below, is important for understanding locomotion generation.

1.1 The Utility of the Present Work

The main difference between classical work on nonholonomic systems and the present work is that this paper develops the *geometry* of mechanical systems with nonholonomic constraints and thereby provides a framework for additional control theoretic development of such systems. This paper is not a shortcut to the equations themselves; traditional approaches (such as those in ROSENBERG [1977]) will yield the equations of motion perfectly adequately. Rather, by exploring the geometry of mechanical systems with nonholonomic constraints, we seek to understand the structure of the equations of motion in a way that aids the analysis and helps to isolate the important geometric objects which govern the motion of the system.

One example of the application of this new theory is in the context of robotic locomotion. For a large class of land-based locomotion systems—included legged robots, snake-like robots, and wheeled mobile robots—it is possible to model the motion of the system using the geometric phase associated with a connection on a principal bundle (see KRISHNAPRASAD [1990], KELLY & MURRAY [1995] and references therein). By modeling the locomotion process using connections, it is possible to more fully understand the behavior of the system and in a variety of instances the analysis of the system is considerably simplified. In particular, this point of view seems to be well suited for studying issues of controllability and choice of gait. Analysis of more complicated systems, where the coupling between symmetries and the kinematic constraints is crucial to understanding locomotion, has been made possible through the basic developments in the present paper.

A specific example in which the theory developed here has been quite crucial is the analysis of locomotion for the snakeboard, which we study in some detail in Section 8.4. The snakeboard is a modified version of a skateboard in which locomotion is achieved using a coupling of the nonholonomic constraints with the symmetry properties of the system. For that system, traditional analysis of the complete dynamics of the system does not readily explain the mechanism of locomotion. Using the momentum equation which we derive in this paper, the interaction between the constraints and the symmetries becomes quite clear and the basic mechanics underlying locomotion is clarified. Indeed, even if one guessed how to add in the extra "constraint" associated with the nonholonomic momentum, without writing everything in the language of connections then things in fact appear to be much more complicated than they really are.

The locomotion properties of the snakeboard were originally studied by LEWIS, OSTROWSKI, BURDICK & MURRAY [1994] using simulations and experiments. They showed that several different gaits are achievable for the system and that these gaits involve periodic inputs to the system at integrally related frequencies. In particular, a 1:1 gait generates forward motion, a 1:2 gait generates rotation about a fixed point and and 2:3 gait generates sideways motion. Recently, using motivation based on the present approach, it has been possible to gain deeper insight into why the 2:1 and 3:2 gaits in the snakeboard generate movement that was first observed only numerically and experimentally. In the traditional framework, without the special structure that the momentum equation provides, this and similar issues would have been quite difficult. In the next subsection we will exhibit the general form of the control systems that result from the present work so that the reader can see these points a little more clearly.

Another instance where the geometry associated with nonholonomic mechanics has been useful is in analyzing controllability properties. For example, in BLOCH AND CROUCH [1994] it is shown that for a nonabelian Chaplygin control system, the principal bundle structure of the system can be used to prove that if the full system is accessible and the system is controllable on the base, the full system is controllable. This result uses earlier work of Crouch and San Martin and is nontrivial in the sense that proving controllability is generally much harder than proving accessibility. In BLOCH, REYHANOGLU & MCCLAMROCH [1992], the nonholonomic structure is used to prove accessibility results as well small time local controllability. Further, the holonomy of the connection given by the constraints is used to design both open loop and feedback controls.

A long term goal of our work is to develop the basic control theory for mechanical systems and Lagrangian systems in particular. There are several reasons why mechanical systems are good candidates for new results in nonlinear control. On the practical end, mechanical systems are often quite well identified and accurate models exist for specific systems, such as robots, airplanes, and spacecraft. Furthermore, instrumentation of mechanical systems is relatively easy to achieve and hence modern nonlinear techniques (which often rely on full state feedback) can be readily applied. We also note that the present set up suggests that some of the traditional concepts such as controllability itself may require modification. For example, one may not always require full state space controllability (in parking a car, you may not care about the orientation of your tire stems).

For ideas in this direction, see KELLY & MURRAY [1995]. These and other results in Lagrangian mechanics, including those described in this paper, have generated new insights into the control problem and are proving to be useful in specific engineering systems.

Despite being motivated by problems in robotics and control theory, the present paper does not discuss the effect of general forces. The control theory we have used as motivation deals largely with "internal forces" such as those that naturally enter into the snakeboard. While we do not systematically deal with general external forces in this paper, we do have them in mind and plan to include them in future publications. As LAM [1994] and JALNAKUPAR [1995] have pointed out, external forces acting on the system have to be treated carefully in the context of the Lagrange d'Alembert principle. Our framework is that of the traditional setup for constraint forces as described in ROSENBERG [1977]. In this framework the forces of constraint do no work and in certain cases (such as for point particles and particles and rigid bodies) the Lagrange d'Alembert equations can be derived from Newton's laws, as the preceding references show.

1.2 Control Systems in Momentum Equation Form[1]

To help clarify the link with control systems, we now discuss the general form of nonholonomic mechanical control systems with symmetry that have a nontrivial evolution of their nonholonomic momentum. The group elements for such systems generally are used to describe the overall position and attitude of the system. The dynamics are described by a system of equations having the form of a reconstruction equation for a group element g, an equation for the nonholonomic momentum p (no longer conserved in the general case), and the equations of motion for the reduced variables r which describe the "shape" of the system. In terms of these variables, the equations of motion (to be derived later) have the functional form

$$\begin{aligned} g^{-1}\dot{g} &= -A(r)\dot{r} + B(r)p && (1.2.1) \\ \dot{p} &= \dot{r}^T\alpha(r)\dot{r} + \dot{r}^T\beta(r)p + p^T\gamma(r)p && (1.2.2) \\ M(r)\ddot{r} &= -C(r,\dot{r}) + N(r,\dot{r},p) + \tau. && (1.2.3) \end{aligned}$$

The first equation describes the motion in the group variables as the flow of a left invariant vector field determined by the internal shape r, the velocity $\dot{r}$, as well as the generalized momentum p. The term $g^{-1}\dot{g}$ is related to the body angular velocity in the case that the symmetry group is the group of rigid transformations. (As we shall see later, this interpretation is not literally correct; the body angular velocity is actually the vertical part of the vector $(\dot{r}, \dot{g})$.) The momentum equation describes the evolution of p and will be shown to be bilinear in $(\dot{r}, p)$. Finally, the bottom (second-order) equation describes the motion of the variables r which describe the configuration up to a symmetry (*i.e.,* the shape). The term $M(r)$ is the mass matrix of the system, C is the Coriolis term which is quadratic in $\dot{r}$ and N is quadratic in $\dot{r}$ and $\dot{p}$. The variable τ represents the potential forces and the external forces applied to the system, which we assume here only affect the shape variables. Note that the evolution of the momentum p and the shape r *decouple* from the group variables. In this paper we shall derive a general form of the reduced Lagrange d'Alembert equations for systems with nonholonomic constraints, which the above equations illustrate. In this form of the equations, the constraints are implicit in the structure of the first equation.

The utility of this form of the equations is that it separates the dynamics into pieces consistent with the overall geometry of the system. This can be quite powerful in the context of control theory. In some locomotion systems one has full control of the shape variables r. Thus, certain questions in locomotion can be reduced to the case where $r(t)$ is specified and the properties of the system are described only by the group and momentum equations. This significantly reduces the complexity of locomotion systems with many internal degrees of freedom (such as snake-like systems).

[1] We thank Jim Ostrowski for his notes on this material, which served as a first draft of this section.

More specifically, consider the problem of determining the controllability of a locomotion system. That is, we would like to determine if it is possible for a given system to move between two specified equilibrium configurations. To understand local controllability of a locomotion system, one computes the Lie algebra of vector fields associated with the control problem. For the full problem represented by the above equations this can be an extremely detailed calculation and is often intractable except in simple examples. However, by exploiting the particular structure of the equations above, one sees that it is sufficient to ignore the details of the dynamics of the shape variables: it is enough to assume that $r(t)$ can be specified arbitrarily, for example by assuming that $\ddot{r} = u$. Using this simplification, one can show, for example, that the Lie bracket $[\,[f, g_i], g_j]$ is given by

$$[\,[f, g_i], g_j] = \begin{bmatrix} 0 \\ \alpha_{ij} \\ 0 \\ 0 \end{bmatrix}$$

where the four slots correspond to the variables $g, p, r, \dot{r}$; f is the drift vector field defined by setting the inputs to zero; g_i and g_j represent input vector fields; and α_{ij} is the ij component of the matrix α. Thus *the term α that appears in the momentum equation is directly related to controllability of the system in the momentum direction.* The fact that the Lie bracket between two of the input vector fields lies in the p direction helps explain the use of the 1:1 gait in the snakeboard example for achieving forward motion, which corresponds to building up momentum.

This point of view is described in Kelly & Murray [1995] for the case where no momentum equation is present and Ostrowski [1995] for the more general case, including the snakeboard. In fact, it was precisely this form of the equations which was used to understand some of the gait behavior present in the snakeboard example.

1.3 Outline of the Paper

In §2 we develop some of the basic features of nonholonomic systems. In particular, we show how to describe constraints using Ehresmann connections and we show how to write the equations of motion using the curvature of this connection. Moreover, a basic geometric setup is laid out that enables one to use the ideas of holonomy and geometric phases in the context of the dynamics of nonholonomic systems for the first time. Our overall philosophy is to start with the general case of Ehresmann connections, *then* add the symmetry group structure, and later specialize, for example, to purely kinematic (Chaplygin) systems or systems where the nonholonomic connection is a principal connection, when appropriate.

In §3, we begin by recalling some basic notions about symmetry of mechanical systems, and show that the Lagrangian and the dynamics drop to quotient spaces, providing the reduced dynamics. Later on, in §7 the reduced *equations* are explicitly computed. We also review principal connections in §3 and relate them to Ehresmann connections.

The equations for the momentum map that replace the usual conservation laws are derived in §4. We distinguish the cases in which one gets conservation and those in which one gets a nontrivial evolution equation for the momentum. For example, for the vertical rolling disk, one has invariance (of the Lagrangian and constraints) under rotation about the disk's vertical axis and this leads to a conservation law for the disk that, in addition to the conservation of energy shows that the system is completely integrable. This example, a constrained particle moving in three space and the snakeboard example are studied in §8. Various representations of the momentum equation are given as well and in particular, the form (1.2.2).

In §5 we review some of the basic ideas from Lagrangian reduction that will provide important motivation and ideas for the nonholonomic case. In rough outline, *Lagrangian reduction* means dropping the Euler-Lagrange equations and the associated variational principles to the quotient of the velocity phase space by the given symmetry group, which generalizes the classical Routh procedure for cyclic variables. On the other hand, in *Hamiltonian reduction* one drops the symplectic form or the Poisson brackets along with the dynamical equations to a quotient space. The reduced Euler-Lagrange equations may be derived by breaking up of the Euler-Lagrange equations into two sets that correspond to splitting variations into horizontal and vertical parts relative to the mechanical connection, a fundamental principal connection associated with the given symmetry group.

In §6, the first of two sections on nonholonomic reduction from the Lagrangian point of view, we study reconstruction and combine the connection determined by the constraints (the "kinematic connection") and that associated to the kinetic energy and the group action (the "mechanical connection"). This results in a new connection called the *nonholonomic connection* that encodes both sorts of information. This process gives equation (1.2.1).

In §7 we develop the reduced Lagrange d'Alembert equations (Theorem 7.5) which gives the equation (1.2.3) . For systems with nonholonomic constraints, the equations of motion are associated with the horizontal variations relative to the Ehresmann connection associated with the constraints. This shows why there is such a similarity between the equations of a nonholonomic system and the first set of reduced Euler-Lagrange equations, as we shall see explicitly. In the general case with both symmetries and nonholonomic constraints we use the nonholonomic connection and relative to it, the reduced equations will break up into *two* sets: a set of reduced Euler-Lagrange equations (1.2.3) (which have curvature terms appearing as "forcing"), and a momentum equation (1.2.2) , which have a form generalizing the components of the Euler-Poincaré equations along the symmetry directions consistent with the constraints. When one supplements these equations with the reconstruction equations (1.2.1), and the constraint equations, one recovers the full set of equations of motion for the system.

In §8 we consider some examples that illustrate the theory, namely the vertical rolling disk, a nonholonomically constrained particle in 3-space, a homogeneous sphere on a rotating table, and the snakeboard. The conclusions give some suggestions for future work in this area.

1.4 Summary of the Main Results

- The development of a general setting for nonholonomic systems using the theory of Ehresmann connections and the derivation of the Lagrange d'Alembert equations as Euler-Lagrange equations on the base space in the presence of curvature forces. The constraints are viewed as a distribution $\mathcal{D} \subset TQ$ and the distribution is regarded as the horizontal space for an Ehresmann connection, which we call the kinematic connection. Both linear and affine constraints are studied.

- Furthering the basic framework for the theory of nonholonomic systems with symmetry with control theoretic goals in mind. In particular, a symmetry group G is that acts on the configuration space and for which the Lagrangian is invariant is systematically studied.

- The derivation of a momentum equation for nonholonomic systems with symmetry. We show that this equation implies, in particular, the standard conservation laws for nonholonomic systems. However, the general momentum equation allows for important cases in which the momentum equation is not conserved. This case is well illustrated by the snakeboard example. The *nonconservation* of momentum plays an important role in locomotion generation.

- The momentum equation is written in a variety of forms that bring out different geometric and dynamic features. For example, some forms involve the covariant derivative (relative to a certain natural connection) of the momentum. The momentum equations are also closely related to the Euler-Poincaré equations.

- A connection, called the nonholonomic connection, which synthesizes the mechanical connection and the kinematic connection is introduced. In many cases of control theoretic interest, even though the kinematic connection is not principal (*i.e.,* the system is *not* Chaplygin), the nonholonomic connection *is* principal and this is the case we concentrate on.

- The reduced equations on the space $\mathcal{D}/G$ are calculated and a comparison with the theory of Lagrangian reduction is made.
- Several examples, including the vertical rolling disk, a constrained particle, the rolling ball on a rotating turntable, and the snakeboard are all treated in some detail to illustrate the theory.

2 Constraint Distributions and Ehresmann Connections

We shall first consider mechanics in the presence of (linear and affine) nonholonomic velocity constraints and develop its geometry. For the moment, no assumptions on any symmetry are made; rather we prefer to add such assumptions separately and will do so in the following sections.

2.1 The Lagrange d'Alembert Principle

The starting point is a configuration space Q and a distribution $\mathcal{D}$ that describes the kinematic constraints of interest. Here, $\mathcal{D}$ is a collection of linear subspaces denoted $\mathcal{D}_q \subset T_qQ$, one for each $q \in Q$. A curve $q(t) \in Q$ will be said to *satisfy the constraints* if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t. This distribution will, in general, be nonintegrable; *i.e.,* the constraints are, in general, nonholonomic. One of our goals below will be to model the constraints in terms of Ehresmann connections (see Cardin & Favretti [1994] and Marle [1994] for some related ideas).

The above setup describes *linear* constraints; for *affine* constraints, for example, a ball on a rotating turntable (where the rotational velocity of the turntable represents the affine part of the constraints), we assume that there is a given vector field V_0 on Q and the constraints are written $\dot{q}(t) - V_0(q(t)) \in \mathcal{D}_{q(t)}$. We will explicitly discuss the affine case at various points in the paper and the example of the ball on a rotating table will be treated in detail.

Consider a Lagrangian $L : TQ \to \mathbb{R}$. In coordinates $q^i, i = 1, \dots, n$, on Q with induced coordinates $(q^i, \dot{q}^i)$ for the tangent bundle, we write $L(q^i, \dot{q}^i)$. The equations of motion are given by the by the Lagrange d'Alembert principle (see, for example, Rosenberg [1977] for a discussion).

Definition 2.1 *The **Lagrange d'Alembert equations of motion** for the system are those determined by*

$$\delta \int_a^b L(q^i, \dot{q}^i)\, dt = 0, \tag{2.1.1}$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(t) \in \mathcal{D}_q(t)$ for each $t, a \leq t \leq b$.

This principle is supplemented by the condition that the curve itself satisfies the constraints. In such a principle, we follow standard procedure and take the variation *before* imposing the constraints; that is, we *do not* impose the constraints on the family of curves defining the variation. The usual arguments in the calculus of variations show that this constrained variational principle is equivalent to the equations

$$-\delta L = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0, \tag{2.1.2}$$

for all variations δq such that $\delta q \in \mathcal{D}_q$ at each point of the underlying curve $q(t)$.

To explore the structure of these equations in more detail, consider a mechanical system evolving on a configuration space Q with a given Lagrangian $L : TQ \to \mathbb{R}$ and let $\{\omega^a\}$ be a set of p independent one forms whose vanishing describes the constraints on the system. The constraints in general are nonintegrable. Choose a local coordinate chart and a local basis for the constraints such that

$$\omega^a(q) = ds^a + A^a_\alpha(r, s) dr^\alpha \qquad a = 1, \dots, p, \tag{2.1.3}$$

where $q = (r, s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$.

The equations of motion for the system are given by (2.1.2) where we choose variations $\delta q(t)$ that satisfy the condition $\omega^a(q) \cdot \delta q = 0$, *i.e.,* where the variation $\delta q = (\delta r, \delta s)$ satisfies $\delta s^a + A^a_\alpha \delta r^\alpha = 0$. Substitution into (2.1.2) gives

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha}\right) = A^a_\alpha \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a}\right), \qquad \alpha = 1, \ldots, n-p. \tag{2.1.4}$$

Equation (2.1.4) combined with the constraint equations

$$\dot{s}^a = -A^a_\alpha \dot{r}^\alpha \qquad a = 1, \ldots, p \tag{2.1.5}$$

gives a complete description of the equations of motion of the system.

We now define the "constrained" Lagrangian by substituting the constraints (2.1.5) into the Lagrangian:

$$L_c(r^\alpha, s^a, \dot{r}^\alpha) = L(r^\alpha, s^a, \dot{r}^\alpha, -A^a_\alpha(r, s)\dot{r}^\alpha).$$

The equations of motion can be written in terms of the constrained Lagrangian in the following way, as a direct coordinate calculation (given in Remark 3 below) shows:

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta}\dot{r}^\beta, \tag{2.1.6}$$

where

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a}\right). \tag{2.1.7}$$

Letting $d\omega^b$ be the exterior derivative of ω^b, another computation (see Remark 4 below) shows that

$$d\omega^b(\dot{q}, \cdot) = B^b_{\alpha\beta}\dot{r}^\alpha dr^\beta$$

and hence the equations of motion have the form

$$-\delta L_c = \left(\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a}\right)\delta r^a = -\frac{\partial L}{\partial \dot{s}^b} d\omega^b(\dot{q}, \delta r).$$

This form of the equations isolates the effects of the constraints, and shows that in the case where the constraints are integrable ($d\omega = 0$) then the correct equations of motion are obtained by substituting the constraints into the Lagrangian and setting the variation of L_c to zero. However in the non-integrable case the constraints generate extra (curvature) forces, which must be taken into account.

2.2 Ehresmann Connections

The above coordinate results can be put into an interesting and useful geometric framework. To carry this out, we first develop the notion of an Ehresmann connection. A general reference for Ehresmann connections is Marsden, Montgomery & Ratiu [1990], where many additional references may be found.

First of all, we *assume* that there is a bundle structure $\pi_{Q,R} : Q \to R$ for our space Q; that is, there is another manifold R called the *base* and a map $\pi_{Q,R}$ which is a submersion (its derivative $T_q\pi_{Q,R}$ is onto at each point $q \in Q$). We call the kernel of $T_q\pi_{Q,R}$ at any point the *vertical space* and denote it by V_q.

Definition 2.2 *An* ***Ehresmann connection*** *A is a vertical valued one form on Q that satisfies*

1. *$A_q : T_qQ \to V_q$ is a linear map for each point $q \in Q$*

2. *A is a projection:* $A(v_q) = v_q$ *for all* $v_q \in V_q$.

Note that these conditions imply that $T_qQ = V_q \oplus H_q$ where $H_q = \ker A_q$ is the *horizontal space at* q. We will sometimes write hor_q for the horizontal space. Thus, an Ehresmann connection gives us a way to split the tangent space to Q at each point into a horizontal and vertical part; for example, we can speak about projecting a tangent vector onto its vertical part using the connection. Notice also that the vertical space at q, namely V_q, is tangent to the *vertical fiber* $\mathcal{V}_q$, which consists of all points that get sent by the projection $\pi_{Q,R}$, to the same point as q. This situation is illustrated in Figure 2.2.1.

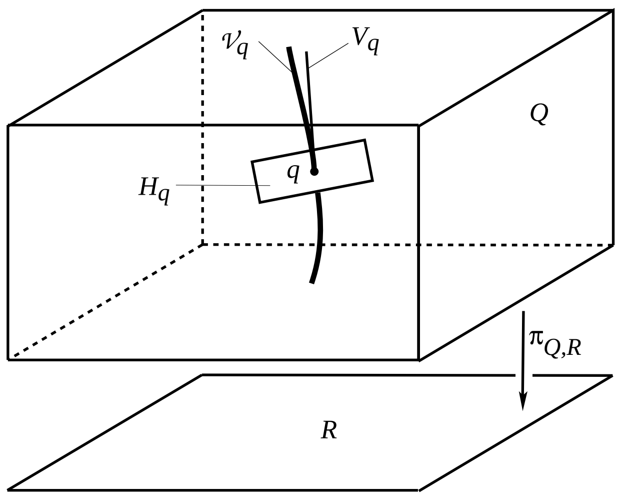

Figure 2.2.1: An Ehresmann connection specifies a horizontal subspace at each point.

We now assume that we choose the Ehresmann connection in such a way that the given constraint distribution $\mathcal{D}$ is the horizontal space of the connection; that is, $H_q = \mathcal{D}_q$. We emphasize that the choice of the bundle $\pi_{Q,R}$ is not unique and that the formulation of the Lagrange d'Alembert principle does not depend on this choice. However, it is clear that once the bundle structure $\pi_{Q,R}$ is chosen (*i.e.,* what the base and fiber variables are), the constraint distribution uniquely determines the connection. We also caution the reader that later on, when the assumption of symmetry is added to this context, it may affect the choice of bundle and the connection will get modified.

Using the bundle coordinates $q^i = (r^\alpha, s^a)$ described earlier, the coordinate representation of the projection $\pi_{Q,R}$ is just projection onto the factor r and the connection A can be represented locally by a vector valued differential form which we shall denote ω^a:

$$A = \omega^a \frac{\partial}{\partial s^a}, \qquad \omega^a(q) = ds^a + A^a_\alpha(r,s)dr^\alpha.$$

The exterior derivative of A is not defined (since it is a vertical valued form, not a differential form), but we can, at least locally in coordinates, take the exterior derivative of ω^a. In fact, this will give an easy way to compute the curvature of the connection A, as we shall see shortly.

Given an Ehresmann connection A, a point $q \in Q$ and a vector $v_r \in T_rR$ tangent to the base at a point $r = \pi_{Q,R}(q) \in R$, we can define the horizontal lift of v_r to be the unique vector v_r^h in H_q that projects to v_r under $T_q\pi_{Q,R}$. If we have a vector $X_q \in T_qQ$, we shall also write its horizontal part as

$$\mathrm{hor}\, X_q = X_q - A(q) \cdot X_q.$$

In coordinates, the vertical projection is the map

$$(\dot r^\alpha, \dot s^a) \mapsto (0, \dot s^a + A^a_\alpha(r,s)\dot r^\alpha) \tag{2.2.1}$$

while the horizontal projection is the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (\dot{r}^\alpha, -A^a_\alpha(r,s)\dot{r}^\alpha). \tag{2.2.2}$$

Next, we recall the basic notion of curvature.

Definition 2.3 *The* ***curvature*** *of A is the vertical valued two form B on Q defined by its action on two vector fields X and Y on Q by*

$$B(X,Y) = -A([\mathrm{hor}\, X, \mathrm{hor}\, Y])$$

where the bracket on the right hand side is the Jacobi-Lie bracket of vector fields obtained by extending the stated vectors to vector fields.

Notice that this definition shows that the curvature exactly measures the failure of the constraint distribution to be an integrable bundle.

A useful standard identity for the exterior derivative $\mathbf{d}\alpha$ of a one form α (which could be vector space valued) on a manifold M acting on two vector fields X, Y is

$$(\mathbf{d}\alpha)(X,Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X,Y]).$$

This identity shows that *in coordinates*, one can evaluate the curvature by writing the connection as a form ω^a in coordinates, computing its exterior derivative (component by component) and restricting the result to horizontal vectors, that is, to the constraint distribution. In other words,

$$B(X,Y) = d\omega^a(\mathrm{hor}\, X, \mathrm{hor}\, Y)\frac{\partial}{\partial s^a},$$

so that the local expression for curvature is given by

$$B(X,Y)^a = B^a_{\alpha\beta} X^\alpha Y^\beta \tag{2.2.3}$$

where the coefficients $B^a_{\alpha\beta}$ are given by (2.1.7).

2.3 Intrinsic Formulation of the Equations

We can now rephrase our coordinate computations from §2.1 in the language of Ehresmann connections. We shall do this first for systems with homogeneous constraints and then treat the affine case.

