Submitted, 2004 Conference on Decision and Control
http://www.cds.caltech.edu/~murray/papers/2004k_bftm04-cdc.html

Ellipsoidal Cones and Rendezvous of Multiple Agents

Raktim Bhattacharya Jimmy Fung Abhishek Tiwari
Richard M. Murray
College of Engineering and Applied Science
California Institute of Technology
Pasadena, CA-91125

***Abstract*— In this paper we use ellipsoidal cones to achieve rendezvous of multiple agents. Rendezvous of multiple agents is shown to be equivalent to ellipsoidal cone invariance and a controller synthesis framework is presented. We first demonstrate the methodology on first order LTI systems and then extend it to rendezvous of mechanical systems, that is systems that are force driven.**

I. Introduction

Invariant sets play an important role in many situations when the behaviour of the closed-loop system is constrained in some way. Blanchini in ref. [1] provides an excellent survey of set invariant control. Invariant sets that are cones have found application in problems related to areas as diverse as industrial growth [2], ecological systems and symbiotic species [3], arms race [4] and compartmental system analysis [5], [6]. In general, cone invariance is an essential component in problems involving competition or cooperation. For those interested in cones and dynamical systems, the book by Berman *et al.* [7] will be useful.

In our earlier work [8], we demonstrated that rendezvous of multiple agents is equivalent to cone invariance. Cones in general could be polyhedral or ellipsoidal and the rendezvous problem can be cast as a cone invariance problem of either type. In this paper we use ellipsoidal cones to develop a framework for controller synthesis that achieves multi-agent rendezvous. In [9] we analyze rendezvous using polyhedral cones.

The paper is organized as follows. We first present mathematical preliminaries that is fundamental to our research work. The equivalence of rendezvous and cone invariance is established next. We then present the controller synthesis framework for multiple vehicles modeled as first order linear time invariant (LTI) systems. This framework is extended to agents with higher order dynamics. The synthesis algorithms presented in this paper are then verified and illustrated by theoretical results and simulations.

Postdoctoral Scholar, raktim@cds.caltech.edu
Graduate Student, fung@caltech.edu
Graduate Student, atiwari@cds.caltech.edu
Professor, murray@cds.caltech.edu

II. Mathematical Preliminaries

A. Ellipsoidal Cones

An ellipsoidal cone in $\mathbb{R}^n$ is the following,

$$\Gamma_n = \{\xi \in \mathbb{R}^n : K_n(\xi, Q) \le 0,\ \xi^T u_n \ge 0\}, \quad (1)$$

where $K_n(\xi, Q) = \xi^T Q \xi$, $Q \in \mathbb{R}^{n,n}$ is a symmetric nonsingular matrix with a *single* negative eigen-value λ_n and u_n is the eigen-vector associated with λ_n.

The boundary of the cone Γ_n is denoted by $\partial\Gamma_n$ and is defined by

$$0 \ne \xi \in \partial\Gamma_n \equiv \{\xi \in \Gamma_n : K_n(\xi, Q) = 0\}.$$

The outward pointing normal is the vector $Q\xi$ for $\xi \in \partial\Gamma_n$.

Lemma 1 (2.7 in [10]). *If Γ_n is an ellipsoidal cone, then there exists a nonsingular transformation matrix $M \in \mathbb{R}^{n,n}$ such that*

$$(M^{-1})^T Q M^{-1} = \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} = Q_n$$

where $P \in \mathbb{R}^{n-1,n-1}$, $P > 0$ and $P = P^T$.

Let the transformed state be $x = M\xi$. The ellipsoidal cone in x is therefore,

$$\Gamma_n = \{x : \begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \le 0\} \quad (2)$$

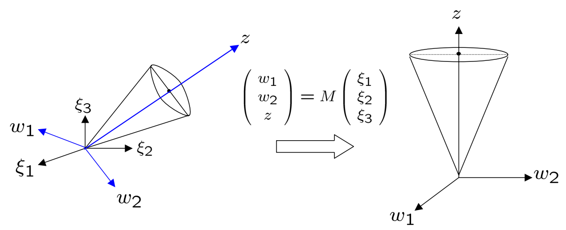


Fig. 1. Ellipsoidal cone in 3-dimension.

where $x = (w \;\; z)^T$, $w \in \mathbb{R}^{n-1}, z \in \mathbb{R}$.

An ellipsoidal cone in three dimension is shown in Fig.(1). The axis of the cone is the eigen-vector associated with the z axis.