Homogeneous Constraints

Let A be an Ehresmann connection on a given bundle such that the constraint distribution $\mathcal{D}$ is given by the horizontal subbundle associated with A. The constrained Lagrangian can be written as

$$L_c(q, \dot{q}) = L(q, \mathrm{hor}\, \dot{q})$$

and we have the following theorem.

Theorem 2.4 *The Lagrange d'Alembert equations may be written as the equations*

$$\delta L_c = \langle \mathbb{F}L, B(\dot{q}, \delta q)\rangle,$$

where $\langle\, , \rangle$ denotes the pairing between a vector and a dual vector and where

$$\delta L_c = \left\langle \delta q^\alpha, \frac{\partial L_c}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L_c}{\partial \dot{q}^\alpha} \right\rangle,$$

in which δq is a horizontal variation (i.e., it takes values in the horizontal space) and B is the curvature regarded as a vertical valued two form, in addition to the constraint equations

$$A(q) \cdot \dot{q} = 0.$$

This theorem follows from the way that the constraints restrict $\dot{q}$ and the fact that the Lagrange d'Alembert principle requires δq to be horizontal. This formulation depends on a specific choice of connection, and there is some freedom in this choice. However, as we will see later, the freedom can be removed in many cases for systems with symmetry.

Affine Constraints

We next consider the modifications necessary to allow affine constraints of the form

$$A(q) \cdot \dot{q} = \gamma(q, t)$$

where A is an Ehresmann connection as described above and $\gamma(q, t)$ is vertical valued. The expression γ here is related to the vector field V_0 given above by $\gamma(q) = A(q) \cdot V_0(q)$. Affine constraints arise, for example, in studying the motion of a ball on a spinning turntable. Since the turntable is moving underneath the ball, the velocity in the constraint directions is not zero, but is instead determined by the position of the ball on the turntable and the angular velocity of the turntable.

Since $\gamma(q, t)$ is vertical, we can define the covariant derivative of γ as

$$D\gamma(X) = \mathrm{ver}\,[\mathrm{hor}\, X, \gamma]$$

(see MARSDEN, MONTGOMERY & RATIU [1990]). Relative to bundle coordinates $q = (r, s)$, we write γ as

$$\gamma(q, t) = \gamma^a(q, t) \frac{\partial}{\partial s^a}$$

and the covariant derivative along a *horizontal* vector field

$$X = X^\alpha \left(\frac{\partial}{\partial r^\alpha} - A^a_\alpha \frac{\partial}{\partial s^a} \right)$$

is given by

$$D\gamma(X) = X^\alpha \left(\frac{\partial \gamma^a}{\partial r^\alpha} - A^b_\alpha \frac{\partial \gamma^a}{\partial s^b} + \gamma^b \frac{\partial A^a_\alpha}{\partial s^b} \right) \frac{\partial}{\partial s^a} =: \gamma^a_\alpha X^\alpha \frac{\partial}{\partial s^a},$$

which defines the symbols γ^a_α.

We now define the constrained Lagrangian as

$$L_c(q, \dot{q}, t) = L\left(q, \mathrm{hor}\, \dot{q} + \gamma(q, t)\right).$$

A long calculation, similar to what we have already carried out in the case of linear constraints, shows that the dynamics have the form

$$\left.\begin{aligned} \delta L_c &= \langle \mathbb{F}L, B(\dot{q}, \delta q) \rangle + \langle \mathbb{F}L, D\gamma(\delta q) \rangle \\ A(q) \cdot \dot{q} &= \gamma(q, t) \end{aligned}\right\} \tag{2.3.1}$$

where the δq are restricted to satisfy $A(q) \cdot \delta q = 0$. In coordinates, the first of these equations reads as follows:

$$\frac{d}{dt} \frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta} \dot{r}^\beta - \frac{\partial L}{\partial \dot{s}^a} \gamma^a_\alpha, \tag{2.3.2}$$

while the second reads as $\dot{s}^a + A^a_\alpha \dot{r}^\alpha = \gamma^a$. Notice that these equations show how, in the affine case, the covariant derivative of the affine part γ enters into the description of the system; in particular, note that the covariant derivative in (2.3.1) is with respect to the configuration variables and not with respect to the time.

Remarks

1. For a mechanical system with homogeneous nonholonomic constraints, conservation of energy holds: along a solution, the energy function

$$E_c(r^\alpha, \dot{r}^\alpha, s^a) = \frac{\partial L_c}{\partial \dot{r}^\alpha}\dot{r}^\alpha - L_c(r^\alpha, \dot{r}^\alpha, s^a)$$

is constant in time as is readily verified. (In the affine case, one requires the condition $(\partial L/\partial \dot{s}^a)\gamma^a_\alpha \dot{r}^\alpha = 0$). On the other hand, unlike the usual Euler-Lagrange equations for systems with holonomic constraints, the Lagrange-d'Alembert equations need not preserve the symplectic form along orbits; its rate of change involves the curvature terms. This phenomenon is related to Hamiltonian formulations of the problem and the failure of the Jacobi identity (see Bates & Sniatycki [1992]); this aspect is not discussed further in the present paper.

2. Dynamics in the presence of external forces, which of course is important for control theoretic purposes, will be treated more fully in a forthcoming article; see also Yang [1992], Yang, Krishnaprasad & Dayawansa [1993] and Bloch, Krishnaprasad, Marsden & Ratiu [1994a]. Briefly, we represent forces as mappings which take values in T^*Q and can depend on configuration, velocity, and time; that is, forces are maps $F : TQ \times \mathbb{R} \to T^*Q$, which are bundle maps (take tangent vectors to q to covectors also at q). Let $F(q, \dot{q}, t) \in T^*Q$ represent the external forces on the system and take all other quantities as described above. From the Lagrange d'Alembert equations, the motion of the system is given by

$$\delta L_c = \langle \mathbb{F}L, B(\dot{q}, \delta q)\rangle - \langle F, \delta q\rangle.$$

Systems with forces can be extended to the case of affine constraints case by adding exactly the extra term in equation (2.3.1).

3. The derivation of the equations of motion in terms of the constrained Lagrangian proceeds as follows: using the relationships

$$\begin{aligned}
\frac{\partial L_c}{\partial \dot{r}^\alpha} &= \frac{\partial L}{\partial \dot{r}^\alpha} - A^b_\alpha \frac{\partial L}{\partial \dot{s}^b}\\
\frac{\partial L_c}{\partial r^\alpha} &= \frac{\partial L}{\partial r^\alpha} - \frac{\partial L}{\partial \dot{s}^b}\left(\frac{\partial A^b_\beta}{\partial r^\alpha}\dot{r}^\beta\right)\\
\frac{\partial L_c}{\partial s^a} &= \frac{\partial L}{\partial s^a} - \frac{\partial L}{\partial \dot{s}^b}\left(\frac{\partial A^b_\beta}{\partial s^a}\dot{r}^\beta\right)
\end{aligned}$$

and substituting L_c into Lagrange's equations (2.1.2) yields

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a}\\
&\quad= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha}\right) - A^a_\alpha\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a}\right) - \frac{\partial L}{\partial \dot{s}^b}\frac{d}{dt}A^b_\alpha + \frac{\partial L}{\partial \dot{s}^b}\frac{\partial A^b_\beta}{\partial r^\alpha}\dot{r}^\beta - A^a_\alpha \frac{\partial L}{\partial \dot{s}^b}\frac{\partial A^b_\beta}{\partial s^a}\dot{r}^\beta\\
&\quad= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha}\right) - A^a_\alpha\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a}\right) + \frac{\partial L}{\partial \dot{s}^b}\left(\frac{\partial A^b_\beta}{\partial r^\alpha} - \frac{\partial A^b_\alpha}{\partial r^\beta} + A^a_\beta\frac{\partial A^b_\alpha}{\partial s^a} - A^a_\alpha\frac{\partial A^b_\beta}{\partial s^a}\right)\dot{r}^\beta.
\end{aligned}$$

Hence the equations of motion can be written as (2.1.6).

Note that L_c is a degenerate Lagrangian in the sense that it does not depend on $\dot{s}$. Also note that thinking of s as a cyclic variable does not lead to conservation laws in the usual way because of the constraints.

4. To see how the right hand side of the constrained Lagrange d'Alembert equation (2.1.6) is related to the curvature of the Ehresmann connection of $A = \omega^a(\partial/\partial s^a)$, let $d\omega^b$ be the exterior derivative of ω^b:

$$\begin{aligned} d\omega^b &= d(ds^b + A^b_\alpha dr^\alpha) \\ &= \frac{\partial A^b_\alpha}{\partial r^\beta} dr^\beta \wedge dr^\alpha - \frac{\partial A^b_\alpha}{\partial s^a} A^a_\beta dr^\beta \wedge dr^\alpha. \end{aligned} \tag{2.3.3}$$

Contracting $d\omega^b$ with $\dot{q}$ yields

$$\begin{aligned} d\omega^b(\dot{q}, \cdot) &= \frac{\partial A^b_\alpha}{\partial r^\beta} \dot{r}^\beta dr^\alpha - \frac{\partial A^b_\alpha}{\partial s^a} A^a_\beta \dot{r}^\beta dr^\alpha - \frac{\partial A^b_\alpha}{\partial r^\beta} \dot{r}^\alpha dr^\beta + \frac{\partial A^b_\alpha}{\partial s^a} A^a_\beta \dot{r}^\alpha dr^\beta \\ &= \left(\frac{\partial A^b_\alpha}{\partial r^\beta} + \frac{\partial A^b_\beta}{\partial s^a} A^a_\alpha - \frac{\partial A^b_\beta}{\partial r^\alpha} - \frac{\partial A^b_\alpha}{\partial s^a} A^a_\beta \right) \dot{r}^\beta dr^\alpha \\ &= B^b_{\alpha\beta} \dot{r}^\alpha dr^\beta. \end{aligned} \tag{2.3.4}$$

Combining all of these calculations, we can write the equations of motion for the constrained system as

$$\frac{d}{dt} \frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^a} d\omega^a \left(\dot{q}, \frac{\partial}{\partial r^\alpha} \right). \tag{2.3.5}$$

The left-hand side of (2.3.5) may be checked to be the variational derivative of the constrained Lagrangian. The right-hand side consists of the forces that maintain the constraints. In the special case that the constraints are holonomic, $d\omega^a = 0$ since $d\omega^a$ represents the curvature and the curvature measures the lack of integrability of the constraints; when they are integrable, we have, by definition, the holonomic case. In this case, equation (2.3.5) reduces to the usual form of Lagrange's equations. This verifies that for holonomic systems it is appropriate to "plug in the constraints" before applying Lagrange's equations.

Specific examples of the computation of the dynamics using the formulation in this section are given in §8.

3 Systems with Symmetry

We now add symmetry to our nonholonomic system. We will begin with some general remarks about symmetry, review some facts about principal connections and then treat a special case that we call the principal kinematic case (sometimes called the Chaplygin case) both for completeness and to set the stage for the more general main results to follow.

3.1 Group Actions and Invariance

We refer the reader to Marsden & Ratiu [1994], Chapter 9 for the basic definitions and examples of Lie groups and group actions for what follows. Assume that we are given a Lie group G and an action of G on Q. The action of G will be denoted $q \mapsto gq = \Phi_g(q)$. The group orbit through a point q, which is always an (immersed) submanifold, is denoted

$$\mathrm{Orb}(q) := \{gq \mid g \in G\}.$$

When there is danger of confusion about which group is meant, we write the orbit as $\mathrm{Orb}_G(q)$.

Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G . For an element $\xi \in \mathfrak{g}$, we write ξ_Q, a vector field on Q for the corresponding infinitesimal generator; recall that this is obtained by differentiating the flow $\Phi_{\exp(t\xi)}$ with respect to t at $t = 0$. The tangent space to the group orbit through a point q is given by the set of infinitesimal generators at that point:

$$T_q(\mathrm{Orb}(q)) = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}.$$

Throughout this paper we make the assumption that the action of G on Q is free (none of the maps Φ_g has any fixed points) and proper (the map $(q, g) \mapsto gq$ is proper; *i.e.,* that is, the inverse images of compact sets are compact). The case of nonfree actions is very important and the investigation of the associated singularities needs to be carried out, but that topic is not the subject of the present paper.

The quotient space $M = Q/G$, whose points are the group orbits, is called ***shape space***. It is known that if the group action is free and proper then shape space is a smooth manifold and the projection map $\pi : Q \to Q/G$ is a smooth surjective map with a surjective derivative $T_q\pi$ at each point. We will denote the projection map by $\pi_{Q,G}$ if there is any danger of confusion. The kernel of the linear map $T_q\pi$ is the set of infinitesimal generators of the group action at the point q, *i.e.,*

$$\ker T_q\pi = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\},$$

so these are also the tangent spaces to the group orbits.

We now introduce some assumptions concerning the relation between the given group action, the Lagrangian, and the constraint distribution.

Definition 3.1

(L1) *We say that the Lagrangian is* ***invariant*** *under the group action if L is invariant under the induced action of G on TQ.*

(L2) *We say that the Lagrangian is* ***infinitesimally invariant*** *if for any Lie algebra element $\xi \in \mathfrak{g}$ we have $dL \circ \dot{\xi}_Q = 0$ where, for a vector field X on Q, $\dot{X}$ denotes the vector field on TQ naturally induced by it (if F_t is the flow of X then the flow of $\dot{X}$ is TF_t).*

(S1) *We say that the distribution $\mathcal{D}$ is* ***invariant*** *if the subspace $\mathcal{D}_q \subset T_qQ$ is mapped by the tangent of the group action to the subspace $\mathcal{D}_{gq} \subset T_{gq}Q$.*

(S2) *An Ehresmann connection A on Q (that has $\mathcal{D}$ as its horizontal distribution) is* ***invariant*** *under G if the group action preserves the bundle structure associated with the connection (in particular, it maps vertical spaces to vertical spaces) and if, as a map from TQ to the vertical bundle, A is G-equivariant.*

(S3) *A Lie algebra element ξ is said to act* ***horizontally*** *if $\xi_Q(q) \in \mathcal{D}_q$ for all $q \in Q$.*

Some relationships between these conditions are as follows: condition (L1) implies (L2), as is obtained by differentiating the invariance condition. It is also clear that condition (S2) implies the condition (S1) since the invariance of the connection A implies that the group action maps its kernel to itself. Condition (S1) may be stated as follows:

$$T_q\Phi_g \cdot \mathcal{D}_q = \mathcal{D}_{gq} \tag{3.1.1}$$

In the case of affine constraints, we will explicitly state when we need the assumption that the vector field γ be invariant under the action.

To help explain condition (S1), we will rewrite it in infinitesimal form. Let $\mathfrak{X}_{\mathcal{D}}$ be the space of sections X of the distribution $\mathcal{D}$; that is, the space of vector fields X that take values in $\mathcal{D}$. The condition (S1) implies that for each $X \in \mathfrak{X}_{\mathcal{D}}$, we have $\Phi_g^* X \in \mathfrak{X}_{\mathcal{D}}$. Here, $\Phi_g^* X$ denotes the pull back of the vector field X under the map Φ_g. Differentiation of this condition with respect to g proves the following result.

Proposition 3.2 *Assume that condition* (S1) *holds and let* X *be a section of* $\mathcal{D}$. *Then, for each Lie algebra element* ξ, *we have*

$$[\xi_Q, X] \in \mathfrak{X}_{\mathcal{D}} \tag{3.1.2}$$

which we also write as

$$[\xi_Q, \mathfrak{X}_{\mathcal{D}}] \subset \mathfrak{X}_{\mathcal{D}}.$$

3.2 Reduced Lagrange d'Alembert Systems

We now explain in general terms how one forms reduced systems by eliminating the group variables. Later on, we will compute the associated reduced equations explicitly and will also show how to reconstruct the group variables. We confine ourselves to linear constraints for the moment.

Proposition 3.3 *Under assumptions (L1) and (S1), we can form the* ***reduced velocity phase space*** TQ/G *and the* ***constrained reduced velocity phase space*** $\mathcal{D}/G$. *The Lagrangian* L *induces well defined functions, the* ***reduced Lagrangian***

$$l : TQ/G \to \mathbb{R}$$

satisfying $L = l \circ \pi_{TQ}$ *where* $\pi_{TQ} : TQ \to TQ/G$ *is the projection, and the* ***constrained reduced Lagrangian***

$$l_c : \mathcal{D}/G \to \mathbb{R},$$

which satisfies $L|\mathcal{D} = l_c \circ \pi_{\mathcal{D}}$ *where* $\pi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}/G$ *is the projection. Also, the Lagrange d'Alembert equations induce well defined* ***reduced Lagrange d'Alembert equations*** *on* $\mathcal{D}/G$. *That is, the vector field on the manifold* $\mathcal{D}$ *determined by the Lagrange d'Alembert equations (including the constraints) is* G*-invariant, and so defines a reduced vector field on the quotient manifold* $\mathcal{D}/G$.

This proposition follows from general symmetry considerations. For example, to get the constrained reduced Lagrangian l_c we restrict the given Lagrangian to the distribution $\mathcal{D}$ and then use its invariance to pass to the quotient. The problem of constrained Lagrangian reduction is the detailed determination of these reduced structures and will be dealt with later. The special case in which there are no constraints (that is, the case in which $\mathcal{D} = TQ$) will be reviewed in §5.

We make a few more general remarks and constructions before proceeding. In studying the reduced Lagrangian l, the space TQ/G, (which was studied in Marsden & Scheurle [1993]) is itself important. As explained above, we let the natural projection map associated with the action of G be denoted $\pi : Q \to Q/G$. We will let bundle coordinates be denoted (r, g) where r is a coordinate in the base, or shape space Q/G, and where g is a group coordinate. Such a local trivialization is characterized by the fact that in such coordinates, the group does not act on the factor r but acts on the group coordinate by *left* translations. Thus, *locally in the base,* the space Q is isomorphic to the product $Q/G \times G$ and in this *local trivialization*, the map π becomes the projection onto the first factor.

The space $(TQ)/G$, is a vector bundle over $T(Q/G)$ with fiber isomorphic to $\mathfrak{g}$, with the projection from $(TQ)/G$ to $T(Q/G)$ being the map induced by $T\pi$, the tangent of the projection. In other words, for $v_q \in T_qQ$, the map $[v_q] \mapsto T\pi(v_q)$ is well defined, independent of the chosen representative v_q of the equivalence class, as is easily checked. In a local trivialization of the bundle π with coordinates $q = (r, g)$, induced coordinates for the bundle $(TQ)/G \to T(Q/G)$ are given by $(r, \dot{r}, \xi)$, where $\xi = g^{-1}\dot{g}$. The bundle projection in these coordinates is simply the projection onto the first two factors.

In these coordinates, the reduced Lagrangian l is easy to understand. Namely, the Lagrangian L as a function $L(r, g, \dot{r}, \dot{g})$ is invariant under the left action of G and so it depends on g and $\dot{g}$ only through the combination $\xi = g^{-1}\dot{g}$. Thus, the induced function l is given in this local trivialization by

$$l(r, \dot{r}, \xi) = L(r, g, \dot{r}, \dot{g}). \tag{3.2.1}$$

To write out the *constrained* reduced Lagrangian l_c in coordinates requires a coordinate description of the constraints, using, for example, an Ehresmann connection, including a choice of bundle $\pi_{Q,R} : Q \to R$. This bundle and the bundle $\pi : Q \to Q/G$ *need not coincide in general*. As we shall see in the next subsection, there is a well developed theory dealing with the bundle $\pi : Q \to Q/G$ with a point of view that is rather different from that we have already presented utilizing Ehresmann connections. One of our goals is to eventually synthesize these two points of view. In the special case in which these two bundles coincide, which we will call the principal kinematic case, there is no ambiguity. To describe it in more detail we will need the notion of a principle connection.

3.3 Principal Connections

We now recall, for the convenience of the reader and to set notation and conventions, the notion of a principal connection. The reader who is consulting Kobayashi & Nomizu [1963] will notice that there are various factors of 2 and minus signs that are different from what we have here. These are due to the different conventions that various authors use for the wedge product and the exterior derivative and the fact that we use left actions for our default, whereas much of the literature assumes one has right actions. We follow the most common "Bourbaki" conventions for the wedge product, as in Abraham, Marsden & Ratiu [1988].

As above, we start with a free and proper group action of a Lie group on a manifold Q and construct the projection map $\pi : Q \to Q/G$; this setup is also referred to as a ***principal bundle***. The kernel $\ker T_q\pi$ (the tangent space to the group orbit through q) is called the vertical space of the bundle at the point q and is denoted by ver_q.

Definition 3.4 *A* ***principal connection*** *on the principle bundle $\pi : Q \to Q/G$ is a map (referred to as the connection form) $\mathcal{A} : TQ \to \mathfrak{g}$ that is linear on each tangent space (i.e., $\mathcal{A}$ is a $\mathfrak{g}$-valued one form) and is such that*

1. $\mathcal{A}(\xi_Q(q)) = \xi$ *for all* $\xi \in \mathfrak{g}$ *and* $q \in Q$, *and*
2. $\mathcal{A}$ *is equivariant:*
$$\mathcal{A}(T_q\Phi_g(v_q)) = \mathrm{Ad}_g \mathcal{A}(v_q)$$
for all $v_q \in T_qQ$ and $g \in G$, where Φ_g denotes the given action of G on Q and where Ad denotes the adjoint action of G on $\mathfrak{g}$.

The *horizontal space* of the connection at a point $q \in Q$ is the linear space

$$\mathrm{hor}_q = \{v_q \in T_qQ \mid \mathcal{A}(v_q) = 0\}.$$

Thus, at any point, we have the decomposition

$$T_qQ = \mathrm{hor}_q \oplus \mathrm{ver}_q.$$

Often one finds connections defined by specifying the horizontal spaces (complementary to the vertical spaces) at each point and requiring that they transform correctly under the group action. In particular, notice that a connection is uniquely determined by the specification of its horizontal spaces, a fact that we will use later on. We will denote the projections onto the horizontal and vertical spaces relative to the above decomposition using the same notation; thus, for $v_q \in T_qQ$, we write

$$v_q = \mathrm{hor}_q v_q + \mathrm{ver}_q v_q.$$

The projection onto the vertical part is given by

$$\mathrm{ver}_q v_q = (\mathcal{A}(v_q))_Q(q)$$

and the projection to the horizontal part is thus

$$\mathrm{hor}_q v_q = v_q - (\mathcal{A}(v_q))_Q(q).$$

The projection map at each point defines an isomorphism from the horizontal space to the tangent space to the base; its inverse is called the *horizontal lift.* Using the uniqueness theory of ODE's one finds that a curve in the base passing through a point $\pi(q)$ can be lifted uniquely to a horizontal curve through q in Q (*i.e.,* a curve whose tangent vector at any point is a horizontal vector).

Since we have a splitting, we can also regard a principal connection as a special type of Ehresmann connection. However, Ehresmann connections are regarded as vertical valued forms whereas principal connections are regarded as Lie algebra valued. Thus, the Ehresmann connection A and the connection one form $\mathcal{A}$ are different and we will distinguish them; they are related in this case by

$$A(v_q) = (\mathcal{A}(v_q))_Q(q).$$

The general notions of curvature and other properties which hold for general Ehresmann connections specialize to the case of principal connections. As in the general case, given any vector field X on the base space (in this case, the shape space), using the horizontal lift, there is a unique vector field X^h that is horizontal and that is π-related to X; that is, at each point q, we have

$$T_q\pi \cdot X^h(q) = X(\pi(q))$$

and the vertical part is zero:

$$(\mathcal{A}(X^h_q))_Q(q) = 0.$$

It is well known (see, for example, Abraham, Marsden & Ratiu [1988]) that the relation of being π-related is bracket preserving; in our case, this means that

$$\mathrm{hor}\,[X^h, Y^h] = [X, Y]^h,$$

where X and Y are vector fields on the base.

Definition 3.5 *The* ***covariant exterior derivative*** $\mathbf{D}$ *of a Lie algebra valued one form* α *is defined by applying the ordinary exterior derivative* $\mathbf{d}$ *to the horizontal parts of vectors:*

$$\mathbf{D}\alpha(X, Y) = \mathbf{d}\alpha(\mathrm{hor}\,X, \mathrm{hor}\,Y).$$

The ***curvature*** *of a connection* $\mathcal{A}$ *is its covariant exterior derivative and it is denoted by* $\mathcal{B}$.

Thus, $\mathcal{B}$ is the Lie algebra valued two form given by

$$\mathcal{B}(X, Y) = \mathbf{d}\mathcal{A}(\mathrm{hor}\,X, \mathrm{hor}\,Y).$$

Using the identity

$$(\mathbf{d}\alpha)(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y]).$$

together with the definition of horizontal shows that for two vector fields X and Y on Q, we have

$$\mathcal{B}(X, Y) = -\mathcal{A}([\mathrm{hor}\,X, \mathrm{hor}\,Y])$$

where the bracket on the right hand side is the Jacobi-Lie bracket of vector fields. The Cartan structure equations say that if X and Y are vector fields that are invariant under the group action, then

$$\mathcal{B}(X, Y) = \mathbf{d}\mathcal{A}(X, Y) - [\mathcal{A}(X), \mathcal{A}(Y)]$$

where the bracket on the right hand side is the Lie algebra bracket. This follows readily from the definitions, the fact that $[\xi_Q, \eta_Q] = -[\xi, \eta]_Q$, the first property in the definition of a connection, and writing $\operatorname{hor} X = X - \operatorname{ver} X$ and similarly for Y, in the preceding formula for the curvature.

Next, we give some useful local formulas for the curvature. To do this, we pick a local trivialization of the bundle; that is, locally in the base, we write $Q = Q/G \times G$ where the action of G is given by left translation on the second factor. We choose coordinates r^α on the first factor and a basis e_a of the Lie algebra $\mathfrak{g}$ of G. We write coordinates of an element ξ relative to this basis as ξ^a. Let tangent vectors in this local trivialization at the point (r, g) be denoted (u, w). We will write the action of $\mathcal{A}$ on this vector simply as $\mathcal{A}(u, w)$. Using this notation, we can write the connection form in this local trivialization as

$$\mathcal{A}(u, w) = \operatorname{Ad}_g(w_b + \mathcal{A}_{\text{loc}}(r) \cdot u), \tag{3.3.1}$$

where w_b is the left translation of w to the identity (that is, the expression of w in "body coordinates"). The preceding equation defines the expression $\mathcal{A}_{\text{loc}}(r)$. We define the connection components by writing

$$\mathcal{A}_{\text{loc}}(r) \cdot u = \mathcal{A}^a_\alpha u^\alpha e_a.$$

Similarly, the curvature can be written in a local representation as

$$\mathcal{B}((u_1, w_1), (u_2, w_2)) = \operatorname{Ad}_g(\mathcal{B}_{\text{loc}}(r) \cdot (u_1, u_2)),$$

which again serves to define the expression $\mathcal{B}_{\text{loc}}(r)$. We can also define the coordinate form for the local expression of the curvature by writing

$$\mathcal{B}_{\text{loc}}(r) \cdot (u_1, u_2) = \mathcal{B}^a_{\alpha\beta} u_1^\alpha u_2^\beta e_a.$$

Then one has the formula

$$\mathcal{B}^b_{\alpha\beta} = \left(\frac{\partial \mathcal{A}^b_\alpha}{\partial r^\beta} - \frac{\partial \mathcal{A}^b_\beta}{\partial r^\alpha} - C^b_{ac} \mathcal{A}^a_\alpha \mathcal{A}^c_\beta \right),$$

where C^b_{ac} are the structure constants of the Lie algebra defined by

$$[e_a, e_c] = C^b_{ac} e_b.$$

3.4 The Principal or Purely Kinematic Case

To illustrate how symmetries affect the equations of motion, we will start with one of the simplest cases in which the group orbits exactly complement the constraints, which we call the *principal or the purely kinematic case*, sometimes called the Chaplygin, or the nonabelian Chaplygin case. This case goes back to Chaplygin [1897], Hamel [1904], and was put into a geometric context by Koiller [1992].

An example of the purely kinematic case is the vertical rolling disk discussed in the examples section below. However, in other examples, such as the snakeboard, this condition is not valid and its failure is crucial to understanding the dynamic behavior of this system, and thus below we will consider the more general case.

Definition 3.6 *The* ***principal kinematic case*** *is the case in which* (L1) *and* (S1) *hold and where at each point $q \in Q$, the tangent space T_qQ is the direct sum of the tangent to the group orbit and to the constraint distribution; that is, we require that, at each point, $\mathcal{S}_q = \{0\}$ and that*

$$T_qQ = T_q\operatorname{Orb}(q) \oplus \mathcal{D}_q =: V_q \oplus \mathcal{D}_q.$$

In other words, we require that the group directions provide a vertical space for the Ehresmann connection introduced earlier; thus, in this situation there is a preferred vertical space and so there is no freedom in choosing the associated Ehresmann connection whose horizontal space is the given constraint distribution. In other words, the nonholonomic kinematic constraints provide a connection on the principal bundle $\pi : Q \to Q/G$, so that we can choose this bundle to be coincident with the bundle $\pi_{Q,R} : Q \to R$ introduced earlier. If the Lagrangian and the constraints are invariant with respect to the group action (assumptions (L1) and (S1)), then as we explained above, the equations of motion in Theorem 2.4 drop to the reduced space $\mathcal{D}/G$. As we shall see, in the principal kinematic case, these reduced equations may be regarded as second order equations on Q/G together with the constraint equations. The connection that describes the constraints provides the information necessary to reconstruct the trajectory on the full space. In essence, the constraints provide a connection that replaces the mechanical connection which is used in the reduction theory of unconstrained systems with symmetry. The general case, described later, requires a synthesis of the two approaches.

From the well known fact that a principal connection is uniquely determined by the specification of its horizontal spaces as an invariant complement to the group orbits, we get the following.

Proposition 3.7 *In the principal kinematic case, there is a unique principal connection on $Q \to Q/G$ whose horizontal space is the given distribution $\mathcal{D}$.*

We now make these considerations more explicit. The vertical space for the principal bundle $\pi : Q \to Q/G$ is $V_q = \ker T_q\pi$, which is the tangent space to the group orbit through q. Thus, each vertical fiber at a point q is isomorphic to the Lie algebra $\mathfrak{g}$ by means of the map $\xi \in \mathfrak{g} \mapsto \xi_Q(q)$. In the principal kinematic case, the splitting of the tangent space to Q given in the preceding definition defines a projection onto the vertical space and hence defines an Ehresmann connection that, as before, we denote by A. If condition (S1) holds then $A : TQ \to V$ will be group invariant (assumption (S2)), and there exists a Lie algebra valued one form $\mathcal{A} : TQ \to \mathfrak{g}$ such that

$$A(q) \cdot \dot{q} = \left(\mathcal{A}(q) \cdot \dot{q}\right)_Q (q) \quad \text{or} \quad A = \mathcal{A}_Q.$$

Thus on a principal bundle we can express our results in terms of $\mathcal{A}$ instead of A. In bundle coordinates, $\mathcal{A}$ can be written as

$$\mathcal{A}(r, g) \cdot (\dot{r}, \dot{g}) = \mathrm{Ad}_g(g^{-1}\dot{g} + \mathcal{A}_{\mathrm{loc}}(r)\dot{r}),$$

as in equation (3.3.1).

Above, we gave the expression (3.2.1) for the reduced Lagrangian in a local trivialization. We now turn to the expression in a local trivialization for the *constrained* reduced Lagrangian l_c. This is obtained by substituting the constraints $\mathcal{A}(q) \cdot \dot{q} = 0$ into the reduced Lagrangian. Thus $l_c : T(Q/G) \to \mathbb{R}$ is given by

$$l_c(r, \dot{r}) = l(r, \dot{r}, -\mathcal{A}_{\mathrm{loc}}(r)\dot{r}). \tag{3.4.1}$$

Alternatively, note that we can write

$$l_c(r, \dot{r}) = L(q, \mathrm{hor}\, \dot{q})$$

where $r = \pi(q)$, and $\dot{r} = T_q\pi(\dot{q})$.

Using this notation, the equations of motion can be read off of Theorem 2.4 to give the following theorem.

Theorem 3.8 *In the principal kinematic case, the equations of motion are*

$$\left.\begin{aligned} \delta l_c &= \left\langle \frac{\partial l}{\partial \xi}, \mathcal{B}_{\mathrm{loc}}(\dot{r}, \delta r) \right\rangle \\ \dot{g} &= -g\mathcal{A}_{\mathrm{loc}}(r)\dot{r}, \end{aligned}\right\} \tag{3.4.2}$$

where $\delta r \in T(Q/G)$ and $\mathcal{B}_{\mathrm{loc}}$ is the curvature of $\mathcal{A}_{\mathrm{loc}}$ and where $\xi = -\mathcal{A}_{\mathrm{loc}}(r)\dot{r}$.

This theorem goes back to the works of Chaplygin starting in 1897 (see the references) for the abelian principal case and was extended to the nonabelian case by Koiller. This result is also a consequence of the results of MARSDEN & SCHEURLE [1993]; indeed, they show that the first of these equations is a consequence of the horizontal variations in the action (*i.e.,* the Lagrange d'Alembert principle) and that in this calculation one can choose any connection, in particular, the principal kinematic connection in this case. Of course the second of the equations is just the condition of horizontality; that is, the kinematic constraints themselves.

We see in local coordinates that the dynamics of the system can be completely written in terms of the dynamics in base coordinates $r \in Q/G$ and the full dynamics are given by reconstruction of $\dot{g}$ using the constraints. Thus, in the purely kinematic case, we recover the process of reduction and reconstruction with the kinematic connection $\mathcal{A}$ replacing the mechanical connection. We stress, in particular, that in the principal kinematic case, something special happens, namely there is no dynamic equation for $\xi = g^{-1}\dot{g}$ but rather ξ can be expressed directly in terms of r and $\dot{r}$ using the constraints and when this is substituted into the first equation, they become second order equations for r. Thus, in this case, the equations actually reduce from equations on $\mathcal{D}/G$ to equations on Q/G. The dynamics of g itself is then recovered by the constraint equation, which may be regarded as similar to the problem of calculating holonomy, as in MARSDEN, MONTGOMERY & RATIU [1990]. In particular, for abelian groups, the dynamics of g can be written in terms of that of r by an explicit quadrature.

The purely kinematic case can easily be extended to allow affine constraints (see YANG [1992] and YANG, KRISHNAPRASAD & DAYAWANSA [1993]). If the constraints are of the form $A(q) \cdot \dot{q} = \gamma(q,t)$ where γ is a vertical valued vector field on Q and is G-invariant, then in the principal kinematic case the constraints can be regarded as being Lie algebra valued and written

$$\mathcal{A}(q) \cdot \dot{q} = \Gamma(q,t) = \mathrm{Ad}_g \Gamma_{\mathrm{loc}}(r,t)$$

where $\Gamma : Q \times \mathbb{R} \to \mathfrak{g}$ is defined by $(\Gamma(q,t))_Q = \gamma(q,t)$ and $\Gamma_{\mathrm{loc}}(r,t) \in \mathfrak{g}$ is the version of Γ in a local trivialization. The Lagrangian is modified as before and the equations of motion become

$$\left.\begin{aligned} \delta l_c &= \left\langle \frac{\partial l}{\partial \xi}, \mathcal{B}(\dot{q}, \delta q) \right\rangle + \left\langle \frac{\partial l}{\partial \xi}, D\Gamma(\delta q) \right\rangle \\ \mathcal{A}(q) \cdot \dot{q} &= \Gamma(q,t), \end{aligned}\right\} \tag{3.4.3}$$

where the variations in r are free; that is, $\delta r = T_q\pi \cdot \delta q$ is free and where $D\Gamma(X) = d\Gamma(\mathrm{hor}\, X)$ is the covariant derivative of Γ. The proof is via a direct coordinate calculation and uses the fact that Γ depends equivariantly on the group variable. As before, relative to a local trivialization, these equations can be written as

$$\left.\begin{aligned} \delta l_c &= \left\langle \frac{\partial l}{\partial \xi}, \mathcal{B}_{\mathrm{loc}}(\dot{r}, \delta r) \right\rangle + \left\langle \frac{\partial l}{\partial \xi}, d\Gamma_{\mathrm{loc}}(\delta r) \right\rangle \\ \dot{g} &= g(-\mathcal{A}_{\mathrm{loc}}(r)\dot{r} + \Gamma_{\mathrm{loc}}(r,t)) \end{aligned}\right\} \tag{3.4.4}$$

which again determines a second order dynamical system on shape space Q/G and where $\xi = -\mathcal{A}_{\mathrm{loc}}(r)\dot{r} + \Gamma_{\mathrm{loc}}(r,t)$.

4 The Momentum Equation

In this section we use the Lagrange d'Alembert principle to derive an equation for a generalized momentum as a consequence of the symmetries. Under the hypotheses that the action of some Lie algebra element is horizontal (that is, the infinitesimal generator is automatically in the constraint distribution), this yields a conservation law in the usual sense. As we shall see, the momentum equation does not directly involve the choice of an Ehresmann connection to describe the distribution $\mathcal{D}$, but the choice of such a connection will be useful for the coordinate versions.

We have already mentioned in the introduction that simple physical systems that have symmetries do not have associated conservation laws, namely the wobblestone and the snakeboard. It is also easy to see why this is not generally the case from the equations of motion. The simplest situation would be the case of cyclic variables. Recall that the equations of motion have the form

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta} \dot{r}^\beta .$$

If this has a cyclic variable, say r^1, this would mean that all the quantities $L_c, L, B^b_{\alpha\beta}$ would be independent of r^1. This is equivalent to saying that there is a translational symmetry in the r^1 direction. Let us also suppose, as is often the case, that the s variables are also cyclic. Then the above equation for the momentum $p_1 = \partial L_c / \partial \dot{r}^1$ becomes

$$\frac{d}{dt} p_1 = -\frac{\partial L}{\partial \dot{s}^b} B^b_{1\beta} \dot{r}^\beta .$$

This fails to be a conservation law in general. Note that the right hand side is linear in $\dot{r}$ (the first term is linear in p_r) and the equation does not depend on r^1 itself. This is a very special case of the momentum equation that we shall develop in this chapter. Even for systems like the snakeboard, the symmetry group is not abelian, so the above analysis for cyclic variables fails to capture the full story. In particular, the momentum eqution is not of the preceding form in that example and thus it must be generalized.

4.1 The Classical Noether Theorem

To derive the momentum equation, it will be useful to first recall Noether's original derivation of the conservation laws directly from Hamilton's variational principle. Consider a Lie group G acting on a configuration manifold Q and lift this action to the tangent bundle TQ using the tangent operation. Given a G-invariant Lagrangian $L : TQ \to \mathbb{R}$, the corresponding momentum map is the mapping $J : TQ \to \mathfrak{g}^*$ defined by

$$\langle J(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle \tag{4.1.1}$$

or, in coordinates,

$$J_a = \frac{\partial L}{\partial \dot{q}^i} K^i_a , \tag{4.1.2}$$

where we define the action coefficients K^i_a relative to a basis $e_a, a = 1, \ldots, k$ of $\mathfrak{g}$ by writing $\xi_Q(q) = K^i_a \xi^a \partial / \partial q^i$ with $\xi = \xi^a e_a$, and a sum on the index a is understood.

Theorem 4.1 Classical Noether Theorem *For a solution of the Euler-Lagrange equations, the quantity J is a constant in time.*

We remark in passing, although we shall not use it, that this result holds even if the Lagrangian is degenerate, that is, the fiber derivative defined by $p_i = \partial L / \partial \dot{q}^i$ is not invertible.

Proof Choose any function $\phi(t,s)$ of two variables such that the conditions $\phi(a,s) = \phi(b,s) = \phi(t,0) = 0$ hold, where a and b are the temporal endpoints of the given solution to the Euler-Lagrange equations. Since L is G-invariant, for each Lie algebra element $\xi \in \mathfrak{g}$, the expression

$$\int_a^b L(\exp(\phi(t,s)\xi) \cdot q, \exp(\phi(t,s)\xi) \cdot \dot{q}) \, dt \tag{4.1.3}$$

is independent of s. Differentiating this expression with respect to s at $s = 0$ and setting $\phi' = \partial\phi/\partial s$ taken at $s = 0$, gives infinitesimal invariance:

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i} \xi_Q^i \phi' + \frac{\partial L}{\partial \dot{q}^i} (T\xi_Q \cdot \dot{q})^i \phi' \right) dt. \tag{4.1.4}$$

Now we consider the variation $q(t,s) = \exp(\phi(t,s)\xi) \cdot q(t)$. The corresponding infinitesimal variation is given by $\delta q(t) = \phi'(t)\xi_Q(q(t))$. Noting that these variations vanish at the endpoints, Hamilton's principle gives

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \dot{\delta q}^i \right) dt. \tag{4.1.5}$$

Note that

$$\dot{\delta q} = \dot{\phi}' \xi_Q + \phi' (T\xi_Q \cdot \dot{q})$$

and subtract (4.1.5) from (4.1.4) to give

$$0 = \int_a^b \frac{\partial L}{\partial \dot{q}^i} (\xi_Q)^i \dot{\phi}' \, dt = -\int_a^b \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \xi_Q^i \right) \phi' \, dt. \tag{4.1.6}$$

Since ϕ' is arbitrary, except for endpoint conditions, it follows that the integrand vanishes, and so the time derivative of the momentum map is zero. ■

The reader will find that this notion of momentum map coincides with the classical notion for Lagrangian systems with symmetry; see, for example, Marsden & Ratiu [1994].

4.2 The Derivation of the Momentum Equation

We now adapt this approach to derive an equation for a generalized momentum map for nonholonomic systems. The number of equations obtained will equal the dimension of the intersection of the orbit with the given constraints. As we will see, this result will give conservation laws as a particular case.

To formulate our result, some additional ideas and notation will be useful. As the examples show, in general the tangent space to the group orbit through q intersects the constraint distribution at q nontrivially. It will be helpful to give this intersection a name.

Definition 4.2 *The intersection of the tangent space to the group orbit through the point $q \in Q$ and the constraint distribution at this point is denoted $\mathcal{S}_q$, as in Figure 4.2.1, and we let the union of these spaces over $q \in Q$ be denoted $\mathcal{S}$. Thus,*

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q(\mathrm{Orb}(q)).$$

Definition 4.3 *Define, for each $q \in Q$, the vector subspace $\mathfrak{g}^q$ to be the set of Lie algebra elements in $\mathfrak{g}$ whose infinitesimal generators evaluated at q lie in $\mathcal{S}_q$:*

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} : \xi_Q(q) \in S_q\}$$

The corresponding bundle over Q whose fiber at the point q is given by $\mathfrak{g}^q$, is denoted $\mathfrak{g}^{\mathcal{D}}$.

Consider a section of the vector bundle $\mathcal{S}$ over Q; *i.e.*, a mapping that takes q to an element of $\mathcal{S}_q = \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))$. Assuming that the action is free, a section of $\mathcal{S}$ can be uniquely represented as ξ_Q^q and defines a section ξ^q of the bundle $\mathfrak{g}^{\mathcal{D}}$. For example, one can construct the section by orthogonally projecting (using the kinetic energy metric) $\xi_Q(q)$ to the subspace $\mathcal{S}_q$. However, as we shall see, in later examples, it is often easy to choose a section by inspection.

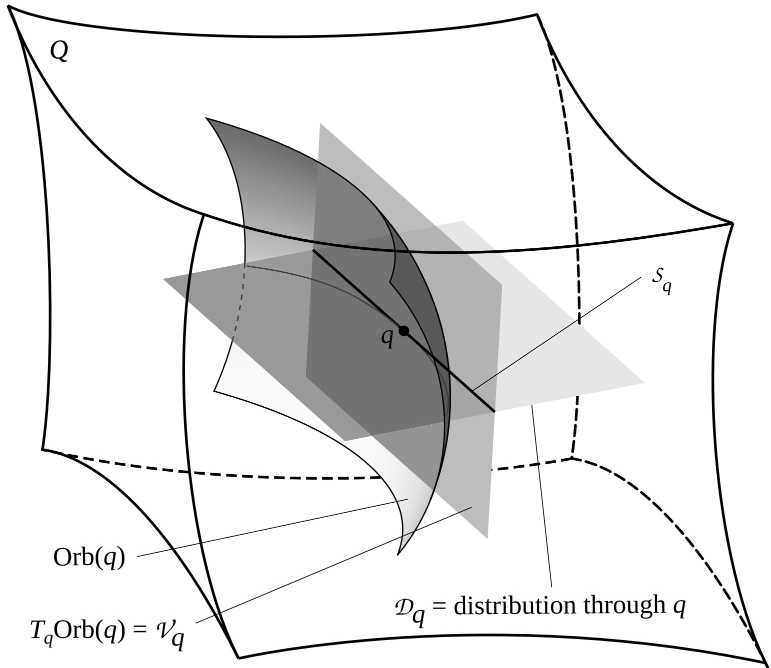

Figure 4.2.1: The intersection of the tangent space to the group orbit with the constraint distribution; here the tangent spaces are superimposed on the spaces themselves.

Next, we choose the variation analogously to what we chose in the case of the standard Noether theorem above, namely, $q(t,s) = \exp(\phi(t,s)\xi^{q(t)}) \cdot q(t)$. The corresponding infinitesimal variation is given by $\delta q(t) = \phi'(t)\xi^q_Q(q(t))$. Letting $\partial \xi^q$ denote the derivative of ξ^q with respect to q, we have

$$\dot{\delta q} = \dot{\phi}' \xi_Q^{q(t)} + \phi' \left[(T\xi_Q^{q(t)} \cdot \dot{q}) + (\partial \xi^{q(t)} \cdot \dot{q})_Q \right].$$

In this equation, the term $T\xi_Q^{q(t)}$ is computed by taking the derivative of the vector field $\xi_Q^{q(t)}$ with $q(t)$ held fixed. By construction, the variation δq satisfies the constraints and the curve $q(t)$ satisfies the Lagrange d'Alembert equations, so that the following variational equation holds:

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \dot{\delta q}^i \right) dt. \tag{4.2.1}$$

In addition, the invariance identity from above holds using ξ^q:

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i} (\xi_Q^{q(t)})^i \phi' + \frac{\partial L}{\partial \dot{q}^i} (T\xi_Q^{q(t)} \cdot \dot{q})^i \phi' \right) dt. \tag{4.2.2}$$

Subtracting equations (4.2.1) and (4.2.2) and using the arbitrariness of ϕ' and integration by parts shows that

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} (\xi^{q(t)})^i_Q = \frac{\partial L}{\partial \dot{q}^i} \left[\frac{d}{dt} (\xi^{q(t)}) \right]^i_Q.$$

The quantity whose rate of change is involved here is the nonholonomic version of the momentum map in geometric mechanics.

Definition 4.4 *The* ***nonholonomic momentum map*** J^{nhc} *is the bundle map taking* TQ *to the bundle* $(\mathfrak{g}^{\mathcal{D}})^*$ *whose fiber over the point* q *is the dual of the vector space* $\mathfrak{g}^q$ *that is defined by*

$$\langle J^{\mathrm{nhc}}(v_q), \xi \rangle = \frac{\partial L}{\partial \dot{q}^i} (\xi_Q)^i .$$

where $\xi \in \mathfrak{g}^q$*. Intrinsically, this reads*

$$\langle J^{\mathrm{nhc}}(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q \rangle ,$$

where $\mathbb{F}L$ *is the fiber derivative of* L *and where* $\xi \in \mathfrak{g}^q$*. For notational convenience, especially when the variable* v_q *is suppressed, we will often write the left hand side of this equation as* $J^{\mathrm{nhc}}(\xi)$*.*

Notice that the nonholonomic momentum map may be viewed as giving just some of the components of the ordinary momentum map, namely along those symmetry directions that are consistent with the constraints.

We summarize these results in the following theorem.

Theorem 4.5 *Assume that condition (L2) of definition 3.1 holds (which is implied by (L1)) and that* ξ^q *is a section of the bundle* $\mathfrak{g}^{\mathcal{D}}$*. Then any solution of the Lagrange d'Alembert equations for a nonholonomic system must satisfy, in addition to the given kinematic constraints, the* ***momentum equation****:*

$$\frac{d}{dt}\left(J^{\mathrm{nhc}}(\xi^{q(t)}) \right) = \frac{\partial L}{\partial \dot{q}^i} \left[\frac{d}{dt} (\xi^{q(t)}) \right]_Q^i . \tag{4.2.3}$$

When the momentum map is paired with a section in this way, we will just refer to it as the momentum. The following is a direct corollary of this result.

Corollary 4.6 *If* ξ *is a horizontal symmetry (see (S3) above), then the following conservation law holds:*

$$\frac{d}{dt} J^{\mathrm{nhc}}(\xi) = 0 \tag{4.2.4}$$

A somewhat restricted version of the momentum equation was given by Koslov & Kolesnikov [1978] and the corollary was given by Arnold [1988, page 82] (see Bloch & Crouch [1992, 1994] for the controlled case).

Remarks

1. The right hand side of the momentum equation (4.2.3) can be written in more intrinsic notation as
$$\left\langle \mathbb{F}L(\dot{q}(t)), \left(\frac{d}{dt} \xi^{q(t)} \right)_Q \right\rangle .$$

2. In the theorem and the corollary, we do not need to assume that the distribution itself is G-invariant; that is, we do not need to assume condition (S1). In particular, as we shall see in the examples, one can get conservation laws in some cases in which the distribution is not invariant.

3. The validity of the form of the momentum equation is not affected by any "internal forces", that is, any control forces on shape space. Indeed, such forces would be invariant under the action of the Lie group G and so would be annihilated by the variations taken to prove the above result.

4. The momentum equation still holds in the presence of affine constraints. We do *not* need to assume that the affine vector field defining the affine constraints is invariant under the group. However, this vector field may appear in the final momentum equation (or conservation law) because the constraints may be used to rewrite the resulting equation. We will see this explicitly in the example of the ball on a rotating table.

5. Assuming that the distribution is invariant (hypothesis (S1)), the nonholonomic momentum map as a bundle map is equivariant with respect to the action of the group G on the tangent bundle TQ and on the bundle $(\mathfrak{g}^{\mathcal{D}})^*$. In fact, since the distribution is invariant and using the general identity $(\mathrm{Ad}_g\xi)_Q = \Phi^*_{g^{-1}}\xi_Q$, valid for any group action, we see that the space $\mathfrak{g}^g$ is mapped to $\mathfrak{g}^{gq}$ by the map Ad_g, and so in this sense, the adjoint action acts in a well defined manner on the bundle $\mathfrak{g}^{\mathcal{D}}$. By taking its dual, we see that the coadjoint action is well defined on $(\mathfrak{g}^{\mathcal{D}})^*$. In this setting, equivariance of the nonholonomic momentum map follows as in the usual proof (see, for example, Marsden & Ratiu [1994], Chapter 11).