B. Ellipsoidal Cone Invariance

Consider a linear autonomous system

$$\dot{\xi} = A\xi. \tag{3}$$

A cone Γ_n is said to be invariant with respect to the dynamics in eqn.(3) if $x(t_0) \in \Gamma_n \Rightarrow x(t) \in \Gamma_n, \forall t \geq t_0$, i.e. if the system starts inside the cone, it stays in the cone for all future time. Such a condition is also known as *exponential non-negativity*, i.e. $e^{At}\Gamma_n \in \Gamma_n$.

It is well known that certain structure in the matrix A imposes constraints on e^{At} [11]. The most well known result is the condition of non-negativity on A which states that if $A_{ij} \geq 0$ for $i \neq j$, then non-negative initial conditions yield non-negative solutions. Schneider and Vidyasagar [12] introduced the notion of *cross-positivity* of A on Γ_n which was shown to be equivalent to exponential non-negativity. Meyer *et al.* [13] extended cross-positivity to nonlinear fields.

Let us characterize $p(\Gamma_n)$ to be the set of matrices $A \in \mathbb{R}^{n,n}$ which are exponentially non-negative on Γ_n. It is defined by the following lemma.

Lemma 2 (3.1 in [10]). *Let Γ_n be an ellipsoidal cone as in eqn.(2). Then,*

$$p(\Gamma_n) = \{A \in \mathbb{R}^{n,n} \; :< A\xi, Q\xi > \leq 0, \, \forall \xi \in \Gamma_n\}. \tag{4}$$

Lemma 2 states that A is such that the flow of the associated vector field is directed towards the interior of Γ_n, i.e. the dot product of the outward normal of Γ_n and the field is negative at the boundary of the cone. This leads to the result on the necessary and sufficient condition for exponential non-negativity of ellipsoidal cones.

Theorem 1 (3.5 in [10]). *A necessary and sufficient condition for $A \in p(\Gamma_n)$ is that there exists $\gamma \in \mathbb{R}$ such that,*

$$Q_n A + A^T Q_n - \gamma Q_n \leq 0.$$

where Q_n is defined in Lemma 1.
Proof *Please refer to pg.162 of [10].*

III. Rendezvous of Multiple Agents

In our earlier work [8] we had defined rendezvous to be the problem of driving multiple agents to a desired point such they all arrive within a small time interval of each other. It is also required that the trajectories of the agents be such that they arrive at the destination point only once. In the rest of the paper, we refer to the destination point as the origin.

A. Rendezvous Interpretation on the Phase Plane

Consider two scalar systems $\mathcal{V}_1$ and $\mathcal{V}_2$ defined by

$$\begin{array}{ll} \mathcal{V}_1: & \dot{x}_1 = f_1(x_1), \quad f_1(0) = 0, \\ \mathcal{V}_2: & \dot{x}_2 = f_2(x_2), \quad f_2(0) = 0. \end{array}$$

If $\mathcal{V}_1$ and $\mathcal{V}_2$ are exponentially stable systems then they will arrive at the origin at eventually as time tends to infinity. The trajectories may also be such that $\mathcal{V}_1$ arrives long before $\mathcal{V}_2$ does, which is undesirable. Therefore, just exponential stability doesn't ensure rendezvous.

To achieve rendezvous in finite time we relaxed the definition of rendezvous to be such that rendezvous is achieved if the agents enter a certain neighborhood around the origin. We defined this region to be the *rendezvous region*. We also demonstrated that rendezvous is best visualized in the phase plane. To interpret rendezvous for $\mathcal{V}_1$ and $\mathcal{V}_2$, we defined the following regions in the phase plane,

$$\begin{array}{rcl} U_1 & = & \{(x_1, x_2) : -\delta \leq x_1 \leq \delta\}, \\ U_1 & = & \{(x_1, x_2) : -\delta \leq x_1 \leq \delta\}, \\ \mathcal{S} & = & U_1 \cup U_2, \\ \mathcal{F} & = & U_1 \cup U_2 - U_1 \cap U_2, \\ \mathcal{W} & = & \mathbb{R}^2 - U_1 \cup U_2. \end{array}$$

The rendezvous problem is well posed if the initial condition of the two agents satisfy $(x_1(0), x_2(0)) \in \mathcal{W}$, i.e. they both start far away from the destination point. The set $\mathcal{F}$ is the set of all points where one agent enters the rendezvous region much before the other. Therefore, trajectories must avoid $\mathcal{F}$ for a valid rendezvous, i.e.