6. One can find an *invariant* momentum if the section is chosen such that

$$(\mathrm{Ad}_{g^{-1}}\xi^{g\cdot q})_Q = \xi^q_Q.$$

This can always be done in the case of trivial bundles; one chooses any ξ^q at the identity in the group variable and translates it around by using the action to get a ξ^q at all points. This direction of reasoning (initiated by remarks of Ostrowski, Lewis, Burdick and Murray) is discussed in §4.4. As we will see later, this point of view is useful in the case of the snakeboard.

7. The form of the momentum equation in this section is valid for any curve $q(t)$ that satisfies the Lagrange d'Alembert principle; we do *not* require that the constraints be satisfied for this curve. The version of the momentum equation given in the next section and later in §7 will explicitly require that the constraints are satisfied. Of course, in examples we always will impose the constraints, so this is really a comment about the logical structure of the various versions of the equation.

8. In some interesting cases, one can get conservation laws *without* having horizontal symmetries, as required in the preceding corollary. These are cases in which, for reasons other than horizontality, the right hand side of the momentum equation vanishes. This may be an important observation for the investigation of completely integrable nonholonomic systems. A specific case in which this occurs is the vertical rolling disk discussed below.

4.3 The Momentum Equation in a Moving Basis

There are several ways of rewriting the momentum equation that are useful; the examples will show that each of them can reveal interesting aspects of the system under consideration. This subsection develops the first of these coordinate formulas, which is in some sense the most naive, but also the most direct. The next subsection will develop a form that is suitable for a local trivialization of the bundle $Q \to Q/G$. Later on, when the nonholonomic connection is introduced, we shall come back to both of these forms and rewrite them in a more sophisticated but also more revealing way.

Introduce coordinates $q^1, \ldots q^n$ in the neighborhood of a given point q_0 in Q. At the point q_0, introduce a basis

$$\{e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_k\}$$

of the Lie algebra such that the first m elements form a basis of $\mathfrak{g}^{q_0}$. Thus, $k = \dim \mathfrak{g}$ and $m = \dim \mathfrak{g}^q$, which, by assumption, is locally constant. We can introduce a similar basis

$$\{e_1(q), e_2(q), \ldots, e_m(q), e_{m+1}(q), \ldots, e_k(q)\}$$

at neighboring points q. For example, one can choose an orthonormal basis (in either the locked inertia metric or relative to a Killing form) that varies smoothly with q. We introduce a change of basis matrix by writing

$$e_b(q) = \sum_{a=1}^{k} \psi_b^a(q) e_a$$

for $b = 1, \ldots, k$. Here, the change of basis matrix $\psi_b^a(q)$ is an invertible $k \times k$ matrix. Relative to the dual basis, we write the components of the nonholonomic momentum map as J_b. By definition,

$$J_b = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i} [e_b(q)]_Q^i.$$

Using this notation, the momentum equation, with the choice of section given by

$$\xi^{q(t)} = e_b(q(t)), \quad 1 \leq b \leq m$$

reads as follows:

$$\frac{d}{dt} J_b = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial \dot{q}^i} \left[\frac{d}{dt} e_b(q(t)) \right]_Q^i \right). \tag{4.3.1}$$

Next, we define Christoffel-like symbols by

$$\Gamma_{bl}^c = \sum_{a=1}^{k} (\psi^{-1})_a^c \frac{\partial \psi_b^a}{\partial q^l} \tag{4.3.2}$$

where the matrix $(\psi^{-1})_a^d$ denotes the inverse of the matrix ψ_b^a. Observe that

$$\frac{d}{dt} e_b(q(t)) = \sum_{c=1}^{k} \sum_{l=1}^{n} \Gamma_{bl}^c \dot{q}^l e_c(q(t)), \tag{4.3.3}$$

which implies that

$$\left[\frac{d}{dt} e_b(q(t)) \right]_Q^i = \sum_{c=1}^{k} \sum_{l=1}^{n} \Gamma_{bl}^c \dot{q}^l [e_c(q(t))]_Q^i. \tag{4.3.4}$$

Thus, we can write the momentum equation as

$$\frac{d}{dt} J_b = \sum_{c=1}^{k} \sum_{i,l=1}^{n} \frac{\partial L}{\partial \dot{q}^i} \Gamma_{bl}^c \dot{q}^l [e_c(q(t))]_Q^i. \tag{4.3.5}$$

Introducing the shorthand notation $e_c^i := [e_c(q(t))]_Q^i$, the momentum equation reads

$$\frac{d}{dt} J_b = \sum_{c=1}^{k} \sum_{i,l=1}^{n} \frac{\partial L}{\partial \dot{q}^i} \Gamma_{bl}^c \dot{q}^l e_c^i \tag{4.3.6}$$

Breaking the summation over c into two ranges and using the definition

$$J_c = \frac{\partial L}{\partial \dot{q}^i} e_c^i \qquad 1 \leq c \leq m$$

gives the following form of the momentum equation.

Proposition 4.7 Momentum equation in a moving basis *The momentum equation in the above coordinate notation reads*

$$\frac{d}{dt} J_b = \sum_{c=1}^{m} \sum_{l=1}^{n} \Gamma_{bl}^{c} J_c \dot{q}^l + \sum_{c=m+1}^{k} \sum_{i,l=1}^{n} \frac{\partial L}{\partial \dot{q}^i} \Gamma_{bl}^{c} \dot{q}^l e_c^i. \tag{4.3.7}$$

Assuming that the Lagrangian is of the form kinetic minus potential energy, the second term on the right hand side of this equation vanishes if the orbit and the constraint distribution are orthogonal; that is, if we can choose the basis so that the vectors $[e_c(q(t))]_Q$ for $c \geq m+1$ are orthogonal to the constraint distribution. In this case, the momentum equation has the form of an equation of parallel transport along the curve $q(t)$. The connection involved is the natural one associated with the the bundle $(\mathfrak{g}^{\mathcal{D}})^*$ over Q, using a chosen decomposition of $\mathfrak{g}$, such as the orthogonal one. In the general case, the momentum equation is an equality between the covariant derivative of the nonholonomic momentum and the last term on the right hand side of the preceding equation. In the next section, we shall write the momentum equation in a body frame, which will be important for understanding how to decouple the momentum equation from the group variables. This will be important for the reduction theory in §7.

4.4 The Momentum Equation in Body Representation

Next, we develop an alternative coordinate formula for the momentum equation that is adapted to a choice of local trivialization. Thus, let a local trivialization be chosen on the principal bundle $\pi : Q \to Q/G$, with the local representation having coordinates denoted (r, g). Let r have components denoted r^α as before, being coordinates on the base Q/G and let g be group variables for the fiber, G. In such a representation, the action of G is the left action of G on the second factor. We calculate the nonholonomic momentum map using well known ideas (see, for example, Marsden & Ratiu [1994], Chapter 12), as follows. Let $v_q = (r, g, \dot{r}, \dot{g})$ be a tangent vector at the point $q = (r, g)$, $\eta \in \mathfrak{g}^q$ and let $\xi = g^{-1}\dot{g}$, *i.e.*, $\xi = T_g L_{g^{-1}} \dot{g}$. Since L is G-invariant, we can define a new function l by writing

$$L(r, g, \dot{r}, \dot{g}) = l(r, \dot{r}, \xi).$$

Use of the chain rule shows that

$$\frac{\partial L}{\partial \dot{g}} = T_g^* L_{g^{-1}} \frac{\partial l}{\partial \xi},$$

and so

$$\begin{aligned} \left\langle J^{\mathrm{nhc}}(v_q), \eta \right\rangle &= \langle \mathbb{F}L(r, g, \dot{r}, \dot{g}), \eta_Q(r, g) \rangle \\ &= \left\langle \frac{\partial L}{\partial \dot{g}}, (0, TR_g \cdot \eta) \right\rangle \\ &= \left\langle \frac{\partial l}{\partial \xi}, \mathrm{Ad}_{g^{-1}} \eta \right\rangle. \end{aligned}$$

The preceding equation shows that we can write the momentum map in a local trivialization by making use of the Ad mapping in much the same way as we did with the connection and the local formulas in the principal kinematic case. We define $J^{\mathrm{nhc}}_{\mathrm{loc}} : TQ/G \to (\mathfrak{g}^{\mathcal{D}})^*$ in a local trivialization by

$$\left\langle J^{\mathrm{nhc}}_{\mathrm{loc}}(r, \dot{r}, \xi), \eta \right\rangle = \left\langle \frac{\partial l}{\partial \xi}, \eta \right\rangle.$$

Thus, as with the previous local forms, J^{nhc} and its version in a local trivialization are related by the Ad map; precisely,

$$J^{\mathrm{nhc}}(r, g, \dot{r}, \dot{g}) = \mathrm{Ad}^*_{g^{-1}} J^{\mathrm{nhc}}_{\mathrm{loc}}(r, \dot{r}, \xi).$$

Secondly, choose a q-dependent basis $e_a(q)$ for the Lie algebra such that the first m elements span the subspace $\mathfrak{g}^q$. In a local trivialization, this is done in a very simple way. First, one chooses, for each r, such a basis at the identity element $g = \mathrm{Id}$, say

$$e_1(r), e_2(r), \ldots, e_m(r), e_{m+1}(r), \ldots, e_k(r).$$

For example, this could be a basis such that the corresponding generators are orthonormal in the kinetic energy metric. (Keep in mind that the subspaces $\mathcal{D}_q$ and $T_q\mathrm{Orb}$ need not be orthogonal but here we are choosing a basis corresponding only to the subspace $T_q\mathrm{Orb}$.) Define the ***body fixed basis*** by

$$e_a(r, g) = \mathrm{Ad}_g \cdot e_a(r);$$

then the first m elements will indeed span the subspace $\mathfrak{g}^q$ provided the distribution is invariant (condition (S1)). Thus, in this basis we have

$$\left\langle J^{\mathrm{nhc}}(r, g, \dot{r}, \dot{g}), e_b(r, g) \right\rangle = \left\langle \frac{\partial l}{\partial \xi}, e_b(r) \right\rangle := p_b, \tag{4.4.1}$$

which defines p_b, a function of r, $\dot{r}$ and ξ. We are deliberately introducing the new notation p for the momentum in body representation to signal its special role. Note that in this body representation, the functions p_b are *invariant* rather than equivariant, as is usually the case with the momentum map. The time derivative of p_b may be evaluated using the momentum equation (4.2.3). This gives

$$\begin{aligned}
\frac{d}{dt} p_b &= \frac{\partial L}{\partial \dot{q}^i} \left[\frac{d}{dt} e_b(r, g) \right]_Q^i \\
&= \left\langle (T_g L_{g^{-1}})^* \frac{\partial l}{\partial \xi}, \left[\frac{d}{dt} \left(\mathrm{Ad}_g \cdot e_b(r) \right) \right]_Q \right\rangle \\
&= \left\langle \frac{\partial l}{\partial \xi}, [\xi, e_b] + \frac{\partial e_b}{\partial r^\alpha} \dot{r}^\alpha \right\rangle .
\end{aligned}$$

We summarize the conclusion drawn from this calculation as follows.

Proposition 4.8 Momentum equation in body representation *The momentum equation in body representation on the principal bundle $Q \to Q/G$ is given by*

$$\frac{d}{dt} p_b = \left\langle \frac{\partial l}{\partial \xi}, [\xi, e_b] + \frac{\partial e_b}{\partial r^\alpha} \dot{r}^\alpha \right\rangle . \tag{4.4.2}$$

Moreover, the momentum equation in this representation is independent of, that is, decouples from, the group variables g.

In this representation, the variable ξ is related to the group variable g by $\xi = g^{-1}\dot{g}$. In particular, in this representation, reconstruction of the group variable g can be done by means of the equation

$$\dot{g} = g\xi \tag{4.4.3}$$

On the other hand, this variable $\xi = g^{-1}\dot{g}$, as in the case of the reduced Euler-Poincaré equations, *is not the vertical part of the velocity vector $\dot{q}$ relative to the nonholonomic connection to be constructed in the next section.* The vertical part is related to the variable ξ by a velocity shift and this velocity shift will make the reconstruction equation look affine, as in the case of the snakeboard (see Lewis, Ostrowski, Murray & Burdick [1994]). In that example, the decoupling of the momentum equation from the group variables played a useful role. We also recall (as in the example of the rigid body with rotors discussed in Marsden & Scheurle [1993]) that it is often the shifted velocity and not ξ that diagonalizes the kinetic energy, so this shift is fundamental for a number of reasons. As we shall see later, the same ideas in this section, combined with the calculations of MARSDEN & SCHEURLE [1993] will show how to calculate the fully reduced equations.

In the above local trivialization form of the momentum equation, we may write the terms $(\partial e_b/\partial r^\alpha)\dot{r}^\alpha$ in terms of a connection, as we did in deriving the momentum equation in a moving basis. We will carry this out later in §6.

Other noteworthy features of this form of the momentum equation are the following direct consequences of the preceding proposition.

Corollary 4.9

1. *If e_b, $b = 1, \ldots, m$ are independent of r, then the momentum equation in body representation is equivalent to the Euler-Poincaré equations projected to the subspace $\mathfrak{g}^q$.*
2. *If $\mathfrak{g}$ is abelian, then the momentum equations reduce to*

$$\frac{d}{dt}p_b = \left\langle \frac{\partial l}{\partial \xi}, \frac{\partial e_b}{\partial r^\alpha}\dot{r}^\alpha \right\rangle. \tag{4.4.4}$$

3. *If $\mathfrak{g}$ is abelian, or more generally, if the bracket of an element of $\mathfrak{g}^q$ with one in $\mathfrak{g}$ is annihilated by $\partial l/\partial \xi$, and if $e_b, b = 1, \ldots, m$, are independent of r, then the quantities $p_b, b = 1, \ldots, m$ are constants of motion.*

Regarding the first item, see MARSDEN & RATIU [1994] for a discussion of the Euler-Poincaré equations; these are also briefly reviewed in the following section. In this case, the spatial form of the momentum is conserved, just as in the case of systems with holonomic constraints. For the snakeboard, $\mathfrak{g}^q$ is abelian, but $\mathfrak{g}$ is not and the second item above does not apply. We shall develop the geometry and notation to study this situation more thoroughly in section 7. As we shall see later in the examples section, the last case occurs for the vertical rolling penny.

5 A Review of Lagrangian Reduction

Lagrangian reduction theory for systems with holonomic constraints was developed by MARSDEN & SCHEURLE [1993a,b].[2] We summarize some of the features of that theory, not only for purposes of comparison, but to exploit areas of commonality. The ultimate picture of a nonholonomic mechanical system with symmetry will involve a synthesis of the reduced Euler-Lagrange equations and the equations for a nonholonomic system, as we mentioned in the introduction.

5.1 Rigid Body Reduction

We begin by recalling a simple case, namely the rotational motion of a free rigid body. Let $R \in SO(3)$ denote the time dependent rotation that gives the current configuration of the rigid body. The body angular velocity Ω is defined in terms of R by

$$R^{-1}\dot{R} = \hat{\Omega},$$

where $\hat{\Omega}$ is the three by three skew matrix defined by $\hat{\Omega}v := \Omega \times v$. Denoting by I the (time independent) moment of inertia tensor, the Lagrangian thought of as a function of R and $\dot{R}$ is given by $L(R, \dot{R}) = \frac{1}{2}\langle I\Omega, \Omega\rangle$ and when we think of it as a function of Ω alone, we write $l(\Omega) = \frac{1}{2}\langle I\Omega, \Omega\rangle$.

A basic fact about rigid body dynamics and reduction is that the following statements are equivalent:

[2] Sign conventions for the curvature in this reference differ from those in the present paper. We have consistently used the conventions in the current paper to avoid confusion.

1. $(R, \dot{R})$ satisfies the Euler-Lagrange equations on $SO(3)$ for L,
2. Hamilton's principle on $SO(3)$ holds:
$$\delta \int L \, dt = 0,$$
3. Ω satisfies the Euler equations
$$I\dot{\Omega} = I\Omega \times \Omega,$$
4. the reduced variational principle holds on $\mathbb{R}^3$:
$$\delta \int l \, dt = 0,$$
where variations in Ω are restricted to be of the form $\delta\Omega = \dot{\eta} + \eta \times \Omega$, with η an arbitrary curve in $\mathbb{R}^3$ satisfying $\eta = 0$ at the temporal endpoints.

An important point is that when one reduces the standard variational principle from $SO(3)$ to its Lie algebra $\mathfrak{so}(3)$, one ends up with a variational principle in which the *variations are constrained*. In this case, the term η represents the infinitesimal displacement of particles in the rigid body. Note that the same phenomenon of constrained variations occurs in the case of nonholonomic systems.

In symplectic reduction, one imposes $J = \mu$ and passes to the quotient phase space, inducing a symplectic form on the quotient. For Poisson reduction on the other hand, one passes directly to the quotient (phase space)/(group) without the imposition of $J = \mu$ using the induced Poisson bracket. (See Marsden & Ratiu [1986] for a more sophisticated version.) The symplectic reduced spaces are the symplectic leaves of the quotient Poisson manifold. For example, in the rigid body, the phase space is $P = T^*SO(3)$ and the quotient space $P/G = \mathfrak{so}(3)^* \cong \mathbb{R}^3$ contains the body angular momentum $p = I\Omega$ as the basic dynamical variable. This body angular momentum space carries the rigid body bracket
$$\{K, L\} = -\langle p, \nabla K \times \nabla L \rangle$$
and the angular momentum spheres $||p|| =$ constant are its symplectic leaves.

5.2 The Euler-Poincaré Equations

To understand the Lagrangian analogue of Poisson reduction, we first consider the equations of generalized rigid bodies, governed by the Euler-Poincaré equations. Poincaré [1901] showed how to generalize Euler's rigid body and fluid equations to *any* Lie algebra. The Euler-Poincaré equations may be described as follows (see Marsden & Scheurle [1993b] and Bloch, Krishnaprasad, Marsden & Ratiu [1994] for more details). Let $\mathfrak{g}$ be a Lie algebra and let $l : \mathfrak{g} \to \mathbb{R}$ be a given Lagrangian. Then the equations are:
$$\frac{d}{dt}\frac{\partial l}{\partial \xi} = \mathrm{ad}_\xi^* \frac{\partial l}{\partial \xi}$$
or, in coordinates,
$$\frac{d}{dt}\frac{\partial l}{\partial \xi^a} = C_{da}^b \xi^d \frac{\partial l}{\partial \xi^b},$$
where the structure constants are defined by
$$[\xi, \eta]^a = C_{de}^a \xi^d \xi^e.$$
If G is a Lie group with Lie algebra $\mathfrak{g}$, we let $L : TG \to \mathbb{R}$ be the left invariant extension of l and let $\xi = g^{-1}\dot{g}$. In this context, ξ reduces to Ω, the body angular velocity in the case of the rigid body.

The basic fact regarding the Lagrangian reduction leading to these equations is:

Theorem 5.1 *A curve $(g(t), \dot{g}(t)) \in TG$ satisfies the Euler-Lagrange equations for L if and only if ξ satisfies the Euler-Poincaré equations for l.*

In this situation, the reduction is implemented by the map $(g, \dot{g}) \in TG \mapsto \xi = g^{-1}\dot{g} \in \mathfrak{g}$.

One proof of this theorem is of special interest, as it shows how to drop variational principles to the quotient. Namely, we transform

$$\delta \int L \, dt = 0$$

under the map $(g, \dot{g}) \mapsto \xi$ to give the reduced variational principle for the Euler-Poincaré equations: ξ satisfies the Euler-Poincaré equations if and only if

$$\delta \int l \, dt = 0,$$

where the variations are all those of the form

$$\delta\xi = \dot{\eta} + [\xi, \eta]$$

and where η is an arbitrary curve in the Lie algebra satisfying $\eta = 0$ at the endpoints. Variations of this form are obtained by calculating what variations are induced by variations on the Lie group itself.

In fluid mechanics (where the Euler equations of ideal flow are Euler-Poincaré equations on the Lie algebra of divergence free vector fields), these restrictions on the variations are related to the so-called "Lin constraints".

One obtains the Lie-Poisson equations on $\mathfrak{g}^*$ by the Legendre transformation:

$$\mu = \frac{\partial l}{\partial \xi}, \quad h(\mu) = \mu \cdot \xi - l(\xi).$$

In Lagrangian mechanics, dropping the variational principle is the analogue of Lie-Poisson reduction in which one drops the Poisson bracket from T^*G to $\mathfrak{g}^*$.

5.3 The Reduced Euler-Lagrange Equations

The Euler-Poincaré equations can be generalized to the situation in which G acts freely on a configuration space Q to obtain the *reduced Euler-Lagrange equations*. This process starts with a G-invariant Lagrangian $L : TQ \to \mathbb{R}$, which induces a reduced Lagrangian $l : TQ/G \to \mathbb{R}$. The Euler-Lagrange equations for L induce the reduced Euler-Lagrange equations on TQ/G. To compute them in coordinates, it is useful to introduce a principal connection on the bundle $Q \to Q/G$. Although any can be picked, a convenient choice is the mechanical connection, so we pause to recall its construction. We will make a similar construction for nonholonomic systems shortly.

The Mechanical Connection

Assume that there is a G-invariant metric on the configuration space. Normally this metric is the one associated with the kinetic energy of our mechanical system.

Definition 5.2 *The **mechanical connection** $\mathcal{A}$ is the connection on Q regarded as a bundle over shape space Q/G that is defined by declaring its horizontal space at a point $q \in Q$ to be the subspace that is the orthogonal complement to the tangent space to the group orbit through $q \in Q$ using the kinetic energy metric. The **locked inertia tensor** $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ (where $\mathfrak{g}^*$ denotes the dual of the vector space $\mathfrak{g}$) is defined by*

$$\langle \mathbb{I}(q)\xi, \eta \rangle = \langle\langle \xi_Q(q), \eta_Q(q) \rangle\rangle$$

where ξ_Q is the infinitesimal generator of $\xi \in \mathfrak{g}$ and where $\langle\langle \, , \rangle\rangle$ is the kinetic energy inner product.

The mechanical connection defines an equivariant $\mathfrak{g}$-valued one form $\mathcal{A}$ on Q. An explicit formula for it (see below and MARSDEN [1992] for further details and references) is

$$\mathcal{A}(v_q) = \mathbb{I}(q)^{-1}J(v_q) \tag{5.3.1}$$

where $J : TQ \to \mathfrak{g}^*$ is the *momentum map* defined, as we saw earlier, by

$$\langle J(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle$$

As a simple example, consider the angular momentum of a single particle moving in 3-space. We let $q, p \in \mathbb{R}^3$, and $J(q,p) = q \times p$. Identifying a tangent vector v_q to $\mathbb{R}^3$ at the point $q \in \mathbb{R}^3$ with the pair (q, v), formula (5.3.1) gives

$$\mathcal{A}(q,v) = \frac{1}{\|q\|^2}\,(q \times v)\,,$$

an $SO(3)$ connection on $\mathbb{R}^3 \backslash \{0\}$. Another characterization of the mechanical connection is that it picks out the "optimal" rotating frame, *i.e.,* it minimizes the kinetic energy subject to the constraint $J = \mu$. The mechanical connection $\mathcal{A}$ plays a fundamental role in the theory of geometric phases (MARSDEN, MONTGOMERY & RATIU [1990]), where holonomy of an associated connection is involved, and in stability theory where it is used to separate internal and rotational modes (SIMO, LEWIS & MARSDEN, [1991]).

The mechanical connection has the physical interpretation for a system of interconnected particles and rigid bodies, of being the *spatial* angular velocity of the instantaneously equivalent rigid body system obtained by locking all the joints. Thus, the phrase *(spatial) locked angular velocity* is sometimes used.

The Reduced Equations

To describe the reduced Lagrange d'Alembert equations, we make use of a connection on the principal G-bundle $Q \to Q/G$; for the Euler-Poincaré-equations, in which $Q = G$, the group structure automatically provides such a connection. For a more general choice of Q one can choose the mechanical connection as defined in the previous subsection.

Thus, assume that the bundle $Q \to Q/G$ has a given (principal) connection $\mathcal{A}$. Divide variations into horizontal and vertical parts — this breaks up the Euler-Lagrange equations on Q into 2 sets of equations that we now describe. Let x^α be coordinates on shape space Q/G and Ω^a be coordinates for vertical vectors in a local bundle chart. Drop L to TQ/G to obtain a reduced Lagrangian $l : TQ/G \to \mathbb{R}$ in which the group coordinates are eliminated. We can represent this reduced Lagrangian in a couple of ways. First, if we choose a local trivialization as we have described earlier, we obtain l as a function of the the variables are $(r^\alpha, \dot{r}^\alpha, \xi^a)$. However, it will be more convenient and intrinsic to change variables from ξ^a to the local version of the locked angular velocity, which has the physical interpretation of the *body* angular velocity, namely $\Omega = \xi + \mathcal{A}_{\mathrm{loc}}\dot{r}$, or in coordinates,

$$\Omega^a = \xi^a + A^a_\alpha(r)\dot{r}^\alpha.$$

We will write $l(r^\alpha, \dot{r}^\alpha, \Omega^a)$ for the local representation of l in these variables.

Theorem 5.3 *A curve $(q^i, \dot{q}^i) \in TQ$, satisfies the Euler-Lagrange equations if and only if the induced curve in TQ/G with coordinates given in a local trivialization by $(r^\alpha, \dot{r}^\alpha, \Omega^a)$ satisfies the* ***reduced Euler-Lagrange equations:***

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{r}^\alpha} - \frac{\partial l}{\partial r^\alpha} = \frac{\partial l}{\partial \Omega^a}\left(-\mathcal{B}^a_{\alpha\beta}\dot{r}^\beta + \mathcal{E}^a_{\alpha d}\Omega^d\right) \tag{5.3.2}$$

$$\frac{d}{dt}\frac{\partial l}{\partial \Omega^b} = \frac{\partial l}{\partial \Omega^a}(-\mathcal{E}^a_{\alpha b}\dot{r}^\alpha + C^a_{db}\Omega^d) \tag{5.3.3}$$

where $\mathcal{B}^a_{\alpha\beta}$ are the coordinates of the curvature $\mathcal{B}$ of $\mathcal{A}$, and $\mathcal{E}^a_{\alpha d} = C^a_{bd}\mathcal{A}^b_\alpha$.

The first of these equations is similar to the equations for a nonholonomic system written in terms of the constrained Lagrangian and that the second is similar to the momentum equation. It is useful to note that the first set of equations results from the variational principle of Hamilton by restricting the variations to be horizontal relative to the given connection. As we shall see, this is very similar to what one has in systems with nonholonomic constraints with the principle of Lagrange d'Alembert.

One other observation is of interest here. If one uses as variables, $(r^\alpha, \dot{r}^\alpha, p_a)$, where p is the body angular momentum, so that $p = \mathbb{I}_{\text{loc}}(r)\Omega = \partial l/\partial\Omega$, then the equations become (using the same letter l for the reduced lagrangian, an admitted abuse of notation):

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{r}^\alpha} - \frac{\partial l}{\partial r^\alpha} = p_a\left(-\mathcal{B}^a_{\alpha\beta}\dot{r}^\beta + \mathcal{E}^a_{\alpha d} I^{de} p_e\right) - p_d \frac{\partial I^{de}}{\partial r^\alpha} p_e \tag{5.3.4}$$

$$\frac{d}{dt}p_b = p_a(-\mathcal{E}^a_{\alpha b}\dot{r}^\alpha + C^a_{db} I^{de} p_e), \tag{5.3.5}$$

where I^{de} denotes the inverse of the matrix I_{ab}.

Connections are also useful in control problems with feedback. For example, Bloch, Krishnaprasad, Marsden & Sánchez de Alvarez [1992] found a feedback control that stabilizes rigid body dynamics about its middle axis using an internal rotor. This feedback controlled system can be described in terms of connections (Marsden & Sánchez de Alvarez [1994]): a shift in velocity (change of connection) turns the free Euler-Poincaré equations into the feedback controlled Euler-Poincaré equations.

6 The Nonholonomic Connection and Reconstruction

In this section we continue to discuss the application of the momentum equation to the problem of reconstructing paths on configuration space Q given a path in the base space Q/G. In many systems the base space Q/G corresponds to the set of variables which are directly controlled by the application of control forces, and hence we can follow any path in Q/G by application of appropriate forces. It is therefore natural to focus on how these paths lift, as described by the constraints, the generalized momenta, and the momentum equation, to the full configuration space. The main new tool to be introduced in this section is that of the *nonholonomic connection*, a synthesis of the mechanical and the kinematic connections; see also Bloch & Crouch [1992] and Yang [1992] for preliminary versions of similar ideas.

6.1 The Unconstrained Case

We will begin by recalling the reconstruction procedure for *unconstrained* mechanical systems. As we discussed in the preceding section, for unconstrained mechanical systems with symmetries, the equations of motion are naturally described in using the principal bundle $Q \to Q/G$. In essence, the dynamical equations split into two pieces by using Hamilton's principle $\delta L = 0$ and dividing the variations into vertical variations and a set of complementary variations. The vertical variations lead to a set of conservation laws of the form

$$\frac{d}{dt}\langle \mathbb{F}L, \eta_Q \rangle = 0,$$

for all $\eta \in \mathfrak{g}$. These equations are equivalent to the Euler-Poincaré equations when the Euler-Lagrange equations are written in a local trivialization. As we mentioned above, the mechanical connection is related to the momentum map and the locked inertia tensor by

$$\mathcal{A}(q) \cdot v_q = \mathbb{I}^{-1}(q) J(v_q).$$

Given a path in the base space Q/G, we can now use the connection to reconstruct the motion of the system in the full space Q. The conservation law can be written as

$$\mathcal{A}(q) \cdot \dot{q} = \mathbb{I}^{-1}(q) J(\dot{q}) = \mathbb{I}^{-1}(q)\mu$$

where $\mu \in \mathfrak{g}^*$ is a (constant) momentum. If we choose a local trivialization of the bundle with coordinates $q = (r, g) \in (Q/G) \times G$ (locally), the conservation law becomes

$$\mathcal{A}(q) \cdot \dot{q} = \mathrm{Ad}_g(g^{-1}\dot{g} + \mathcal{A}_{\mathrm{loc}}(r)\dot{r}) = (\mathrm{Ad}_g \mathbb{I}_{\mathrm{loc}}^{-1}(r) \mathrm{Ad}_g^*) \cdot \mu$$

where $\mathbb{I}_{\mathrm{loc}}$ is the local expression for the locked inertia tensor written as a function over Q/G. Rearranging this equation, we see that the group variables evolve according to

$$\dot{g} = g\left(-\mathcal{A}_{\mathrm{loc}}(r)\dot{r} + \Omega\right) \tag{6.1.1}$$

where $\Omega = \mathbb{I}_{\mathrm{loc}}^{-1}(r)p$ is the body angular velocity and where $p = \mathrm{Ad}_g^* \cdot \mu$ is the body angular momentum. Note that the variables p (or Ω if one is doing the Lagrangian point of view) are to be included amongst the variables in the reduced phase space. Thus, given a path $r(t)$ in the base variables, a motion in the body angular momentum (p) space or velocity (Ω) space, and an initial condition for the group variables, we can reconstruct the motion in the group and hence on the entire space, as in Marsden, Montgomery & Ratiu [1990]. Finally, we reiterate a basic fact from this discussion: *the body angular velocity $\Omega = \xi + \mathcal{A}_{\mathrm{loc}}(r)\dot{r}$ (where $\xi = g^{-1}\dot{g}$) is the local representative of the vertical part of the velocity vector $\dot{q}$.*

If nonholonomic constraints are present, it is still possible to reconstruct the path in the group variables given the path in the base. This is useful in control applications since it allows us to study the motion of the system without considering the full equations of motion. We break the following discussion into three cases: purely kinematic constraints, horizontal symmetries, and the general case. Examples of each of the cases are given in the §8. The purely kinematic case occurs when the constraint distribution complements the symmetry group orbit. In this case, it is clear that we do not get any nontrivial components to the momentum equation and that the constraint distribution itself defines a principal connection.

6.2 The Principal or Purely Kinematic Case

Recall that a set of nonholonomic constraints is said to be *purely kinematic* if the constraints define a connection on a principal bundle and that this situation occurs when the constraint distribution is G-invariant and the tangent space to the group orbit forms a complement to the constraint distribution; that is, the subbundle with the fibers $\mathcal{S}_q = \mathcal{D}_q \cap T_q(\mathrm{Orb}(q)) = \{0\}$ for all $q \in Q$. What this really means is that there are no momentum equations in this case and that correspondingly there is no analogue of the body angular momentum or velocity, as there was in the preceding discussion of unconstrained systems. In particular, relative to a local trivialization $q = (r, g)$ the constraints can be written as

$$A(q)\dot{q} = \left[\mathrm{Ad}_g(g^{-1}\dot{g} + \mathcal{A}_{\mathrm{loc}}(r)\dot{r})\right]_Q = 0.$$

The motion in the fibers is thus given by

$$\dot{g} = -g\mathcal{A}_{\mathrm{loc}}(r)\dot{r}$$

and we can reconstruct the group motion given the trajectory in Q/G. In this case, as we saw previously, the equations reduce to second order equations for r; that is, to second order equations on Q/G. The motion on the full space is then determined by the solution to these reduced equations followed by first order equations for the group variables.

This can be said a slightly different way: in the case of purely kinematic constraints, the kinematic connection replaces the mechanical connection to determine the motion in the fibers. This situation occurs only when the constraint distribution $\mathcal{D}$ and the vertical subspace $T_q(\mathrm{Orb}(q))$ are such that $T_qQ = T_q(\mathrm{Orb}(q)) \oplus \mathcal{D}_q$, so that $\mathcal{D}_q$ can be taken as the horizontal space for a connection. Thus the conservation law which would govern the motion in the group variables if no constraints were present is replaced by the motion dictated by the constraints. See KOILLER [1992] for a further discussion of the purely kinematic case, including a description of reduction in that context. As we also noted already, this reduction result can be obtained as a special case of the results of MARSDEN & SCHEURLE [1993] where it is shown how to reduce the horizontal part of the variational principal relative to any connection.

6.3 The Case of Horizontal Symmetries

A second case in which it is possible to lift the motion from the base Q/G to the fibers using a connection is when there are enough horizontal symmetries such that they and the constraints interact in a complementary fashion. This situation occurs, for example, when there is a subgroup of G whose action on the configuration space satisfies the constraints. We call a symmetry of this type a *horizontal symmetry* (relative to the kinematic constraints). When horizontal symmetries are present, the motion in the group variables can be reconstructed by combining the kinematic constraints with the conservation laws corresponding to the horizontal symmetries. This is the case for the ball on the rotating plate.

We begin by restricting ourselves to the case when $\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_qQ$ for all $q \in Q$ and we assume that there exists a a subgroup $H \subset G$ such that $\xi_Q \in \mathcal{D}$ for all $\xi \in \mathfrak{h}$ and $\mathcal{D}_q \cap T_q(\mathrm{Orb}_G(q)) = T_q(\mathrm{Orb}_H(q))$. We call H the group of horizontal symmetries and define the momentum map $J_H : TQ \to \mathfrak{h}^* \subset \mathfrak{g}^*$ as

$$\langle J_H(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q \rangle \qquad \xi \in \mathfrak{h}.$$

For a Lagrangian of the form kinetic energy minus potential energy we can write the generalized momenta as linear functions of the velocity and these generalized momenta are constant along solution curves since $\xi^q = \xi \in \mathfrak{h}$ is constant. Thus we have

$$\langle J_H(q) \cdot \dot{q}, \xi \rangle = \langle \mu, \xi \rangle \qquad \xi \in \mathfrak{h}$$

where $\mu \in \mathfrak{h}^*$ is a constant and we see that the generalized momentum has the form of an affine constraint

$$J_H(q) \cdot \dot{q} = \mu. \tag{6.3.1}$$

To reconstruct the motion in the fibers, we build a connection on $Q \to Q/G$ by augmenting the kinematic constraints with the conservation law. Let $\mathbb{I}(q) : \mathfrak{h} \to \mathfrak{h}^*$ be the locked inertia tensor relative to $\mathfrak{h}$, defined, as in definition 5.3, by

$$\langle \mathbb{I}(q)\xi, \eta \rangle = \langle\!\langle \xi_Q, \eta_Q \rangle\!\rangle \qquad \xi, \eta \in \mathfrak{h}.$$

We define a map $A^{\mathrm{sym}} : TQ \to \mathcal{S}$ as

$$A^{\mathrm{sym}}(v_q) = \left(\mathbb{I}^{-1}(q) J_H(v_q)\right)_Q \tag{6.3.2}$$

and the conservation law (6.3.1) can be rewritten as an affine constraint

$$A^{\mathrm{sym}}(\dot{q}) = \left(\mathbb{I}^{-1}(q)\mu\right)_Q. \tag{6.3.3}$$

The one form A^{sym} takes values in $\mathcal{S}_q = T_q(\mathrm{Orb}_H(q))$ and is equivariant with respect to the full group action since the kinetic energy metric is invariant and the momentum map is equivariant. It also follows from the definition of the momentum map that A^{sym} is a projection onto $\mathcal{S}$ and hence it maps vectors on $\mathcal{S}_q$ to themselves.

By assumption, the constraint distribution $\mathcal{D}$ is invariant and if we choose a subspace $\mathcal{U}_q \subset T_q(\mathrm{Orb}(q))$ such that $T_q(\mathrm{Orb}(q)) = \mathcal{U}_q \oplus \mathcal{S}_q$ then we can represent the constraints using a $\mathcal{U}$ valued one form $A^{\mathrm{kin}} : TQ \to \mathcal{U}_q$ where A^{kin} satisfies the following conditions:

$$\left.\begin{array}{c} A^{\mathrm{kin}}(v_q) = 0 \quad \text{if and only if} \quad v_q \in \mathcal{D}_q \\ A^{\mathrm{kin}}(v_q) = v_q \quad \text{for all} \quad v_q \in \mathcal{U}_q \\ \Phi_{g*} A^{\mathrm{kin}} = A^{\mathrm{kin}} \Phi_{g*}. \end{array}\right\} \tag{6.3.4}$$

We now combine the two mappings to form a new mapping $A : TQ \to T\mathrm{Orb}$, where $T\mathrm{Orb}$ denotes the union of the tangent spaces to the group orbits, that is, to the vertical bundle for the projection $Q \to Q/G$, as follows:

$$A = A^{\mathrm{kin}} + A^{\mathrm{sym}}. \tag{6.3.5}$$

The mapping $A : TQ \to T\mathrm{Orb}$ is an equivariant Ehresmann connection on the bundle $Q \to Q/G$ and hence we can write

$$A(v_q) = (\mathcal{A}(v_q))_Q,$$

where $\mathcal{A} : TQ \to \mathfrak{g}$ is a principal connection. To see that A is an Ehresmann connection it suffices to show that it is a projection on $\mathcal{U}_q$ and $\mathcal{S}_q$. This follows immediately from the fact that A^{sym} and A^{kin} are equivariant projections onto $\mathcal{S}$ and $\mathcal{U}$ respectively and $T_q(\mathrm{Orb}(q)) = \mathcal{S}_q \oplus \mathcal{U}_q$. Equivariance follows directly from the equivariance of $\mathcal{U}_q$ and $\mathcal{S}_q$ and the existence of $\mathcal{A}$ follows from general properties of equivariant Ehresmann connections.

Definition 6.1 *We call the map $A : TQ \to T_q(\mathrm{Orb}(q))$ defined by equations (6.3.3)-(6.3.5) the* ***nonholonomic connection*** *(in the case of horizontal symmetries).*

Notice that the nonholonomic connection in the case of horizontal symmetries reduces to the kinematic connection in the purely kinematic case and the mechanical connection in the unconstrained case.

The overall motion of the system satisfies

$$A(q) \cdot \dot{q} = (\mathbb{I}^{-1}(q)\mu)_Q \tag{6.3.6}$$

which has the form of an affine constraint. The locked inertia tensor relative to $\mathfrak{h}$ satisfies

$$\mathbb{I}(g \cdot q) = \mathrm{Ad}^*_{g^{-1}} \mathbb{I}(q) \mathrm{Ad}_{g^{-1}}$$

and hence in general the affine part of the constraint (6.3.6) is *not* equivariant since

$$\mathbb{I}^{-1}(g \cdot q)\mu = \mathrm{Ad}_g \mathbb{I}^{-1}(q) \mathrm{Ad}^*_g \mu \neq \mathrm{Ad}_g(\mathbb{I}^{-1}(q)\mu).$$

This lack of invariance of the affine portion, as in the unconstrained case, would cause problems in the construction of a principal connection if one tried to make full use of the conservation laws by holding μ fixed. On the other hand, the actual reduced variables correspond to the body angular velocity or momentum, and in these variables, equivariance is restored. Let us be more specific: Equation (6.3.6) describes how to lift paths from the base space Q/G to the full space Q. This is most easily seen relative to a local trivialization $q = (r, g)$, where the constraints can be written as

$$\mathcal{A}(q) \cdot \dot{q} = \mathrm{Ad}_g(g^{-1}\dot{g} + \mathcal{A}_{\mathrm{loc}}(r)\dot{r}) = \mathrm{Ad}_g \mathbb{I}^{-1}_{\mathrm{loc}}(r) \mathrm{Ad}^*_g \mu$$

where $\mathrm{Ad}_g \mathbb{I}^{-1}_{\mathrm{loc}}(r) \mathrm{Ad}^*_g \mu$ is the $\mathfrak{g}^{\mathcal{D}}$-valued function associated with the constant momentum $\mu \in \mathfrak{h}^*$. This equation can be rewritten as

$$\dot{g} = g(-\mathcal{A}_{\mathrm{loc}}(r)\dot{r} + \mathbb{I}^{-1}_{\mathrm{loc}}(r) \mathrm{Ad}^*_g \mu)$$

which shows how the path $r(t) \in Q/G$ lifts to the fibers.

Noting that $\mathrm{Ad}^*_g \mu = p$ is the body angular momentum and $\mathbb{I}^{-1}_{\mathrm{loc}}(r)p = \Omega$ is the corresponding body angular velocity, which may be regarded as a dynamical variable in its own right, then the reconstruction equation takes the form

$$\dot{g} = g(-\mathcal{A}_{\mathrm{loc}}(r)\dot{r} + \Omega). \tag{6.3.7}$$

This equation again has the form $\dot{g} = g\xi$ and where $\xi = -\mathcal{A}_{\mathrm{loc}}(r)\dot{r} + \Omega$ has been determined by equations of motion that themselves are independent of the group variable. This form, rather than the form in which the momentum has been set equal to a constant shows the decoupling from the group variables most clearly. As we saw before, and will do more generally below, it is the variable ξ rather than the body angular velocity variable that evolves by means of a component of the Euler-Poincaré equation. On the other hand, it is Ω that is the vertical variable relative to the nonholonomic connection

$$\mathrm{ver}_q \dot{q} = \Omega = \mathcal{A}_{\mathrm{loc}}(r)\dot{r} + g^{-1}\dot{g},$$

which is an instance of the general coordinate expression for the vertical part of a principal connection. As we shall see in a moment, this point of view generalizes to the case of nonhorizontal symmetries.

The preceding equations only hold when $\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_qQ$ and $\mathcal{D}_q \cap T_q(\mathrm{Orb}_G(q)) = T_q(\mathrm{Orb}_H(q))$. If we drop the second restriction, then the reconstruction procedure must be modified to account for the interaction between the constraints and the symmetries. The developments below will include this more general situation.

Finally we end this section with a notational remark. In the general nonholonomic case, as we have seen, the momentum map need not be conserved. In any case, even if it is, the momentum in body representation, p is not constant.

6.4 The Nonholonomic Connection

We now consider the most general case, where the symmetries are not necessarily horizontal. Although it is not needed for everything that we will be doing, the examples and the theory are somewhat simplified if we make the following assumption:

Dimension Assumption *The constraints and the orbit directions span the entire tangent space to the configuration space:*

$$\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_qQ \tag{6.4.1}$$

If this condition holds, we shall say that we are in the ***principal case****.*

In this case, the momentum equation can be used to augment the constraints and provide a connection on $Q \to Q/G$. Let $J^{\mathrm{nhc}} : TQ \to (\mathfrak{g}^{\mathcal{D}})^*$ be the nonholonomic momentum map,

$$\langle J^{\mathrm{nhc}}(q) \cdot \dot{q}, \xi^q \rangle = \langle \mathbb{F}L, \xi^q_Q \rangle$$

and define, as before, a map $A^{\mathrm{sym}}_q : T_qQ \to \mathcal{S}_q = \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))$ given by

$$A^{\mathrm{sym}}_q(v_q) = (\mathbb{I}^{-1} J^{\mathrm{nhc}}(v_q))_Q \tag{6.4.2}$$

This map is equivariant and a projection onto $\mathcal{S}_q$. Here $\mathbb{I} : \mathfrak{g}^{\mathcal{D}} \to (\mathfrak{g}^{\mathcal{D}})^*$ is the locked inertia tensor relative to $\mathfrak{g}^{\mathcal{D}}$; it is defined in the same way as before.

If we now choose $\mathcal{U}_q \subset T_q(\mathrm{Orb}(q))$ such that $T_q(\mathrm{Orb}(q)) = \mathcal{S}_q \oplus \mathcal{U}_q$ then we can synthesize a connection which encodes both the constraints and the momenta, as before. This splitting of subspaces is shown in Figure 6.4.1. Let $A^{\mathrm{kin}}_q : T_qQ \to \mathcal{U}_q$ be a $\mathcal{U}_q$ valued form that projects $\mathcal{U}_q$ onto itself and maps $\mathcal{D}_q$ to zero; for example, it can be given by orthogonal projection relative to the kinetic energy metric (this will be our default choice). The constraints plus momentum equation can thus be written as

$$\begin{aligned} A^{\text{kin}}(q) \cdot \dot{q} &= 0 && \text{(constraints)} \\ A^{\text{sym}}(q) \cdot \dot{q} &= (\mathbb{I}^{-1}(q)p)_Q && \text{(momenta)}, \end{aligned}$$

where $p \in (\mathfrak{g}^{\mathcal{D}})^*$ is the time dependent momentum defined by $p = \left\langle J^{\text{nhc}}(q) \cdot \dot{q}, \xi^q \right\rangle$.

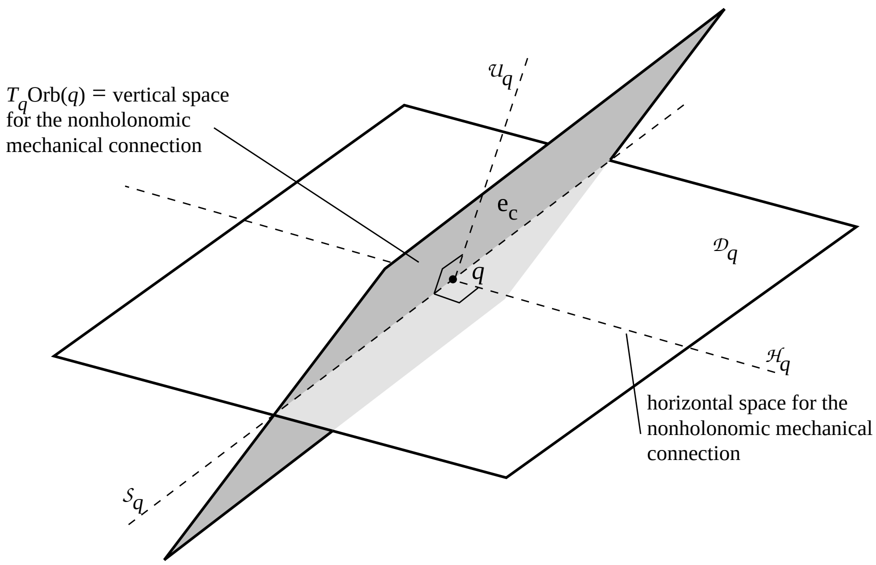

Figure 6.4.1: Subspace definitions for the nonholonomic connection

Definition 6.2 *Under the dimension assumption in equation (6.4.1), and the assumption that the Lagrangian is of the form kinetic minus potential energies, the* ***nonholonomic connection*** $\mathcal{A}$ *is the connection on the principal bundle* $Q \to Q/G$ *whose horizontal space at the point* $q \in Q$ *is given by the orthogonal complement to the space* $\mathcal{S}_q$ *within the space* $\mathcal{D}_q$*; see Figure 6.4.1.*

Under the assumption that the distribution is invariant (condition (S1)), and from the fact that the group action preserves orthogonality, it follows that the distribution $\mathcal{S}$ and the horizontal spaces transform to themselves under the group action. Thus, we get:

Proposition 6.3 *Under the assumptions in the previous definition and the condition (S1), the nonholonomic connection is a principal connection.*

Using the preceding expressions, an expression for the nonholonomic connection as an Ehresmann connection (and hence also as a principal connection) is given by our earlier calculations. In fact, one can readily check that the following proposition holds:

Proposition 6.4 *The nonholonomic connection regarded as an Ehresmann connection is given by*

$$A = A^{\text{kin}} + A^{\text{sym}}. \tag{6.4.3}$$

When the connection is regarded as a principal connection (i.e., takes values in the Lie algebra rather than the vertical space) we will use the symbol $\mathcal{A}$*.*

The nonholonomic connection defined here agrees with the definition in the horizontal case. (In making this comparison, note that in the general definition of the connection, we do *not* fix the value of μ but rather let it be determined by the point v_q at which the connection is evaluated.)

The affine constraint $A(q) \cdot \dot{q} = (\mathbb{I}^{-1}(q) \cdot p)_Q$ describes the lifting of paths from the base. The formula for the nonholonomic connection is given in terms of A^{kin}, which depends on the choice of complement $\mathcal{U}_q$ to $\mathcal{S}_q$ within the tangent space to the orbit. However, it is easily seen that $A : TQ \to T\mathrm{Orb}$ is independent of this choice, as it must be since the definition of the nonholonomic connection was manifestly independent of this choice.

We shall compute the equations of motion in terms of the nonholonomic connection in a local trivialization of the bundle $Q \to Q/G$ in the following sections.

6.5 Special Cases

Various special cases can be conveniently classified by the generic and extreme ways the subspaces in the preceding figure intersect. For example, the purely kinematic case is when the space $\mathcal{S}_q$ is zero dimensional. The extreme case in which the tangent space to the orbit is a subset of the space of constraints is itself an extreme case of that of horizontal symmetries, etc. These different cases we have discussed are summarized in Table 6.5.1.