$$x_1(t), x_2(t)) \notin \mathcal{F} \ \forall t. \tag{5}$$

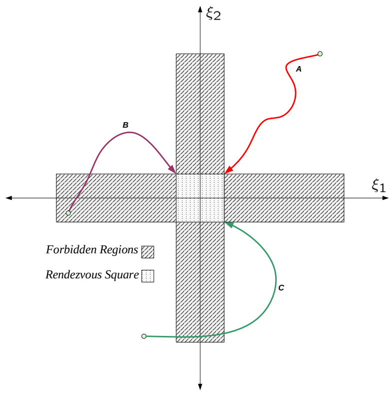


Fig. 2. Rendezvous in Phase Plane

In Fig. 2, trajectory B starts from an invalid initial condition and trajectory C enters the rendezvous region prior to the final entry. Such trajectories are not valid rendezvous trajectories. Trajectory A is an example of a valid rendezvous trajectory.

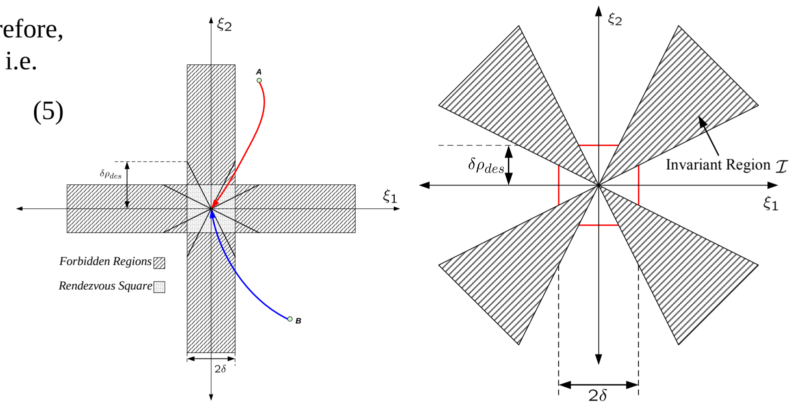
With the constraint defined in eqn.(5), the only way trajectories can approach the origin is through the corners of $\mathcal{S}$, i.e. through one of the points (δ, δ), $(\delta, -\delta)$, $(-\delta, \delta)$ or $(-\delta, -\delta)$. This also restricts the trajectories to be confined to the quadrant they originate from. Entering $\mathcal{S}$ from one of its corners also implies that the agents enter the rendezvous region at precisely the same time. In reality it may be acceptable to allow agents to arrive within ΔT seconds of each other. We distinguished between the two cases by referring to rendezvous with $\Delta T = 0$ as *perfect rendezvous* and rendezvous with ΔT small as *real* or *approximate* rendezvous. The notion of approximate rendezvous led to a design parameter ρ_{des} and a measure of rendezvous ρ defined by

$$\rho = \frac{max(|x_1(t_a)|, |x_2(t_a)|)}{\delta} \tag{6}$$

where t_a is arrival time of the first agent. From the definition of ρ it is clear that for a given trajectory $\rho \geq 1$. Therefore a specification of rendezvous is meaningful if and only if $\rho_{des} \geq 1$.

The notion of approximate rendezvous is illustrated in Fig.3(a).

Approximate rendezvous allows trajectories in phase plane to enter $U_1 \cup U_2$ as long as they are within

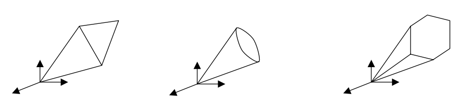


(a) Approximate rendezvous (b) Invariant wedge, the region $\mathcal{I}$

Fig. 3.

the wedge defined by the points $(\delta, \delta\rho_{des})$, $(0, 0)$ and $(\delta\rho_{des}, \delta)$. In Fig. 3(a), trajectory A achieves approximate rendezvous, but trajectory B does not.

Therefore the only admissible trajectories for approximate rendezvous are those that arrive at the origin while remaining in the wedge-like region $\mathcal{I}$ as shown in Fig. 3(b). For n agents achieving rendezvous, the region $\mathcal{I}$ becomes a cone in n-dimensional space. Depending on the norm used to define ρ in eqn.(6), the cone could be either polyhedral or ellipsoidal. For ∞-norm, as is in eqn.(6), the cone is a polyhedral cone with $2^N - 2$ sides. The measure ρ can also be defined using 1-norm or 2-norm. The wedge in that case becomes a polyhedral cone with n sides for 1-norm or an ellipsoidal cone for 2-norm. This is shown in Fig.4.



Desired region of invariance	Polyhedral Cone with n-sides	Quadratic Cone in n-dimension	Polyhedral Cone with 2^n-2 sides
Norm	$\|\|\cdot\|\|_1$	$\|\|\cdot\|\|_2$	$\|\|\cdot\|\|_\infty$
Complexity	n constraints	1 constraint	2^n-2 constraints

Fig. 4. Region $\mathcal{I}$ in 3 dimensional state space.

B. Rendezvous as Cone Invariance

In our earlier work [8] we showed that agents achieve rendezvous if they rendered the region $\mathcal{I}$ invariant. As mentioned before, the region $\mathcal{I}$ in a higher dimensional phase space becomes a cone. Therefore for n agents, rendezvous is guaranteed if the cone in the n-dimensional state space is invariant with respect to their dynamics.

Cone invariance alone does not guarantee that the agents reach the origin. Figure 5 shows trajectories A, B and C. Trajectory A achieves cone invariance but does not reach the origin. Trajectory B reaches the origin but escapes the cones. Trajectory C is the only trajectory that reaches the origin and stays within the cone. We are interested in trajectories such as C.

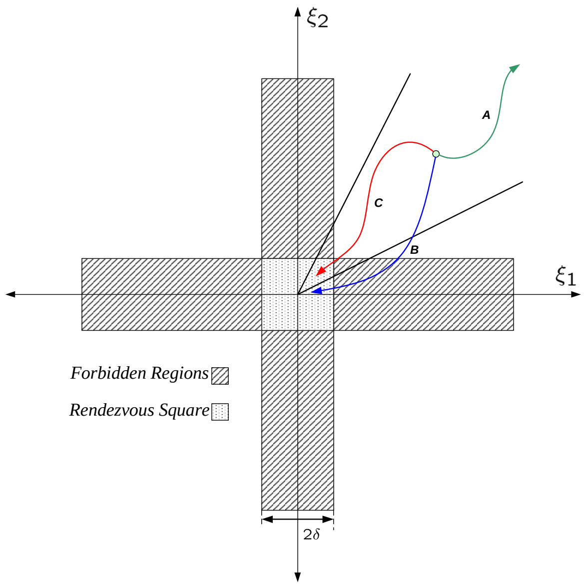


Fig. 5. Possible trajectories in $\mathcal{I}$

In this paper we assume that ρ is defined using the 2-norm and hence we are interested in the invariance of ellipsoidal cones.

C. Problem Formulation

Let us assume that there are n agents for which rendezvous is desired. Let us also assume that the agents are modeled as *first order* LTI systems. Collectively, they can be written as

$$\begin{aligned}\dot{\xi} = \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_n \end{pmatrix} &= \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} \\ &+ \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \\ &= A\xi + Bu.\end{aligned}$$

We also assume that we are given an ellipsoidal cone Γ_n as defined by eqn.(2), where Q depends on the specified measure of rendezvous ρ_{des}.

Therefore, given a cone Γ_n and n agents modeled as first order LTI systems, we are interested in determining control $u(t)$ such that the following are true,

$$\begin{aligned}&\xi(t_0) \in \Gamma_n \Rightarrow \xi(t) \in \Gamma_n,\ \forall t \geq t_0, \text{ and} \\ &\xi(t) \rightarrow 0 \text{ as } t \rightarrow \infty\end{aligned} \tag{7}$$

D. Controller Synthesis in the Lyapunov Function Framework

In this section we address ellipsoidal cone invariance in the framework of Lyapunov functions. Given a cone Γ_n as in eqn.(2), let us define two Lyapunov functions $V_w(w)$ and $V_z(z)$ as

$$V_w(w) = w^T P w \tag{8}$$
$$V_z(z) = z^2 \tag{9}$$

Note that V_w is a valid candidate for Lyapunov function as $P > 0$.

Cone invariance in this context is defined by

$$V_w(w(t)) \leq V_z(z(t)),\ \forall t \geq t_0.$$

This is guaranteed if and only if

$$\dot{V}_w < \dot{V}_z \text{ when } V_w = V_z. \tag{10}$$

which is the condition for exponentially non-negativity on Γ_n.