Case	Conditions	Connection
Unconstrained	$\mathcal{D}_q = T_qQ$	$\mathcal{A}^{\mathrm{sym}}(\dot{q}) = \mathbb{I}^{-1}J(\dot{q})$
Purely kinematic	$\mathcal{D}_q \cap T_q(\mathrm{Orb}(q)) = \{0\}$	$\mathcal{A}^{\mathrm{kin}}(\dot{q}) = 0$
Horizontal symmetries	$\mathcal{D}_q \cap T_q(\mathrm{Orb}_G(q)) = T_q(\mathrm{Orb}_H(q))$	$\mathcal{A}^{\mathrm{sym}}(\dot{q}) + \mathcal{A}^{\mathrm{kin}}(\dot{q}) = \mathbb{I}^{-1}J_H(\dot{q})$
General principal bundle case	$\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_qQ$	$\mathcal{A}^{\mathrm{sym}}(\dot{q}) + \mathcal{A}^{\mathrm{kin}}(\dot{q}) = \mathbb{I}^{-1}J^{\mathrm{nhc}}(\dot{q})$

Table 6.5.1: Special cases of the nonholonomic connection (principal case).

In addition to these possibilities, one can also consider the case where $\mathcal{D}_q + T_q(\mathrm{Orb}(q)) \neq T_qQ$. When this happens the base space for the Ehresmann connection can no longer be chosen as Q/G and hence a bigger base space must be chosen. However, the basic constructions still hold with the momentum augmenting the constraints to give a synthesized connection.

Within this overall framework, reduction is also possible in certain cases. For example, in the purely kinematic case, Koiller [1992] showed that the dynamics of the system drop to the base space Q/G. Similarly, in the case of horizontal symmetries, we have discussed the situation above. The general case will be discussed in the following section and the reduced equations computed. In the general case, the reduced equations will define a dynamical system on the space $\mathcal{D}/G$, and the reconstruction problem, which we have largely discussed already, will be the problem of lifting the dynamics from $\mathcal{D}/G$ back to the space $\mathcal{D} \subset TQ$.

7 The Reduced Lagrange d'Alembert Equations

The goal of this section is to compute the equations on the reduced space $\mathcal{D}/G$. The strategy is to explore the equations of motion, split according to the nonholonomic connection that was constructed in the preceding section. Throughout this section we make the dimension assumption so that the nonholonomic connection is a principal connection. Without this assumption, one would have to assume an additional bundle structure. We avoid this for simplicity and because the dimension assumption holds in all our examples and other related ones we know about (such as the bicycle, the rolling ellipsoid, etc.). We will begin, however, with a second form of the momentum equation that makes use of the geometry associated with the nonholonomic connection. We will follow this with the computation of the reduced equations.

7.1 The Momentum Equation in an Orthogonal Body Frame

We shall first compute the reduced form of the momentum equation that will be one of the sets of equations comprising the reduced Lagrange d'Alembert equations. This splitting of the equations is associated with breaking up the variations that go into the Lagrange d'Alembert principle into vertical and horizontal parts relative to the nonholonomic connection. To do this, we shall make one further assumption, namely that the initial Lagrangian is of the form of kinetic minus potential energy; in particular, the metric structure defined by the kinetic energy will be used. Using the kinetic energy metric, we choose our moving basis $e_c(q)$ to be orthogonal; that is, the corresponding generators $[e_c(q)]_Q$ are orthogonal in the given kinetic energy metric. (Actually, all that is needed is that the vectors in the set of basis vectors corresponding to the subspace $\mathcal{S}_q$ be orthogonal to the remaining basis vectors.) The metric tensor will be denoted by g_{ij}.

We begin by recalling the decompositions defined by the nonholonomic connection described in the preceding section. Refer to Figure 7.1.

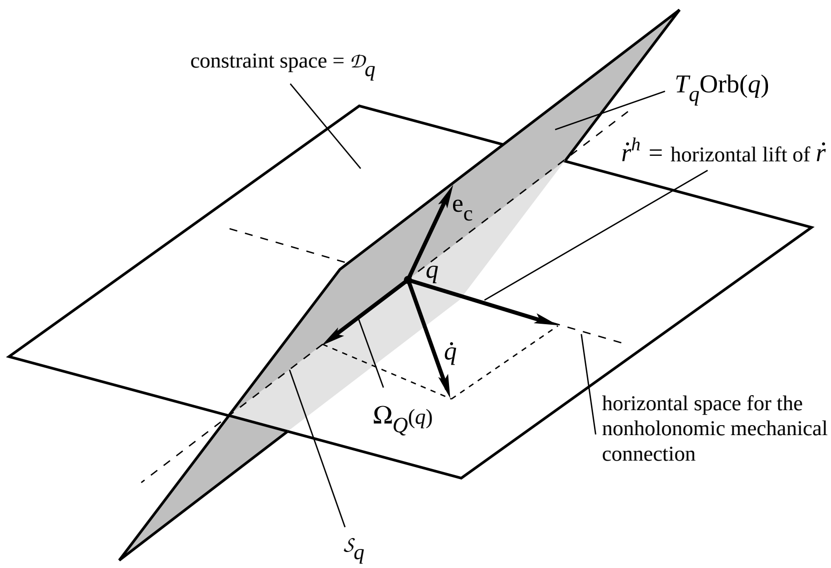

Figure 7.1.1: The decomposition of $\dot q$ into vertical and horizontal pieces relative to the nonholonomic connection.

Given a velocity vector $\dot q$ that satisfies the constraints, we orthogonally decompose it into a piece in $\mathcal{S}_q$ and an orthogonal piece denoted $\dot r^h$. We regard $\dot r^h$ as the horizontal lift of a velocity vector $\dot r$ on shape space; recall that in a local trivialization, the horizontal lift to the point (r, g) is given by (compare equation (2.2.2)

$$\dot r^h = (\dot r, -\mathcal{A}_{\mathrm{loc}}\dot r) = (\dot r^\alpha, -\mathcal{A}^a_\alpha \dot r^\alpha)$$

where $\mathcal{A}^a_\alpha$ are the components of the nonholonomic connection (recall that it is a principal connection) in a local trivialization.

We will denote the decomposition of $\dot q$ by

$$\dot q = \Omega_Q(q) + \dot r^h,$$

so that for each point q, ω is an element of the Lie algebra and represents the spatial angular velocity of the locked system. Note that in this expression, the constraints are implicitly included. In a local trivialization, we can write, at a point (r, g)

$$\Omega = \mathrm{Ad}_g(\Omega_{\mathrm{loc}})$$

so that Ω_{loc} represents the body angular velocity. Thus,

$$\Omega_{\mathrm{loc}} = \mathcal{A}_{\mathrm{loc}}\dot{r} + \xi$$

and, at each point q, the constraints are that Ω belongs to $\mathfrak{g}^q$, *i.e.*,

$$\Omega \in \mathrm{span}\{e_1(r), e_2(r), \ldots, e_m(r)\}.$$

As noted above, the vector $\dot{r}^h$ need not be orthogonal to the whole orbit, just to the piece $\mathcal{S}_q$. Even if $\dot{q}$ does not satisfy the constraints we can decompose it into three parts according to the figure and write

$$\dot{q} = \Omega_Q(q) + \dot{r}^h = \Omega_Q^{\mathrm{nh}}(q) + \Omega_Q^{\perp}(q) + \dot{r}^h,$$

where Ω_Q^{nh} lies in the space $\mathcal{S}_q$, that is, it satisfies the constraints, and is perpendicular within $T_q\mathrm{Orb}$ to $\Omega_Q^{\perp}$. The relation $\Omega_{\mathrm{loc}} = \mathcal{A}_{\mathrm{loc}}\dot{r} + \xi$ is valid even if the constraints do not hold; also note that this decomposition of Ω corresponds to the decomposition of the nonholonomic connection given by $A = A^{\mathrm{kin}} + A^{\mathrm{sym}}$ that was given in Proposition (6.4.3).

We begin with the momentum equation (4.4.2) in body representation, which we recall here for convenience:

$$\frac{d}{dt}p_b = \left\langle \frac{\partial l}{\partial \xi}, [\xi, e_b] + \frac{\partial e_b}{\partial r^\alpha}\dot{r}^\alpha \right\rangle. \tag{7.1.1}$$

This equation is one of the reduced equations since it manifestly decouples from the group variables. We shall now work out this equation in coordinates.

To avoid confusion, we will make the following index and summation conventions

1. The first batch of indices range from 1 to m corresponding to the symmetry directions along constraint space. These indices will be denoted $a, b, c, d, \ldots$ and a summation from 1 to m will be understood.

2. The second batch of indices range from $m+1$ to k corresponding to the symmetry directions not aligned with the constraints. Indices for this range or for the whole range 1 to k will be denoted by $a', b', c', \ldots$ and the summations will be given explicitly.

3. The indices $\alpha, \beta, \ldots$ on the shape variables r range from 1 to σ. Thus, σ is the dimension of the shape space Q/G and so $\sigma = n - k$. The summation convention for these indices will be understood.

We shall need the following calculation:

Proposition 7.1 *In a local trivialization we have*

$$\left\langle \frac{\partial l}{\partial \xi}, \eta \right\rangle = I_{ac}(r)\Omega^a\eta^c + \sum_{a'=m+1}^{k} \lambda_{a'\alpha}\eta^{a'}\dot{r}^\alpha = p_c\eta^c + \sum_{a'=m+1}^{k} \lambda_{a'\alpha}\eta^{a'}\dot{r}^\alpha. \tag{7.1.2}$$

In this equation, the partial derivatives of l are evaluated at a point $(r, \dot{r}, \xi)$ satisfying the constraints (that is, the corresponding $\Omega_{\mathrm{loc}} = \xi + \mathcal{A}_{\mathrm{loc}}\dot{r}$ lies in $\mathfrak{g}^q$) and η is an arbitrary element of $\mathfrak{g}$. Also,

$$p_b = I_{ab}(r)\Omega^a,$$

where $I_{ab}(r)$ are the coefficients of the locked inertia tensor $\mathbb{I}_{\rm loc}(r)$ in a local trivialization (recall from the last section that the locked inertia tensor has indices that range only over the first batch), and where

$$\lambda_{a'\alpha} = l_{a'\alpha} - \sum_{b'=1}^{k} l_{a'b'}\mathcal{A}_\alpha^{b'} := \frac{\partial l}{\partial \xi^{a'}\partial \dot{r}^\alpha} - \sum_{b'=1}^{k} \frac{\partial l}{\partial \xi^{a'}\partial \xi^{b'}}\mathcal{A}_\alpha^{b'}, \tag{7.1.3}$$

for $a' = m+1, \ldots k$.

Proof We denote the kinetic energy metric on T_qQ by $\langle\langle\, , \rangle\rangle_q$. The corresponding metric on $\mathfrak{g}$ restricted to the subspace $\mathfrak{g}^q$ gives the locked inertia tensor as we saw before.

The kinetic energy is given as follows, without the assumption that $\dot{q}$ satisfies the constraints:

$$\begin{aligned} K(q,\dot{q}) &= \frac{1}{2}\left\langle\left\langle \Omega_Q^{\rm nh} + \Omega_Q^\perp + \dot{r}^h, \Omega_Q^{\rm nh} + \Omega_Q^\perp + \dot{r}^h \right\rangle\right\rangle_q \\ &= \frac{1}{2}\left\langle\left\langle \Omega_Q^{\rm nh}, \Omega_Q^{\rm nh} \right\rangle\right\rangle_q + \left\langle\left\langle \Omega_Q^\perp, \dot{r}^h \right\rangle\right\rangle_q + \frac{1}{2}\left\langle\left\langle \Omega_Q^\perp, \Omega_Q^\perp \right\rangle\right\rangle_q + \frac{1}{2}\left\langle\left\langle \dot{r}^h, \dot{r}^h \right\rangle\right\rangle_q, \end{aligned} \tag{7.1.4}$$

where we have suppressed the q dependence of $\Omega_Q(q)$ for simplicity.

Now we pass to a local trivialization and remove the explicit g dependence. We change variables to $(r, \dot{r}, \Omega)$ by the transformation $\Omega = \xi + \mathcal{A}_{\rm loc}\dot{r}$, which is valid even if the constraints are not satisfied. The partial derivatives with respect to Ω equal those with respect to ξ (evaluated at the corresponding points).

To form the reduced Lagrangian, we substitute $\dot{r}^h = (\dot{r}^\alpha, -\mathcal{A}_\alpha^a \dot{r}^\alpha)$ into the second term and arrive at

$$\frac{1}{2} I_{ac}\Omega^a\Omega^c + \sum_{a'=m+1}^{k} l_{a',\alpha}\Omega^{a'}\dot{r}^\alpha - \sum_{a',c'=m+1}^{k} l_{a'c'}\Omega^{a'}\mathcal{A}_\alpha^{c'}\dot{r}^\alpha + \chi$$

where Ω^a and $\Omega^{a'}$ are the components of $\Omega^{\rm nh}$ and $\Omega^\perp$ respectively, where the subscripts on the l denote the corresponding partial derivatives, as above, and where χ corresponds to the last two terms in (7.1.4), which will vanish when the partials are taken with respect to Ω at $\Omega^\perp = 0$. It should now be clear that the derivatives of this expression evaluated at $\Omega^\perp = 0$ are as stated in the proposition. ■

The coefficients $\lambda_{a'\alpha}$ measure the failure of the horizontal space for the nonholonomic connection to be orthogonal to the tangent space to the orbit.

Next, for each b such that $1 \leq b \leq m$, we write out the components of the remaining expression in (7.1.1):

$$\begin{aligned} [\xi, e_b] + \frac{\partial e_b}{\partial r^\alpha}\dot{r}^\alpha &= [\Omega - \mathcal{A}_{\rm loc}\dot{r}, e_b] + \frac{\partial e_b}{\partial r^\alpha}\dot{r}^\alpha \\ &= \sum_{c'=1}^{k} C_{ab}^{c'}\Omega^a e_{c'} - \sum_{a',c'=1}^{k} C_{a'b}^{c'}\mathcal{A}_\alpha^{a'}\dot{r}^\alpha e_{c'} + \sum_{c'=1}^{k}\gamma_{b\alpha}^{c'}\dot{r}^\alpha e_{c'} \end{aligned} \tag{7.1.5}$$

where the symbols such as $C_{a'b}^{c'}$ are the corresponding components of the structure constants in the given basis and where we have written (compare equation (4.3.3))

$$\frac{\partial e_b}{\partial r^\alpha} = \sum_{c'=1}^{k}\gamma_{b\alpha}^{c'} e_{c'} \tag{7.1.6}$$

and

$$\mathcal{A}_{\rm loc}\dot{r} = \sum_{a'=1}^{k}\mathcal{A}_\alpha^{a'}\dot{r}^\alpha e_{a'}. \tag{7.1.7}$$

Substituting (7.1.2), (7.1.5), (7.1.6) and (7.1.7) into (7.1.1), we arrive at the following.

Proposition 7.2 *The momentum equation in an orthogonal body frame is given as follows:*

$$\frac{d}{dt}p_b = C^c_{ab} I^{ad} p_c p_d + \mathcal{D}^c_{b\alpha} \dot{r}^\alpha p_c + \mathcal{D}_{\alpha\beta b} \dot{r}^\alpha \dot{r}^\beta, \tag{7.1.8}$$

where

$$\mathcal{D}^c_{b\alpha} = \sum_{a'=1}^{k} -C^c_{a'b} \mathcal{A}^{a'}_\alpha + \gamma^c_{b\alpha} + \sum_{a'=m+1}^{k} \lambda_{a'\alpha} C^{a'}_{ab} I^{ac}, \tag{7.1.9}$$

$$\mathcal{D}_{\alpha\beta b} = \sum_{a'=m+1}^{k} \lambda_{a'\alpha} \left(-C^{a'}_{ab} \mathcal{A}^a_\beta + \gamma^{a'}_{b\beta} \right). \tag{7.1.10}$$

In the case of the snakeboard, the subspace $\mathfrak{g}^q$ is one dimensional as we shall see, and the following corollary applies.

Corollary 7.3 *If the subspace $\mathfrak{g}^q$ is either one dimensional or abelian, then the first term on the right hand side of (7.1.8), which is quadratic in p, is zero.*

Another notable special case is the following, which will be used in the example of a constrained particle in $\mathbb{R}^3$ to produce a nontrivial parallel transport equation.

Corollary 7.4 *If $\mathfrak{g}$ is abelian, and if the horizontal space is (kinetic energy metric) orthogonal to the group orbit, then the momentum equation is in the form of a parallel transport equation over the curve $r(t)$ in shape space:*

$$\frac{d}{dt}p_b = \gamma^c_{b\alpha} \dot{r}^\alpha p_c.$$

We observe that the parallel transport form of the equations is characterized by the vanishing of the terms in the momentum equation that are purely quadratic in $\dot{r}$ and in p. This situation is important in understanding the complete integrability of some systems, such as Routh's problem of the rolling ball in a surface of revolution; cf Zenkov [1995]. In all the examples considered later, the momentum equation does not have the terms quadratic in p, but many of them do have the terms quadratic in $\dot{r}$. An example of one with the momentum equations quadratic in p is the rolling nonhomogeneous ball, which has symmetry group including the nonabelian group $SO(3)$. Note that these terms quadratic in p are exactly those appearing in the Euler-Poincaré equations.

7.2 The Reduced Equations

We now are in a position to put several parts of the preceding discussions together. As we saw above, the momentum equation in body representation decouples from the group variables themselves, which is important for the reconstruction strategy. On the other hand, this is a local representation for the intrinsic equations on the space $\mathcal{D}/G$. As we mentioned before, it is convenient to write them in local representation in terms of the variables Ω and $\dot{r}$ for several reasons:

1. This split of the equations corresponds to a global intrinsic split of the Lagrange d'Alembert principle according to the nonholonomic connection (we emphasize that there is some freedom here; other connections can be used in its place).

2. This split enables us to use the (locked) body angular velocity Ω as a basic variable instead of ξ since it has better diagonalization properties for the kinetic energy and will ultimately be more useful for purposes of stability analyses; these two variables are related by the velocity shift given by the nonholonomic connection:

$$\Omega_{\text{loc}} = \mathcal{A}_{\text{loc}} \dot{r} + \xi.$$

We will show that the equations of motion can be written (using a local trivialization) as three systems of equations, namely

- The constraint equations
- The reduced Euler-Lagrange equations using the nonholonomic connection for the variable $\dot{r}$
- The momentum equation (of Euler-Poincaré type) in body representation.

We formulate the reduced Lagrange d'Alembert equations under the assumptions of Proposition 6.3. In this context, the Lagrange d'Alembert principle may be broken up into two principles by breaking the variations δq into two parts, namely parts that are horizontal with respect to the nonholonomic connection and parts that are vertical (but still in $\mathcal{D}$). We will use as variables, $(r^\alpha, \dot{r}^\alpha, \Omega^a)$ where $(r, \dot{r})$ are variables in the base and where Ω is the vertical part (the locked body angular velocity). Let $l_c(r, \dot{r}, \Omega)$ denote the reduced Lagrangian written in terms of these variables as before; the subscript c is used to indicate the fact that Ω is confined to the constraint subspace $\mathfrak{g}^q$. Use the *orthogonal* basis $e_1(r), e_2(r), \dots, e_m(r), e_{m+1}(r), \dots e_k(r)$ introduced for the momentum equation in body representation (recall that this means that the first m elements are orthogonal as are the second $k-m$ elements but that the two sets need not be orthogonal to each other). Let

$$p_b(r, \dot{r}, \Omega) = \left\langle \frac{\partial l_c}{\partial \Omega}, e_b(r) \right\rangle, \quad b = 1, \dots, m.$$

Theorem 7.5 *Using the preceding notation, the following* ***reduced nonholonomic Lagrange d'Alembert equations*** *hold for each* $1 \le \alpha \le \sigma$ *and* $1 \le b \le m$*:*

$$\frac{d}{dt}\frac{\partial l_c}{\partial \dot{r}^\alpha} - \frac{\partial l_c}{\partial r^\alpha} = p_a\left(-\mathcal{B}^a_{\alpha\beta}\dot{r}^\beta + \mathcal{E}^a_{\alpha d} I^{dc} p_c\right) \tag{7.2.1}$$

$$\frac{d}{dt}p_b = C^c_{ab} I^{ad} p_c p_d + \mathcal{D}^c_{b\alpha}\dot{r}^\alpha p_c + \mathcal{D}_{\alpha\beta b}\dot{r}^\alpha \dot{r}^\beta, \tag{7.2.2}$$

where $\mathcal{B}^a_{\alpha\beta}$ *are the local coordinates of the curvature* $\mathcal{B}$ *of the nonholonomic connection* $\mathcal{A}$*, where* $\mathcal{E}^a_{\alpha d} = C^a_{bd}\mathcal{A}^b_\alpha$*, and where the coefficients* $\mathcal{D}^c_{b\alpha}$ *and* $\mathcal{D}_{\alpha\beta b}$ *are defined in Proposition 7.2.*

Proof The second set of equations, which are the momentum equations, were derived in the preceding proposition. To get the first set of equations, one can proceed in three ways. First, one can invoke the calculations earlier in this paper for the motion relative to a general Ehresmann connection, restricting oneself to the variations that are horizontal; this is a straightforward, although somewhat tedious calculation. Alternatively, one can make use of the horizontal part of the calculations in Marsden & Scheurle [1993], which as we remarked, are valid for any choice of connection. In particular, one can use the nonholonomic connection. A third method is to write the equations in a "vector" form similar to those for the momentum equation in body representation that we derived earlier by using the local form of the equations regarding the momentum terms as affine constraints (see (2.3.1):

$$-\delta l_c = -\left\langle \frac{\partial l}{\partial \xi}, d\mathcal{A}_{\mathrm{loc}}(\dot{r}, \delta r) - [\mathcal{A}_{\mathrm{loc}}(\dot{r}), \mathcal{A}_{\mathrm{loc}}(\delta r)] \right\rangle - \left\langle \frac{\partial l}{\partial \xi}, (D\mathbb{I}^{-1}_{\mathrm{loc}} p)(\delta r) \right\rangle. \tag{7.2.3}$$

When these equations are converted to coordinate form and the basic dynamical variables are taken to be $(r, \dot{r}, \Omega = \mathbb{I}^{-1}_{\mathrm{loc}} p)$ (as in (5.3.4) , one recovers the coordinate form above. ■

The above equations become the reduced Euler-Lagrange equations described earlier in case there are no constraints. Notice also that the reduced equations are decoupled from the group variables, which is important for the reconstruction process. We summarize what we have already established on reconstruction as follows:

Proposition 7.6 *The group variables are reconstructed by means of the equation*

$$\dot{g} = g \cdot \xi$$

where $\xi = \Omega - \mathcal{A}_{\text{loc}}\dot{r}$.

Of course we could also write this equation in terms of the nonholonomic momentum p_b. As before, let $\mathcal{A} : TQ \to \mathfrak{g}$ be the Lie algebra valued one form corresponding to $A_q : T_qQ \to T_q(\text{Orb}(q))$. Since the nonholonomic momentum map is equivariant, we can write it in a local trivialization, as before:

$$J^{\text{nhc}}(g, r, \dot{g}, \dot{r}) = \text{Ad}^*_{g^{-1}}(J^{\text{nhc}}_{\text{loc}}(r, \dot{r}, \xi)).$$

This is a form similar to that for the local expression for a connection and its curvature. Then the reconstruction equation becomes

$$\dot{g} = g(-\mathcal{A}_{\text{loc}}(r)\dot{r} + \mathbb{I}^{-1}_{\text{loc}}(r)p)$$

where $\mathcal{A}_{\text{loc}} : T(Q/G) \to \mathfrak{g}$ is the local version of $\mathcal{A}$ and $\mathbb{I}^{-1}_{\text{loc}}$ is the local version of the locked inertia tensor, as was defined before.

Note that $\dot{g}$ depends linearly on $\dot{r}$ and also linearly on p. In the case of horizontal symmetries, the term $-\mathcal{A}_{\text{loc}}\dot{r}$ defines the *geometric phase* and the term $\Omega_{\text{loc}} = \mathbb{I}^{-1}_{\text{loc}}(r)p := \Gamma(r)p$ determines the *dynamic phase*. We adopt the same terminology in the general case. If the dynamic phase term is zero then the motion in the group variables is determined solely by the *path* in the base space, not its time parametrization. On the other hand, the dynamic phase determines the motion of the system when $\dot{r} = 0$ and hence corresponds to unforced motions of the system. For a system with horizontal symmetries, p is a constant.

As we have shown, it is possible to choose a basis of sections for $S_q = \mathcal{D}_q \cap T_q\text{Orb}$ such that the momentum map and the locked inertia tensor is group invariant (independent of g). This was also shown by Ostrowski, Burdick, Lewis & Murray [1995], who write the momentum and reconstruction equations in the form

$$\begin{aligned} \dot{g} &= g(-\mathcal{A}_{\text{loc}}(r)\dot{r} + \mathbb{I}(r)^{-1}p) \\ \dot{p} &= \sigma(r, \dot{r}, p) \end{aligned}$$

To reiterate, the reconstruction process now decouples as follows: given an initial condition and a path in the base space, we first integrate the momentum equation to determine $p(t)$ for all time. We then use $r(t)$ and $p(t)$ to determine the motion in the fiber. This decoupling is only possible when $\dot{p}$ is independent of g, since otherwise the equations for p and g are coupled. Of course, this whole process can be read in many different ways depending on the dynamics and control objectives. We will be exploring a number of these avenues in future publications.