For controller synthesis we transform Γ_n as defined by lemma 1. Therefore,

$$x = M\xi \Rightarrow \dot{x} = MAM^{-1}x + MBu.$$

If we consider a *full state feedback* control framework, then

$$u = F\xi = FM^{-1}x$$

and the closed-loop system is therefore

$$\dot{x} = M(A + BF)M^{-1}x. \tag{11}$$

With respect to the partition $x = (w\ \ z)^T$, the closed-loop system in eqn.(11) can be written as

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \left[\begin{array}{c|c} A_{ww} & A_{wz} \\ \hline A_{zw} & A_{zz} \end{array}\right] \begin{pmatrix} w \\ z \end{pmatrix}. \tag{12}$$

Therefore, the inequality $\dot{V}_w < \dot{V}_z$ in eqn.(10) implies

$$\begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + P A_{ww} & P A_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & -2A_{zz} \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} < 0.$$

Substituting $w^T P w = z^2$ to impose the condition $V_w =$

V_z, $\dot{V}_w < \dot{V}_z$ at the boundary of the cone implies

$$\begin{bmatrix} A_{ww}^T P + PA_{ww} - 2A_{zz}P & PA_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & 0 \end{bmatrix} < 0. \tag{13}$$

To ensure that V_w and V_z reaches zero as time goes to infinity, it is sufficient to constrain $A_{zz} \in \mathbb{R} < -\alpha_{des}$. The parameter α_{des} is a positive real number that governs the decay rate of $z(t)$. Therefore, the controller synthesis problem in this framework is the following LMI feasibility problem in the state feedback gain matrix F:

$$\begin{bmatrix} A_{ww}^T P + PA_{ww} - 2A_{zz}P & PA_{wz} - A_{zw} & 0 \\ A_{wz}^T P - A_{zw}^T & 0 & 0 \\ 0 & 0 & A_{zz} \end{bmatrix} < 0. \tag{14}$$

Therefore, if there exists an F such that the LMIP in eqn.(14) is feasible, then the control law $u(t) = F\xi(t)$ solves the problem posed by eqn.(7).

The constraint in eqn.(13) is a necessary and sufficient condition for cone invariance and can be proved as follows. Theorem 1 states that the necessary and sufficient condition for exponential non-negativity is the existence of $\gamma \in \mathbb{R}$ such that

$$Q_n A + A^T Q_n - \gamma Q_n \leq 0.$$

This is equivalent to

$$\begin{bmatrix} A_{ww}^T P + PA_{ww} - \gamma P & PA_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & \gamma - 2A_{zz} \end{bmatrix} < 0,$$

which is eqn.(13) for $\gamma = 2A_{zz}$.

The next theorem states the Lyapunov certificate theorem for the rendezvous problem defined by eqn.(7).

Theorem 2. *The LMI in eqn.(14) implies that*

$$V(w, z) = w^T P w + z^2$$

is a Lyapunov function for the closed-loop system in eqn.(12).

Proof For $P > 0$, $V(w, z)$ is a valid Lyapunov function. $\dot{V}(w, z)$ for the system in eqn.(12) is,

$$\dot{V}(w,z) = \begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + PA_{ww} & PA_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & 2A_{zz} \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix}$$

Equation (14) implies,

$$\begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + PA_{ww} & PA_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & 0 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} < 2A_{zz} w^T P w$$

for all $w(t), z(t)$. Adding $2A_{zz}z^2$ to both sides gives us

$$\begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + PA_{ww} & PA_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & 2A_{zz} \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix}^T < 2A_{zz}(w^T P w + z^2)$$

or

$$\dot{V}(w, z) < 2A_{zz} V(w, z).$$

This implies that the largest exponent of the closed-loop system is A_{zz}. With $A_{zz} < 0$, we can conclude $\dot{V}(w, z) < 0$, hence the proof.

□

We next analyze trajectories that start outside the cone Γ_n. It is interesting to note that the condition of cone invariance implies that all trajectories that start outside the cone enter the cone. This is given by the following theorem.

Theorem 3. *For a system $\dot{x} = Ax$ as in eqn.(11),*

$$\begin{aligned} & \dot{K}_n(x, Q_n) < 0, \forall x \in \partial\Gamma_n \\ \Rightarrow \quad & \dot{K}_n(x, Q_n) < 0, \forall x : K_n(x, Q_n) > 0. \end{aligned}$$

Proof Equation (14) implies,

$$\begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + PA_{ww} & PA_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & 0 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} < 2A_{zz} w^T P w$$

for all $w(t), z(t)$. Adding $-2A_{zz}z^2$ to both sides gives us

$$\begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + PA_{ww} & PA_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & -2A_{zz} \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix}^T < 2A_{zz}(w^T P w - z^2).$$

Recalling that $A_{zz} < 0$ and $K_n(x, Q_n) > 0$ implies $w^T P w - z^2 > 0$, we can conclude that

$$\begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + PA_{ww} & PA_{wz} - A_{zw} \\ A_{wz}^T P - A_{zw}^T & -2A_{zz} \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} < 0, \forall x : K_n(x, Q_n) > 0,$$

which is equivalent to $\dot{K}_n(x, Q_n) < 0$ for x outside the cone $K_n(x, Q_n)$. This condition implies that all trajectories that start outside the cone will arrive at the boundary of the cone.

□

Example 1. Figure 6 demonstrates rendezvous for three agents modeled as first order *open-loop unstable* systems with $a_i = 1, b_i = 1$. The trajectory of ξ_3 is worth noting. The initial condition for ξ_3 is closer to the origin relative to that for ξ_1 and ξ_2. It is interesting to observe that ξ_3 initially moves away from the origin before making the final entry, along with ξ_1 and ξ_2. Therefore the control formulation presented in this paper allows agents to *procrastinate*, as demonstrated by ξ_3, in order to achieve rendezvous. Figure 7(a) shows that $V_w < V_z$ for all times. Figure 7(b) shows the trajectory inside the cone Γ_n.

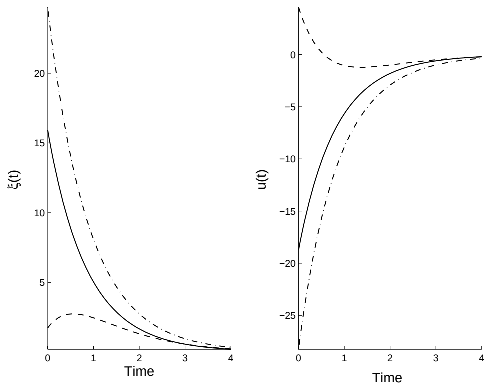
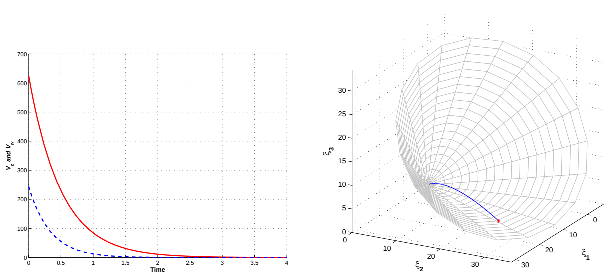


Fig. 6. State and control trajectories - solid(ξ_1, u_1), dash-dot(ξ_2, u_2), dash(ξ_3, u_3)



(a) Solid(V_z), dash(V_w)

(b) Trajectory inside $K_n(\xi, Q) < 0$.

Fig. 7. Cone invariance and asymptotic stability.

IV. Rendezvous of Agents with Higher Order Dynamics

In the previous section we formulated the control synthesis problem for agents that were modeled as *first order systems*. In reality agents will have higher order dynamics. We continue to restrict our interest to linear systems and consider the problem of multi-agent rendezvous for agents that are mechanical systems i.e. agents whose dynamics can be represented by the linear second order differential equation

$$m_i\ddot{\xi}_i(t) + d_i\dot{\xi}_i(t) + k_i\xi_i(t) = u_i(t)$$

where m_i, d_i and k_i are mass, damping and stiffness respectively. In matrix-vector notation the dynamics can be represented by

$$\frac{d}{dt}\begin{pmatrix} \xi_i \\ \dot{\xi}_i \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}\begin{pmatrix} \xi_i \\ \dot{\xi}_i \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u_i.$$

For n agents the collective dynamics can be represented by the equation

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{bmatrix} 0 & I_N \\ A_{\eta\xi} & A_{\eta\eta} \end{bmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{bmatrix} 0 \\ B_\eta \end{bmatrix} u \quad (15)$$

and we assume that the system is controllable.

For dynamical systems given by eqn.(15), the cone Γ_n defined on position states ξ is not closed-loop *holdable* (pg.65 [7]). A cone Γ_n is said to be closed-loop holdable if there exists control $u(t)$ such that the condition of exponential non-negativity can be enforced, i.e.

$$\exists\, u(t) : \dot{K}_n(\xi, Q) < 0,\ \forall \xi \in \partial\Gamma$$

For the system in eqn.(15) and the cone in eqn.(1),

$$\begin{aligned} \dot{K}_n(\xi, Q) &= \dot{\xi}^T Q\xi + \xi^T Q\dot{\xi} \\ &= \eta^T Q\xi + \xi^T Q\eta \end{aligned}$$

which is *independent* of u. Therefore, the condition of exponential non-negativity cannot be enforced by any choice of u.