Some Additional History

We have discussed quite a few of the earlier historical points in the introduction, and here we make a few more remarks on the more recent history relevant to this paper. Additional information, especially on the Russian literature can be found in Sumbatov [1992]. First of all, Koiller [1992] established the reduction procedure in what we call here the purely kinematic case. Thus, our results can be viewed as a generalization of his. Also, Bloch and Crouch [1992] treated the case which we would call here that of horizontal symmetries. Yang 1992] treated the purely kinematic case together with horizontal symmetries, as well as considering affine constraints. Marsden and Scheurle [1993a,b] treated the case of Lagrangian reduction for holonomic systems and introduced the idea of dividing the variations into horizontal and vertical parts in the variational principle and worked out the reduced Euler-Lagrange equations for this case. The present work can also be viewed as the generalization of that work to include the case of nonholonomic constraints.

8 Examples

We now consider several detailed examples to illustrate the theory developed above. Although most of these examples are of strictly academic interest, they illustrate the basic concepts and indicate how more complicated examples should be attacked. The examples which we present are a vertical disk rolling on a plane, a nonholonomically constrained particle in $\mathbb{R}^3$, a ball on a spinning plate, and the snakeboard, a variant of the skateboard. This last example is perhaps the richest of these examples and it uses the full momentum equation to describe its motion. Indeed, it was primarily this example which helped guide the theory and led to the development of the momentum equation and the nonholonomic connection.

In addition to the examples presented here, there are many other nonholonomic systems of both academic and practical importance. We already mentioned the falling disk and the wobblestone in the introduction; another is the controlled bicycle (Getz [1994a] and Getz & Marsden [1995]). Many robotic locomotion systems can be modeled in terms of nonholonomic constraints; see Kelly & Murray [1995]. Additional examples can be found in robotic manipulation, particularly robotic grasping; see Murray, Li & Sastry [1994].

In all of the examples that follow, we will assume that the kinematic constraints hold exactly and that the Lagrange d'Alembert principle holds; in particular, the forces of constraint do no (virtual) work. In practice, one modifies the resulting equations to take into account friction, slipping, and other effects.

8.1 The Vertical Rolling Disk

Although the vertical rolling disk is very simple and classical, it nonetheless illustrates the ideas. We mention that the falling disk can be treated by similar methods; see Vierkandt [1892], O'Reilly [1994] and Getz & Marsden [1994b].

We begin by developing the equations of motion using the Ehresmann connection given by the constraints and deriving the reduced Lagrangian, thus illustrating the material of §2. The equations are then written explicitly in terms of the reduced Lagrangian and the curvature of the connection. We then discuss the momentum equation of §4 and §7. Using different subgroups of the full symmetry group we show how one gets conservation laws from both horizontal and nonhorizontal symmetries. The different forms of the conservation laws are also illustrated.

Consider a vertical disk free to roll on the xy-plane and to rotate about its vertical axis. Let x and y denote the position of contact of the disk in the xy-plane. The remaining variables are θ and φ denoting the orientation of a chosen material point P with respect to the vertical and the "heading angle" of the disk, as in Figure 8.1.

Thus, the unconstrained configuration space for the vertical rolling disk is $Q = \mathbb{R}^2 \times S^1 \times S^1$. The velocities associated with the coordinates x, y, θ, φ are denoted $\dot{x}, \dot{y}, \dot{\theta}$ and $\dot{\varphi}$, which provide the remaining coordinates for the velocity phase space TQ. The Lagrangian for the problem is taken to be the kinetic energy

$$L(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 \tag{8.1.1}$$

where m is the mass of the disk, and I and J are its moments of inertia. Note that so far, we use the full configuration space, ignoring the constraints and that the Lagrangian is the standard "free" Lagrangian.

The rolling constraints (assuming the disk has radius R) may be written as

$$\left.\begin{aligned} \dot{x} &= R(\cos\varphi)\dot{\theta} \\ \dot{y} &= R(\sin\varphi)\dot{\theta} \end{aligned}\right\} \tag{8.1.2}$$

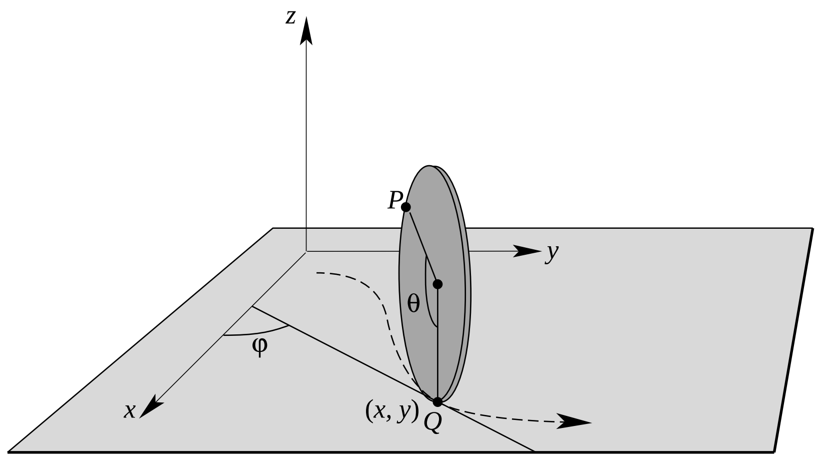

Figure 8.1.1: Geometry of the vertical rolling disk.

At first one can close one's eyes to the symmetry of the problem and just think of the constraints as the horizontal space of an Ehresmann connection, as in §3. To do this, one must choose a bundle $Q \to R$. Given the nature of the constraints and the fact that one imagines that eventually controls would be added to either the θ or the φ variable, one is motivated to choose the base R to be $S^1 \times S^1$ parameterized by θ and φ with the projection to R being the naive one $(s^1, s^2, r^1, r^2) = (x, y, \theta, \varphi) \mapsto (r^1, r^2) = (\theta, \varphi)$. From the constraints one can read off the components of the Ehresmann connection (see (2.1.3) :

$$\left.\begin{aligned} A_1^1 &= -R(\cos\varphi) \\ A_1^2 &= -R(\sin\varphi) \end{aligned}\right\} \tag{8.1.3}$$

and the remaining A_α^a are zero. If we choose to regard the bundle $Q \to R$ as a principal bundle with group $G = \mathbb{R}^2$, we get an abelian purely kinematic system (see Bloch, Reyhanoglu & McClamroch [1992] and Bloch & Crouch [1992]). Indeed, note that using the obvious action of G, we get

$$T_q\mathrm{Orb}(q) = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}. \tag{8.1.4}$$

Notice that $\mathcal{D}_q \cap T_q(\mathrm{Orb}(q)) = \{0\}$ and that the components of A are independent of x and y.

Proceeding with the analysis of §3, the constrained Lagrangian L_c is given by substituting (8.1.2) into (8.1.1):

$$L_c(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}) = \frac{1}{2}(mR^2 + I)\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2. \tag{8.1.5}$$

Note that if the mass density of the disk were constant, then $I = \frac{1}{2}mR^2$ and we could simplify the coefficient of $\dot{\theta}^2$ to $\frac{3}{4}mR^2\dot{\theta}^2$, but we need not make this assumption. The curvature of the connection A is computed using formula (2.1.7) to be

$$B_{21}^1 = -B_{12}^1 = -R\sin\varphi, \quad B_{12}^2 = -B_{21}^2 = -R\cos\varphi, \tag{8.1.6}$$

with the remaining $B_{\alpha\beta}^a$ zero. The equations of motion

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{r}^\alpha}\right) - \frac{\partial L_c}{\partial r^\alpha} = -\left(\frac{\partial L}{\partial \dot{s}^a}\right) B_{\alpha\beta}^a \dot{r}^\beta \tag{8.1.7}$$

become

$$(mR^2 + I)\ddot{\theta} = (mR(\cos\varphi)\dot{\theta})(-R(\sin\varphi)\dot{\varphi}) + (mR(\sin\varphi)\dot{\theta})(R(\cos\varphi)\dot{\varphi}) = 0 \tag{8.1.8}$$

$$J\ddot{\varphi} = (mR(\cos\varphi)\dot{\theta})(R(\sin\varphi)\dot{\theta}) + (mR(\sin\varphi)\dot{\theta})(-R\cos\varphi)\dot{\theta} = 0. \tag{8.1.9}$$

Thus, $\dot{\theta} = \Omega$ and $\dot{\varphi} = \omega$ are constants, so $\theta = \Omega t + \theta_0, \varphi = \omega t + \varphi_0$ and equation (8.1.2) gives

$$\begin{aligned} \dot{x} &= \Omega R\cos(\omega t + \varphi_0) \\ \dot{y} &= \Omega R\sin(\omega t + \varphi_0). \end{aligned}$$

Hence

$$x = \frac{\Omega}{\omega} R\sin(\omega t + \varphi_0) + x_0$$

and

$$y = -\frac{\Omega}{\omega} R\cos(\omega t + \varphi_0) + y_0.$$

We now turn to the momentum equation. It is clear that in the example as presented, one has the whole group $SE(2) \times S^1$ as a symmetry group. In such a case, the orbit of the group spans the entire constraint distribution. While this is certainly allowed by the theory, it is an extreme case that one does not have in general. In the presence of controls some of the symmetry will be broken, so it is appropriate to consider a smaller symmetry group, namely a subgroup of $SE(2) \times S^1$ to be the group G in the general theory. We will, for illustrative purposes, make two choices, namely the subgroup $SE(2)$ and the *direct* product $\mathbb{R}^2 \times S^1$. To keep things clear, we will write these two choices as

$$G_1 = SE(2) \quad \text{and} \quad G_2 = \mathbb{R}^2 \times S^1.$$

It is interesting that, as we shall see, the actions of G_1 and G_2 give rise to the two conservation laws $\dot{\theta} = \Omega$ and $\dot{\varphi} = \omega$ respectively, one being induced by a horizontal symmetry, the other not.

The action of $G_1 = SE(2)$ on $\mathbb{R}^4$ is given by

$$(x, y, \theta, \varphi) \to (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \theta, \varphi + \alpha) \tag{8.1.10}$$

where $(a, b, \alpha) \in SE(2)$. The $\mathbb{R}^2 \times S^1$ action is given by

$$(x, y, \theta, \varphi) \to (x + \lambda, y + \mu, \theta + \beta, \varphi). \tag{8.1.11}$$

The tangent space to the orbits of the $SE(2)$ action is given by

$$T_q\mathrm{Orb}(q) = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \varphi}\right\}, \tag{8.1.12}$$

while for the $G_2 = \mathbb{R}^2 \times S^1$ action, they are

$$T_q\mathrm{Orb}(q) = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\right\}. \tag{8.1.13}$$

One checks that the Lagrangian and the constraints are invariant under each of these actions.

We now consider the momentum equations corresponding to these two actions. The preceding calculations show that the constraint distribution $\mathcal{D}_q$ is given by

$$\mathcal{D}_q = \mathrm{span}\left\{\frac{\partial}{\partial \varphi}, R\cos\varphi\frac{\partial}{\partial x} + R\sin\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}\right\}. \tag{8.1.14}$$

Recall that the space $\mathcal{S}_q$ is given by the intersection of the tangent space to the orbit with the constraint distribution itself. Hence, for the $SE(2)$ action we have

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q \mathrm{Orb}_{G_1}(q) = \mathrm{span}\left\{\frac{\partial}{\partial \varphi}\right\} \tag{8.1.15}$$

and for the $\mathbb{R}^2 \times S^1$ action we have

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q \mathrm{Orb}_{G_2}(q) = \mathrm{span}\left\{R\cos\varphi\frac{\partial}{\partial x} + R\sin\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}\right\}. \tag{8.1.16}$$

To obtain the corresponding momentum equations, we consider the bundles whose fibers are the span of the tangent vectors in the preceding two equations in the respective cases, and choose sections of these bundles. The bundles are of course trivial. In the case of the $G_1 = SE(2)$ action, note that the generators corresponding to the Lie algebra elements represented by the standard basis in $\mathbb{R}^3$ (with translations being the first two components and rotations the third) are given by

$$(1,0,0)_Q = \frac{\partial}{\partial x}, \qquad (0,1,0)_Q = \frac{\partial}{\partial y}, \qquad (0,0,1)_Q = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial \varphi}.$$

To obtain the section of $\mathcal{S}_q$ given by vector field

$$\xi^q_Q = \frac{\partial}{\partial \varphi}, \tag{8.1.17}$$

we thus choose the Lie algebra element

$$\xi^q = (y, -x, 1), \tag{8.1.18}$$

while for the $G_2 = \mathbb{R}^2 \times S^1$ action we take the section to be the vector field

$$\xi^q_Q = R\cos\varphi\frac{\partial}{\partial x} + R\sin\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta} \tag{8.1.19}$$

with corresponding Lie algebra element

$$\xi^q = (R\cos\varphi, R\sin\varphi, 1). \tag{8.1.20}$$

For the $SE(2)$ action, the nonholonomic momentum map is

$$J^{\mathrm{nhc}}(\xi^q) = \frac{\partial L}{\partial \dot{q}^i}(\xi^q_Q)^i = J\dot{\varphi} \tag{8.1.21}$$

and hence the momentum equation becomes

$$\frac{d}{dt}J^{\mathrm{nhc}}(\xi^q) = \frac{d}{dt}(J\dot{\varphi}) = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}(\xi^q)\right]^i_Q = m\dot{x}(\dot{y}) + m\dot{y}(-\dot{x}) + 0 = 0. \tag{8.1.22}$$

This is of course an ordinary conservation law and is one of the equations of motion.

Note that corresponding to this action, $\mathcal{D}_q \cap T_q(\mathrm{Orb}_{G_1}(q)) = T_q(\mathrm{Orb}_H(q))$ where $H = S^1$ and we obtain a conservation law corresponding to the horizontal action of S^1. This law can of course also be obtained by directly considering the S^1 action.

For the G_1 action, a straightforward calculation shows that the third part of Corollary 4.9 applies and so this is one way to find the constants of motion. Rather than giving the details of this calculation, we will give them for the $G_2 = \mathbb{R}^2 \times S^1$ action.

Using the G_2 action, the nonholonomic momentum map is

$$J^{\mathrm{nhc}}(\xi^q) = \frac{\partial L}{\partial \dot{q}^i}(\xi^q_Q)^i = m\dot{x}R\cos\varphi + m\dot{y}R\sin\varphi + I\dot{\theta}, \tag{8.1.23}$$

and so the momentum equation becomes

$$\frac{d}{dt}(m\dot{x}R\cos\varphi + m\dot{y}R\sin\varphi + I\dot{\theta}) = m\dot{x}\frac{d}{dt}(R\cos\varphi) + m\dot{y}\frac{d}{dt}(R\sin\varphi), \tag{8.1.24}$$

i.e.,

$$R\cos\varphi\, m\ddot{x} + R\sin\varphi\, m\ddot{y} + I\ddot{\theta} = 0. \tag{8.1.25}$$

Using the constraints to eliminate $\ddot{x}$ and $\ddot{y}$ from this equation we get

$$(mR^2 + I)\ddot{\theta} = 0, \tag{8.1.26}$$

which we derived in (8.1.8). Alternatively, observe that, after imposing the constraints, the right hand side of equation (8.1.24) is zero and the left hand side reduces to the left hand side of (8.1.26). Thus the two momentum equations yield the reduced equations of motion.

We now illustrate, for the case of the $G_2 = \mathbb{R}^2 \times S^1$ action, the momentum equation in a moving basis (4.3.7) and the momentum equation in a body frame (4.4.2). The latter is equivalent to the reduced form of the momentum equation given in Theorem 7.5. We first treat the version (4.3.7). Choose a fixed basis for the Lie algebra of $G_2 = \mathbb{R}^2 \times S^1$, namely $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. From $\xi^q = \xi^a e_a$, we have

$$\xi^1 = R\cos\varphi, \qquad \xi^2 = R\sin\varphi, \qquad \xi^3 = 1.$$

Choose the moving basis

$$e_1(q) = (R\cos\varphi, R\sin\varphi, 1), \qquad e_2(q) = (1, 0, 0), \qquad e_3(q) = (0, 1, 0),$$

and write $e_b(q) = \psi^a_b(q)e_a$. We find that

$$\psi^1_1 = R\cos\varphi, \qquad \psi^2_1 = R\sin\varphi, \qquad \psi^3_1 = 1, \qquad \psi^1_2 = 1, \qquad \psi^2_3 = 1$$

and $\psi^a_b = 0$ otherwise. Writing $\xi^i_a = K^i_a\xi^a$, we find that the infinitesimal generator coefficients are given by $K^1_1 = K^2_2 = K^3_3 = 1$ and $K^i_a = 0$, otherwise. From the formula $J_b = (\partial L/\partial \dot{q}^i)K^i_a\psi^a_b$, we find

$$J_1 \quad = \quad m\dot{x}R\cos\varphi + m\dot{y}R\sin\varphi + I\dot{\theta},$$

noting that $\mathcal{S}_q$ is one dimensional, so the range of the index b in the nonholonomic momentum map is simply $b = 1$. We find that

$$\begin{aligned} \Gamma^2_{14} &= (\psi^{-1})^2_a\frac{\partial \psi^a_1}{\partial \varphi} = -R\sin\varphi \\ \Gamma^3_{14} &= (\psi^{-1})^3_a\frac{\partial \psi^a_1}{\partial \varphi} = R\cos\varphi \\ \Gamma^d_{1k} &= 0 \quad \text{otherwise.} \end{aligned}$$

With $r = 1, n = 4$, and $k = 3$, these calculations show that the momentum equation (4.3.7) becomes

$$\frac{d}{dt}J_1 = \sum_{l=1}^{4}\Gamma^1_{1l}J_1\dot{q}^l + \sum_{i,l=1}^{4}\frac{\partial L}{\partial \dot{q}^i}\Gamma^2_{1l}\dot{q}^l[e_2(q(t))]^i_Q + \sum_{i,l=1}^{4}\frac{\partial L}{\partial \dot{q}^i}\Gamma^3_{1l}\dot{q}^l[e_3(q(t))]^i_Q. \tag{8.1.27}$$

The first term is zero and the momentum equation simplifies to

$$\frac{d}{dt}J_1 = m\dot{x}\frac{d}{dt}(R\cos\varphi) + m\dot{y}\frac{d}{dt}(R\sin\varphi), \tag{8.1.28}$$

which is indeed the correct momentum equation.

We now discuss the version (4.4.2) of the momentum equation, continuing with the G_2 action. Here the shape variable r is φ and $\xi = (\dot{x}, \dot{y}, \dot{\theta})$ and so the reduced Lagrangian is

$$l(\varphi, \dot{\varphi}, \dot{x}, \dot{y}, \dot{\theta}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2.$$

We choose $e_1(\varphi) = (R\cos\varphi, R\sin\varphi, 1)$, $e_2(\varphi) = (1, 0, 0)$, and $e_3(\varphi) = (0, 1, 0)$. Then (4.4.1) gives

$$p_1 = \left\langle \frac{\partial l}{\partial \xi}, e_1 \right\rangle = m\dot{x}R\cos\varphi + m\dot{y}R\sin\varphi + I\dot{\theta}$$

which, when the constraints are substituted, gives

$$p_1 = (mR^2 + I)\dot{\theta}.$$

The momentum equation (4.4.2) now becomes, since the group is abelian,

$$\begin{aligned}\frac{d}{dt}p_1 &= \left\langle \frac{\partial l}{\partial \xi}, \frac{\partial e_1}{\partial \varphi}\dot{\varphi} \right\rangle \\ &= \left\langle (m\dot{x}, m\dot{y}, I\dot{\theta}), (-R\sin\varphi, R\cos\varphi, 0) \right\rangle \\ &= -m\dot{x}R\sin\varphi + m\dot{y}R\cos\varphi,\end{aligned}$$

which vanishes in view of the constraints. Thus, we recover $dp_1/dt = 0$, as before. Observe that this formulation directly gives us a conservation law even though the symmetry is *not* horizontal.

8.2 A Nonholonomically Constrained Particle

An instructive example due to Rosenberg [1977] that illustrates the momentum equation is the following example of a nonholonomically constrained free particle. This example was also used to illustrate the theory in Bates & Sniatycki [1993]. We show here that the momentum equation in an orthogonal body frame is a pure parallel transport equation with respect to the nonmetric connection for the particle observed by Bates and Sniatycki. We thus provide a general method for deriving such a connection.

Consider a particle with the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{8.2.1}$$

and the nonholonomic constraint

$$\dot{z} = y\dot{x}. \tag{8.2.2}$$

The constraints and Lagrangian are invariant under the $\mathbb{R}^2$ action on $\mathbb{R}^3$ given by

$$(x, y, z) \mapsto (x + \lambda, y, z + \mu). \tag{8.2.3}$$

The tangent space to the orbits of this action is given by

$$T_q\mathrm{Orb}(q) = \mathrm{span}\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right\}, \tag{8.2.4}$$

and the kinematic constraint distribution is given by

$$\mathcal{D}_q = \text{span}\left\{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right\} \tag{8.2.5}$$

and thus

$$\mathcal{D}_q \cap T_q(\text{Orb}(q)) = \text{span}\left\{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right\}. \tag{8.2.6}$$

Consider the bundle $\mathcal{S}$ with fibers the span of these tangent vectors. To obtain the momentum equations we begin by taking an arbitrary section of this bundle. The bundle is of course trivial and for simplicity we take the section to be the vector field

$$\xi_Q^q = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}. \tag{8.2.7}$$

The corresponding Lie algebra element $\xi^q \in \mathbb{R}^2$ is

$$\xi^q = (1, y) \tag{8.2.8}$$

The nonholonomic momentum map in this case is

$$J^{\text{nhc}}(\xi^q) = \frac{\partial L}{\partial \dot{q}^i}(\xi_Q^q)^i = \langle(\dot{x}, \dot{y}, \dot{z}), (1, 0, y)\rangle = \dot{x} + y\dot{z}. \tag{8.2.9}$$

Hence the momentum equation becomes

$$\frac{dJ^{\text{nhc}}(\xi^q)}{dt} = \frac{d}{dt}(\dot{x} + y\dot{z}) = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}(\xi^q)\right]_Q^i = \langle(\dot{x}, \dot{y}, \dot{z}), (0, 0, \dot{y})\rangle = \dot{z}\dot{y}. \tag{8.2.10}$$

i.e.,

$$\ddot{x} + y\ddot{z} = 0. \tag{8.2.11}$$

Using the constraint $\dot{z} = y\dot{x}$, the momentum equation may be rewritten as

$$\ddot{x} + \frac{y}{1 + y^2}\dot{x}\dot{y} = 0. \tag{8.2.12}$$

Together with the Lagrangian equation of motion $\ddot{y} = 0$, this completely specifies the motion, and in fact these two equations are a (non-metric) geodesic flow as pointed out in Bates & Sniatycki [1993]. In this example we note that the momentum equation is the total derivative of a first order conservation law:

$$\dot{x} - \frac{c}{(1 + y^2)^{\frac{1}{2}}} = 0 \tag{8.2.13}$$

for c an arbitrary constant. Note, however, that this equation, which is used in the Bates-Sniatycki reduction, is a conservation law, but is not directly a component of a conserved momentum map. In other words, the fact that the second order momentum equation here is the derivative of a first order conservation law is not due to considerations of symmetry.

Note also that if one chooses the right base and fiber this system is again an abelian Chaplygin system. Here we take $\mathbb{R}^2$ with coordinates $\{x, y\}$ to be the base and $\mathbb{R}$ with coordinate z to be the fiber. Then

$$T_q(\text{Orb}(q)) = \text{span}\left\{\frac{\partial}{\partial z}\right\} \tag{8.2.14}$$

and $\mathcal{D}_q \cap T_q(\text{Orb}(q)) = 0$.

We again illustrate both coordinate versions of the momentum equation, namely (4.3.7) and (4.4.2), first treating the version (4.3.7). We choose a fixed basis for $\mathfrak{g} = \mathbb{R}^2$, namely $e_1 = (1, 0)$ and $e_2 = (0, 1)$, then $\xi^q = \xi^1 e_1 + \xi^2 e_2$, where $\xi^1 = 1, \xi^2 = y$. As before, choose a moving basis

$$e_1(q) = (1, y), \qquad e_2(q) = (0, 1).$$

Then if

$$e_b(q) = \sum_{a=1}^{2} \psi_b^a(q) e_a,$$

clearly

$$\psi_1^1 = 1, \qquad \psi_1^2 = y, \qquad \psi_2^1 = 0, \qquad \psi_2^2 = 1.$$

Writing

$$\xi_Q^i = K_a^i \xi^a,$$

we find $K_1^1 = 1$, $K_2^3 = 1$, and $K_a^i = 0$ otherwise. Hence,

$$J_1 \quad = \quad \frac{\partial L}{\partial \dot{q}^i} K_a^i \psi_1^a = \dot{x} + y\dot{z}$$

noting that $\mathcal{S}_q$ is one dimensional, so the range of the index b in the nonholonomic momentum map is simply $b = 1$.

Next we compute the connection coefficients. We find

$$(\psi^{-1})_1^1 = 1, \qquad (\psi^{-1})_1^2 = -y, \qquad (\psi^{-1})_2^1 = 0, \qquad (\psi^{-1})_2^2 = 1,$$

and hence $\Gamma_{12}^2 = 1$, and $\Gamma_{bk}^1 = 0$ otherwise. These calculations with $r = 1, n = 3$, and $k = 2$, show that the momentum equation (4.3.7) becomes

$$\frac{d}{dt} J_1 = \sum_{l=1}^{3} \Gamma_{1l}^1 J_1 \dot{q}^l + \sum_{i,l=1}^{3} \frac{\partial L}{\partial \dot{q}^i} \Gamma_{1l}^2 \dot{q}^l [e_2(q(t))]_Q^i. \tag{8.2.15}$$

The first term is zero and so the momentum equation simplifies to

$$\frac{d}{dt} J_1 = \dot{z}\dot{y}, \tag{8.2.16}$$

the correct momentum equation.

Now we discuss the version (4.4.2) of the momentum equation. First, we must orthogonalize the preceding moving basis. Here the shape variable r is y and $\xi = (\dot{x}, \dot{z})$ and so the reduced Lagrangian is

$$l(r, \dot{r}, \xi) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

We choose

$$e_1(r) = (1, y), \qquad e_2(r) = (-y, 1).$$

Then (4.4.1) gives

$$p_1 = \dot{x}(1 + y^2).$$

Again the group is abelian, and so the momentum equation (4.4.2) becomes

$$\frac{d}{dt} p_1 = \left\langle \frac{\partial l}{\partial \xi}, \frac{\partial e_1}{\partial y} \dot{y} \right\rangle = \langle (\dot{x}, \dot{z}), (0, 1) \rangle = \dot{z}\dot{y},$$

as before. Writing

$$\frac{\partial e_1}{\partial y} = (0,1) = \gamma^1_{11} e_1 + \gamma^2_{11} e_2,$$

we see that

$$\gamma^1_{11} = \frac{y}{1+y^2}, \quad \gamma^2_{11} = \frac{1}{1+y^2}$$

and so the momentum equation (see Corollary 7.4) becomes

$$\frac{d}{dt} p_1 = \frac{y\dot{y}}{1+y^2} p_1$$

which is in parallel transport form. Note that the connection we have just constructed using the general principles of the momentum equation is the same nonmetric connection as in Bates & Snyiatycki [1992].

8.3 A Homogeneous Ball on a Rotating Plate

An example which illustrates the theory in the case of affine constraints is a model of a homogeneous ball on a rotating plate (see Neimark & Fufaev [1972] and Yang [1992] for the affine case and, for example, Bloch & Crouch [1992], Brockett & Dai [1992] and Jurdjevic [1993] for the linear case). As we mentioned in the introduction, Chaplygin [1897b,1903] studied the motion of a *nonhomogeneous* rolling ball. Here we illustrate the derivation of the equations of motion for a homogeneous ball in the affine case as well the structure of the nonholonomic momentum map in this setting.

Consider the system shown in Figure 8.3.1. Fix coordinates in inertial space and the let the plane rotate with constant angular velocity Ω about the z-axis. The configuration space of the sphere is $Q = \mathbb{R}^2 \times SO(3)$, parameterized by $(x, y, R), R \in SO(3)$, all measured with respect to the inertial frame. Let $\omega = (\omega_x, \omega_y, \omega_z)$ be the angular velocity vector of the sphere measured also with respect to the inertial frame, let m be the mass of the sphere, mk^2 its inertia about any axis, and let a be its radius.

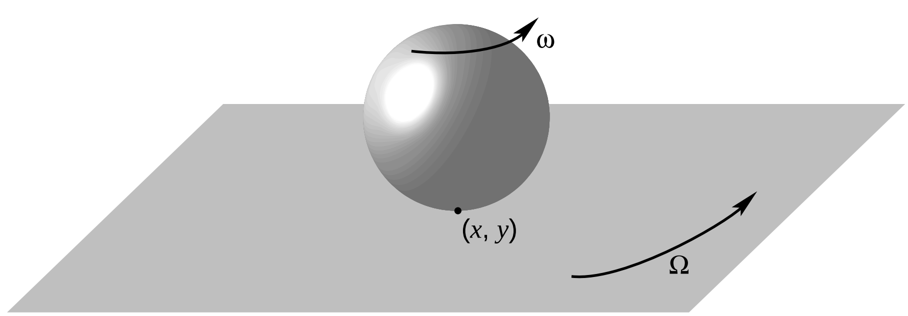

Figure 8.3.1: The ball rolling on a plate

The Lagrangian of the system is

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} mk^2({\omega_x}^2 + {\omega_y}^2 + {\omega_z}^2) \tag{8.3.1}$$

with the affine nonholonomic constraints

$$\left.\begin{aligned} \dot{x} - a\omega_y &= -\Omega y \\ \dot{y} + a\omega_x &= \Omega x. \end{aligned}\right\} \tag{8.3.2}$$

Note that the Lagrangian here is a metric on Q which is bi-invariant on $SO(3)$ as the ball is homogeneous. Note also that $\mathbb{R}^2 \times SO(3)$ is a principal bundle over $\mathbb{R}^2$ with respect to the right $SO(3)$ action on Q given by

$$(x, y, R) \mapsto (x, y, RS) \tag{8.3.3}$$

for $S \in SO(3)$. The action is on the *right* since the symmetry is a material symmetry.

Observe, as in Yang [1992] and Bloch & Crouch [1992], that the angular momentum of the ball about the z-axis is conserved since the Lagrangian is invariant under rotations about the z-axis and the infinitesimal generator of these rotations clearly lies in $\mathcal{D}_q$. That is, we have horizontal symmetries. The conservation laws together with the constraints, namely

$$\left.\begin{aligned} \omega_x + \frac{1}{a}\dot{y} &= \frac{\Omega x}{a} \\ \omega_y - \frac{1}{a}\dot{x} &= \frac{\Omega y}{a} \\ \omega_z &= c, \end{aligned}\right\} \tag{8.3.4}$$

where c is a constant, thus determine the nonholonomic connection.

To compute the equations of motion we use the following notation. Let $r^1 = x$, $r^2 = y$, and let s^1, s^2, s^3 denote the angles corresponding to rotation about the x, y and z axes respectively. The constrained Lagrangian is given by (eliminating ω_x, ω_y and ω_z)

$$L_c = \frac{1}{2a^2}\{m(a^2 + k^2)(\dot{x}^2 + \dot{y}^2) + mk^2\Omega^2(x^2 + y^2) + 2mk^2(\Omega y\dot{x} - \Omega x\dot{y})\} \tag{8.3.5}$$

(up to an irrelevant constant). Using the above definitions of the variables r, s we see that

$$A_2^1 = -\frac{1}{a} \qquad A_1^2 = \frac{1}{a} \tag{8.3.6}$$

with all other A_α^a equal to zero. Hence all the curvature terms $B_{\alpha\beta}^a$ are zero.

On the other hand, recalling that the affine constraints are written as $\omega(\dot{q}) = \gamma(r)$ with $d\gamma^a = \gamma_\alpha^a dr^\alpha$ we have

$$\gamma_1^1 = \frac{\Omega}{a} \qquad \gamma_2^2 = \frac{\Omega}{a} \tag{8.3.7}$$

with all other γ_α^a equal to zero. Then the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{r}^\alpha}\right) - \frac{\partial L_c}{\partial r^\alpha} = -\frac{\partial L}{\partial \dot{s}^a}\gamma_\alpha^a \tag{8.3.8}$$

become

$$\frac{1}{a^2}\{m(a^2 + k^2)\ddot{x} + 2mk^2\Omega\dot{y} - mk^2\Omega^2 x\} = -mk^2\omega_x\frac{\Omega}{a} = -\frac{mk^2}{a^2}(\Omega^2 x - \Omega\dot{y})$$

and

$$\frac{1}{a^2}\{m(a^2 + k^2)\ddot{y} - 2mk^2\Omega\dot{x} - mk^2\Omega^2 y\} = -mk^2\omega_y\frac{\Omega}{a} = -\frac{mk^2}{a^2}(\Omega^2 y + \Omega\dot{x})$$

i.e.,

$$\left.\begin{aligned} \ddot{x} + \frac{k^2\Omega}{a^2 + k^2}\dot{y} &= 0 \\ \ddot{y} - \frac{k^2\Omega}{a^2 + k^2}\dot{x} &= 0. \end{aligned}\right\} \tag{8.3.9}$$

These equations may also be derived by considering the momentum equations associated with the system.

Note firstly that the constraint distribution given by the two kinematic constraints is (modulo the affine piece)

$$\mathcal{D}_q = \operatorname{span}\left\{ a\frac{\partial}{\partial x} + \xi_y, -a\frac{\partial}{\partial y} + \xi_x, \xi_z \right\}, \tag{8.3.10}$$

where ξ_x, ξ_y, and ξ_z denote the infinitesimal generators of rotations about the x, y and z axes of the ball respectively. Now consider the action of the full group $\mathbb{R}^2 \times SO(3)$ on the configuration space. Clearly the Lagrangian is invariant under this action. Also, we see that in this case we then have

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q\mathrm{Orb}(q) = \mathcal{D}_q\,.$$

Thus the nonholonomic momentum map J^{nhc} has three components corresponding to the three independent generators of $\mathcal{D}_q$.

We have

$$J_i = \frac{\partial L}{\partial \dot{q}^j}(\xi_i)^j_Q$$

which gives

$$\left.\begin{aligned} J_1 &= \left\langle (m\dot{x}, m\dot{y}, mk^2\omega_x, mk^2\omega_y, mk^2\omega_z),\, (a,0,0,1,0)\right\rangle = am\dot{x} + mk^2\omega_y \\ J_2 &= \left\langle (m\dot{x}, m\dot{y}, mk^2\omega_x, mk^2\omega_y, mk^2\omega_z),\, (0,-a,1,0,0)\right\rangle = -am\dot{y} + mk^2\omega_x \\ J_3 &= \left\langle (m\dot{x}, m\dot{y}, mk^2\omega_x, mk^2\omega_y, mk^2\omega_z),\, (0,0,0,0,1)\right\rangle = mk^2\omega_z\,. \end{aligned}\right\} \tag{8.3.11}$$

In all cases ξ^q is independent of q, so the momentum equations are simply

$$\left.\begin{aligned} \frac{dJ_1}{dt} &= \frac{d}{dt}(am\dot{x} + mk^2\omega_y) = 0 \\ \frac{dJ_2}{dt} &= \frac{d}{dt}(-am\dot{y} + mk^2\omega_x) = 0 \\ \frac{dJ_3}{dt} &= \frac{d}{dt}(mk^2\omega_z) = 0\,. \end{aligned}\right\} \tag{8.3.12}$$

The third equation is of course nothing but the conservation of angular momentum mentioned earlier. However the first two equations can easily be seen to be equivalent to the equations of motion (8.3.9)—one simply carries out the differentiation and solves for the derivatives of ω_x and ω_y using the derivatives of the constraint equations (8.3.2).

The dynamics of this system are particularly interesting—for example, the ball will generally not roll off a finite plate (see Yang [1992] and the experimental work of Lewis & Murray [1994]). Note also that all the interesting behavior arises from the affine nature of the constraints as there is zero curvature.

8.4 The Snakeboard

The snakeboard is a modified version of a skateboard in which the front and back pairs of wheels are independently actuated. The extra degree of freedom enables the rider to generate forward motion by twisting his or her body back and forth, while simultaneously moving the wheels with the proper phase relationship. A diagram of the snakeboard is shown in Figure 8.4.1. A detailed description was first presented by Lewis, Ostrowski, Murray & Burdick [1994]; see also Ostrowski, Burdick, Lewis & Murray [1995] and Ostrowski [1995].

One of the interesting features of the snakeboard is that it leads to a nontrivial momnentum equation which, in the general notation has terms that are linear in p and also quadratic in $\dot{r}$. Other examples with a similar structure for the momentum equation are the roller racer (see Tsikiris [1995] and the bicycle (see Getz and Marsden [1995]).

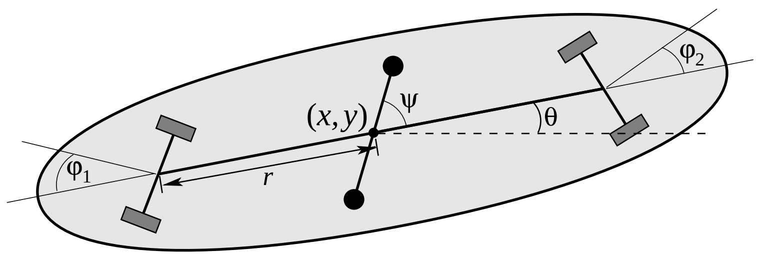

Figure 8.4.1: The variables in the snakeboard

We model the snakeboard as a rigid body (the board) with two sets of independently actuated wheels, one on each end of the board. The human rider is modeled as a momentum wheel which sits in the middle of the board and is allowed to spin about the vertical axis. Spinning the momentum wheel causes a counter-torque to be exerted on the board. The configuration of the board is given by the position and orientation of the board in the plane, the angle of the momentum wheel, and the angles of the back and front wheels. Thus the configuration space is $Q = SE(2) \times S^1 \times S^1 \times S^1$. We let (x, y, θ) represent the position and orientation of the center of the board, ψ the angle of the momentum wheel relative to the board, and ϕ_1 and ϕ_2 the angles of the back and front wheels, also relative to the board. We take the distance between the center of the board and the wheels as r.

The Lagrangian for the snakeboard consists only of kinetic energy terms. We take the simplest possible model for the various mass distributions and write the Lagrangian as

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_0(\dot{\theta} + \dot{\psi})^2 + \frac{1}{2}J_1(\dot{\theta} + \dot{\phi}_1)^2 + \frac{1}{2}J_2(\dot{\theta} + \dot{\phi}_2)^2,$$

where m is the total mass of the board, J is the inertia of the board, J_0 is the inertia of the rotor and J_i, $i = 1, 2$, is the inertia corresponding to ϕ_i. The Lagrangian is independent of the configuration of the board and hence it is invariant to all possible group actions.

The rolling of the front and rear wheels of the snakeboard is modeled using nonholonomic constraints which allow the wheels to spin about the vertical axis and roll in the direction that they are pointing. The wheels are not allowed to slide in the sideways direction. This gives constraint one forms

$$\left.\begin{aligned} \omega_1(q) &= -\sin(\theta + \phi_1)dx + \cos(\theta + \phi_1)dy - r\cos\phi_1 d\theta \\ \omega_2(q) &= -\sin(\theta + \phi_2)dx + \cos(\theta + \phi_2)dy + r\cos\phi_2 d\theta. \end{aligned}\right\} \tag{8.4.1}$$

These constraints are invariant under the $SE(2)$ action given by

$$(x, y, \theta, \psi, \phi_1, \phi_2) \mapsto (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \theta + \alpha, \psi, \phi_1, \phi_2),$$

where $(a, b, \alpha) \in SE(2)$, and also under the S^1 action defined by

$$(x, y, \theta, \psi, \phi_1, \phi_2) \mapsto (x, y, \theta, \psi + \delta, \phi_1, \phi_2).$$

We consider here only the $SE(2)$ symmetry since we have in mind the situation in which the S^1 symmetry is destroyed by the controls. The constraints determine the kinematic distribution $\mathcal{D}_q$:

$$\mathcal{D}_q = \operatorname{span}\left\{\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\phi_1}, \frac{\partial}{\partial\phi_2}, a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial\theta}\right\}, \tag{8.4.2}$$

where a, b, and c, are given by

$$\begin{aligned} a &= -r(\cos\phi_1\cos(\theta + \phi_2) + \cos\phi_2\cos(\theta + \phi_1)) \\ b &= -r(\cos\phi_1\sin(\theta + \phi_2) + \cos\phi_2\sin(\theta + \phi_1)) \\ c &= \sin(\phi_1 - \phi_2). \end{aligned} \tag{8.4.3}$$

The tangent space to the orbits of the $SE(2)$ action is given by

$$T_q(\mathrm{Orb}(q)) = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\right\} \tag{8.4.4}$$

(note that this is not a left invariant basis). The intersection between the tangent space to the group orbits and the constraint distribution is thus given by

$$\mathcal{D}_q \cap T_q(\mathrm{Orb}(q)) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta}. \tag{8.4.5}$$

We construct the momentum by choosing a section of $\mathcal{D} \cap T\mathrm{Orb}$ regarded as a bundle over Q. Since $\mathcal{D}_q \cap T_q\mathrm{Orb}(q)$ is one-dimensional, we choose the section to be

$$\xi_Q^q = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta}, \tag{8.4.6}$$

which is invariant under the action of $SE(2)$ on Q. The corresponding Lie algebra element $\xi^q \in se(2)$ is

$$\xi^q = (a + yc)e_x + (b - xc)e_y + ce_\theta$$

where e_x is the basis element of the Lie algebra corresponding to translations in the x direction (and whose corresponding infinitesimal generator is $\partial/\partial x$), etc. Physically, ξ^q corresponds to planar rotation about the point P where the axles of the front and back wheels intersect. When $\phi_1 = \phi_2$, this rotation becomes a translation (rotation about a point at infinity). See Figure 8.4.2. We notice that there is are singularities in the distribution when P is at the center of the snakeboard (when the axles of the wheels are aligned along the center line of the board).

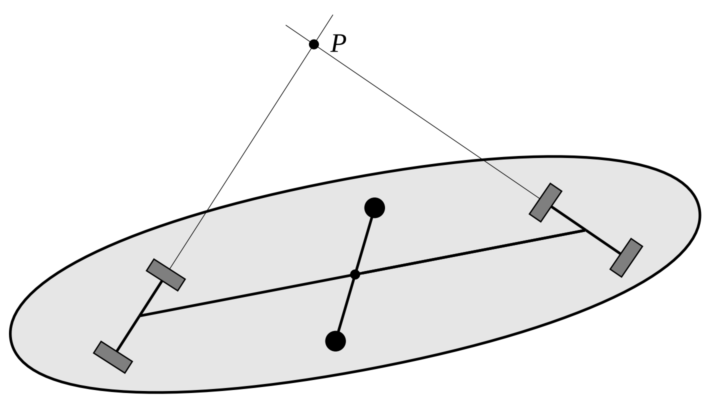

Figure 8.4.2: The momentum map component is the angular momentum about P

Equation (8.4.6) gives a nonholonomic momentum map

$$\begin{aligned} p &= J^{\mathrm{nhc}}(\xi^q) = \frac{\partial L}{\partial \dot{q}^i}(\xi_Q^q)^i \\ &= ma\dot{x} + mb\dot{y} + Jc\dot{\theta} + J_0c(\dot{\theta} + \dot{\psi}) + J_1c(\dot{\theta} + \dot{\phi}_1) + J_2c(\dot{\theta} + \dot{\phi}_2) \end{aligned}$$

We now specialize to the case in which $\phi_1 = -\phi_2$ and $J_1 = J_2$. We also choose the parameter J such that $J + J_0 + J_1 + J_2 = mr^2$. We do this following Ostrowski [1995] for simplicity; it eliminates some terms in the derivation but does not affect the essential geometry of the problem.

We now compute the nonholonomic connection for this case. Setting $\phi = \phi_1 = -\phi_2$, the constraints plus the momentum are given by

$$
\begin{aligned}
0 &= -\sin(\theta+\phi)\dot{x} + \cos(\theta+\phi)\dot{y} - r\cos\phi\dot{\theta} \\
0 &= -\sin(\theta-\phi)\dot{x} + \cos(\theta-\phi)\dot{y} + r\cos\phi\dot{\theta} \\
p &= -2mr\cos^2(\phi)\cos(\theta)\dot{x} - 2mr\cos^2(\phi)\sin(\theta)\dot{y} + mr^2\sin(2\phi)\dot{\theta} + J_0\sin(2\phi)\dot{\psi}.
\end{aligned}
$$

Adding, subtracting, and scaling these equations, we can write (away from $\phi = \pi/2$)

$$
\begin{bmatrix} \cos(\theta)\dot{x} + \sin(\theta)\dot{y} \\ -\sin(\theta)\dot{x} + \cos(\theta)\dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} -\dfrac{J_0}{2mr}\sin(2\phi)\dot{\psi} \\ 0 \\ \dfrac{J_0}{mr^2}\sin^2(\phi)\dot{\psi} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{2mr}p \\ 0 \\ \dfrac{\tan\phi}{2mr^2}p \end{bmatrix}. \tag{8.4.7}
$$

These equations have the form

$$
g^{-1}\dot{g} + \mathcal{A}_{\mathrm{loc}}(r)\dot{r} = \Upsilon(r)p
$$

where

$$
\begin{aligned}
\mathcal{A}_{\mathrm{loc}} &= -\frac{J_0}{2mr}\sin(2\phi)e_x\, d\psi + \frac{J_0}{mr^2}\sin^2(\phi)e_\theta\, d\psi \\
\Upsilon(r) &= \frac{-1}{2mr}e_x + \frac{1}{2mr^2}\tan(\phi)\, e_\theta.
\end{aligned}
$$

These are precisely the terms which appear in the nonholonomic connection relative to the (global) trivialization (r, g). Note that Υ contains the same information as the local (body) form of the locked inertia tensor, $\mathbb{I}_{\mathrm{loc}}$, as was explained earlier. We also note that $\Upsilon(r)$ is not parallel to the vector ξ^q written above, since Υ is in body representation; when an Ad_g is factored out of ξ^q, it *is* parallel.

The momentum equation, which governs the evolution of p, is given by

$$
\begin{aligned}
\dot{p} &= \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}\xi^q\right]_Q^i \\
&= 4mr\cos(\theta)\cos(\phi)\sin(\phi)\dot{x}\dot{\phi} + 4mr\sin(\theta)\cos(\phi)\sin(\phi)\dot{y}\dot{\phi} + 2J_0\cos(2\phi)\dot{\phi}\dot{\psi} \\
&\quad + 2mr^2\cos(2\phi)\dot{\theta}\dot{\phi} - 2mr\cos(\theta)\cos^2(\phi)\dot{y}\dot{\theta} + 2mr\sin(\theta)\cos^2(\phi)\dot{x}\dot{\theta}
\end{aligned} \tag{8.4.8}
$$

Using equation (8.4.7) to solve for the group velocities $\dot{x}, \dot{y}, \dot{\theta}$, the momentum equation can be rewritten as

$$
\dot{p} = 2J_0\cos^2(\phi)\,\dot{\phi}\dot{\psi} - \tan(\phi)\,p\dot{\phi} \tag{8.4.9}
$$

This version of the momentum equation corresponds to the coordinate form in body representation, equation (4.4.1). Note that equation (8.4.9) contains no terms which are quadratic in p, due to the fact that $\mathfrak{g}^q$ is one dimensional.

Equations (8.4.7) and (8.4.8) describe how paths in the base space, parameterized by $r \in S^1 \times S^1 \times S^1$, are lifted to the fiber $SE(2)$. Notice that even if $\dot{r} = 0$ it is still possible to get motion in the group variables if $p \neq 0$. Indeed, the essential property of the snakeboard is that it is possible to build up p so that the board can build up forward momentum without being directly pushed. The utility of equation (8.4.7) is that it greatly simplifies the process of solving for the motion of the system given the base space trajectory. One alternative, used for example, in Lewis, Ostrowski, Murray & Burdick [1994], involves completely solving the dynamics of the system without taking into account the special geometric structures developed here.

The second order equations which describe the evolution of the base space variables are quite complex and are not given here. See Ostrowski [1995] for additional details.

9 Conclusions

In this paper we have established some basic properties of nonholonomic systems from the Lagrangian point of view; in particular, we have shown how Ehresmann connections can be used to write the kinematic constraints as the condition of horizontality with respect to the connection and shown how the equations of motion can be written in terms of base variables and that these equations involve the curvature of the connection. We then regard symmetry properties of such systems and develop one of the main consequences of this symmetry, namely the momentum equation. The geometry and coordinate structure of this momentum equation is explored in detail. The process of reduction and reconstruction for these nonholonomic systems is worked out making use of a new connection which we call the nonholonomic connection. This connection is obtained by synthesizing the mechanical connection and the constraint connection. Building on the analogy between this theory and the theory of Lagrangian reduction, the reduced Lagrange d'Alembert equations are developed. Several examples are worked out in detail, including the vertical rolling penny, the spherical ball on a rotating table and the snakeboard.

Some interesting topics for future work are as follows:

- The setting of VERSHIK & FADDEEV [1981] for nonholonomic systems with symmetry combined with the results of the present paper. This allows one to better understand the presence of forces and allows one to consider more general constraints and ones that may do work.
- The Hamiltonian formalism for nonholonomic systems with symmetry and the failure of Jacobi's identity as related to curvature (see BATES & SNIATYCKI [1992] and VAN DER SCHAFT & MASCHKE [1994]).
- Additional work on control theoretic issues in the present context.
- Energy-momentum integrators for nonholonomic systems.
- Energy-momentum methods for stability and eventually bifurcation using a nonholonomic version of the work of SIMO, LEWIS & MARSDEN [1991]. The literature in this area is already extensive (see, for example, NEIMARK & FUFAEV [1966]) and for one result in the spirit of the present paper, see Zenkov [1995].

Acknowledgments

We thank Joel Burdick, Philip Holmes, Harvey Lam, Naomi Leonard, Andrew Lewis, Oliver O'Reilly, Jim Ostrowski and Juergen Scheurle for very helpful comments concerning this paper. In particular, the snakeboard example studied by Ostrowski, Lewis, and Burdick was a crucial ingredient in the theoretical development of the present paper. We also thank Neil Getz for his comments on the paper and on the controlled bicycle and Hans Duistermaat and Jost Hermans for their very helpful remarks, both scientific and historical.

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