Controller Synthesis

We propose to solve the rendezvous problem by the following two-step controller synthesis algorithm.

Step 1 - We first consider the dynamical system $\dot{\xi} = \eta$. In the first step we determine $\eta(t)$ such that $\xi(t)$ achieves rendezvous. We assume that

$$\eta(t) = F\xi(t). \quad (16)$$

That is, we treat $\eta(t)$ as a control variable and determine the state feedback gain F such that Γ_n is invariant with respect to the system

$$\begin{aligned} \dot{\xi} &= \eta \\ &= F\xi \end{aligned}$$

This synthesis can be achieved by solving the LMIP in eqn.(14).

Step 2 - Once F is known we treat

$$\eta_r(t) = F\xi(t)$$

as the reference signal and design a tracking controller so that

$$||\eta(t) - \eta_r(t)|| \to 0 \text{ as } t \to \infty.$$

Any control design methodology can be used to design this controller. Note that the dynamics of η need not necessarily be linear. In general, the methodology can be used to achieve rendezvous for agents with dynamics

$$\begin{aligned} \dot{\xi} &= \eta \\ \dot{\eta} &= f(\xi, \eta) + g(\xi, \eta)u. \end{aligned}$$

If $f(\xi, \eta)$ and $g(\xi, \eta)$ are nonlinear functions, one would have to adopt a nonlinear control design framework to obtain a tracking controller.

Equation (16) defines a desired $\eta(t)$ with respect to position $\xi(t)$. Even if the agents start inside Γ_n, it is likely that the initial condition of $\eta(t)$ will not be the desired value. Depending on how large this initial offset is, it is possible that the agents escape the cone while $\eta_r(t)$ is being tracked. Once the agents leave the cone, it is important to analyze if $\eta_r(t) = F\xi(t)$ is still a valid reference for rendezvous.

Theorem 3 states that all trajectories starting outside Γ_n will intersect the surface of Γ_n, i.e. the controller obtained by solving eqn.(14) will drive agents into the cone for all initial conditions outside the cone. Hence, agents with dynamics as in eqn.(15) will achieve rendezvous if they track the reference $\eta_r(t) = F\xi(t)$.

Example 2. In this example we demonstrate the application of the proposed control methodology to rendezvous of three double integrators. Figure 8 shows the position and the velocity of the three systems. In the subplots for velocities v_1, v_2 and v_3, the solid line is the desired velocity and the dashed line is the velocity of the agents.

The three agents started from position $(35, 5, 20)$ with velocity $(41.4, 12.2, 21.5)$. The tracking controller for velocity reference was designed as a *linear quadratic regulator* using the formulation

$$\min_K \int_0^\infty \left[(\eta - \eta_r)^T M(\eta - \eta_r) + u^T N u\right] dt,$$

such that

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \left(\begin{bmatrix} 0 & I_N \\ A_{\eta\xi} & A_{\eta\eta} \end{bmatrix} + \begin{bmatrix} 0 \\ B_\eta \end{bmatrix} K\right) \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and

$$\eta_r = F\xi.$$

The initial condition was deliberately chosen so that the agents started outside the cone with large error in velocity. Figure 9(a) shows the $V_z > V_w$ before the agents arrive at the origin, implying trajectories outside the cone enter the cone before arriving at the origin. This is clearly visible in Fig.9(b). Figure 9(b) also demonstrates that the reference η_r is valid outside the cone. The initial loop in the trajectory indicates that the agents were initially heading towards the wrong direction. This is due to a large initial error in the velocity. By reducing this error to zero, the tracking controller was able to achieve rendezvous.

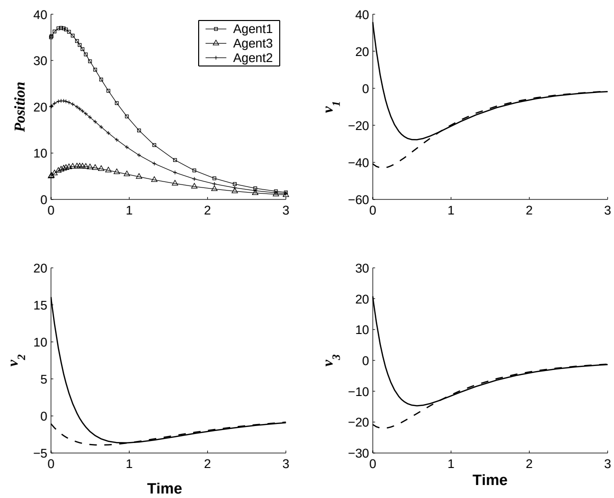


Fig. 8. Position and tracked velocity

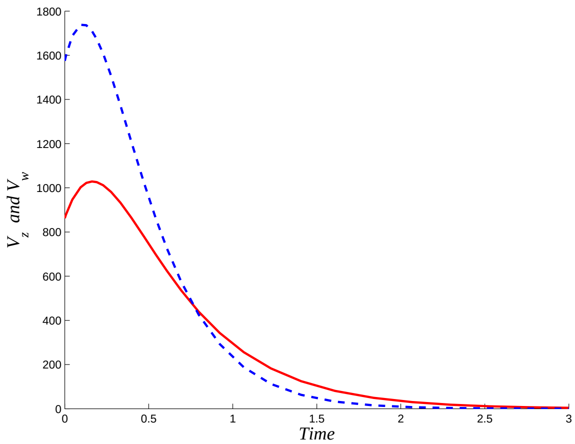
V. Summary

In this paper we presented a control synthesis framework for multi-agent rendezvous problem. It was shown that the problem of rendezvous of multiple agents can be cast as a cone invariance problem. We restricted our attention to ellipsoidal cones. The proposed synthesis algorithm is based on determining control such that the closed-loop system renders a given ellipsoidal cone invariant. We first demonstrated this on agents modeled as first order LTI systems and extended it to agents that are mechanical systems with second order dynamics.

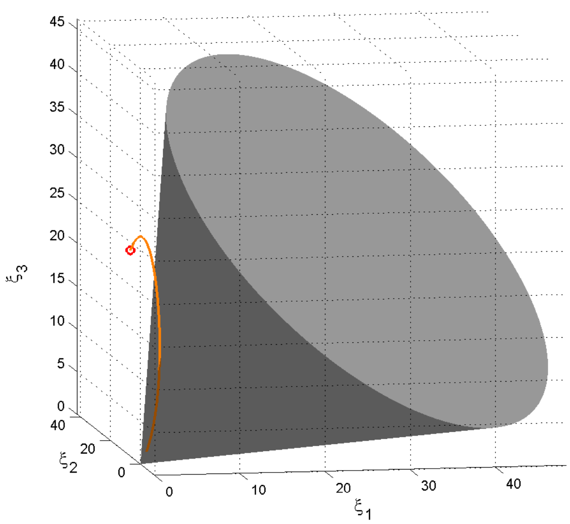
The framework presented in the paper is still restricted to rendezvous on a line, i.e. it can achieve rendezvous on a single state variable of the agent. In reality, it is desired that rendezvous be achieved on multiple state variables. Extension of this framework to such cases is a subject of our current research.

Acknowledgements

The authors are grateful for support through: AFOSR F49620-01-1-0460 for Abhishek Tiwari; NSF-ITR-ACI-0204932 for Jimmy Fung; CIT Special Institute Fellow-



(a) Solid(V_z), dash(V_w)



(b) Trajectory outside the cone enters the cone Γ_n.

Fig. 9. $V_z > V_w$ close to the origin.

ship and CIT EASD Fellowship for John Carson; and DARPA F30602-01-0577 for Raktim Bhattacharya.

REFERENCES

[1] F. Blanchini. Set Invariance in Control. *Automatica*, 35:1747–1767, 1999.

[2] J. Stiglitz, editor. *A Collected Scientific Papers of Paul A. Samuelson*. MIT Press, Cambridge, Massachusetts, 1966.

[3] D. G. Luenberger. *Introduction to Dynamic Systems*, chapter 6. Wiley, New York, 1979.

[4] L. F. Richardson. *Arms and Insecurity*. Boxwood Press, Pittsburgh Quadrangle Books, Pittsburgh,Pennsylvania, 1960.

[5] J. A. Jacquez. *Compartmental Analysis in Biology and Medicine*, volume 50. Elsevier, New York, 1972.

[6] D. H. Anderson. Compartmental Modeling and Tracer Kinetics. *Lecture Notes in Biomathematics*, 1983.

[7] A. Berman, M. Neumann, and R. J. Stern. *Nonnegative Matrices in Dynamic Systems*. A Wiley-Interscience Publication, New York, 1989.

[8] A. Tiwari, J. Fung, J. M. Carson III, R. Bhattacharya, and R. M. Murray. A Framework for Lyapunov Certificates for Multi-Vehicle Rendezvous Problems. In *Proceedings of the American Control Conference*, Boston, 2004.

[9] A. Tiwari, J. Fung, R. Bhattacharya, and R. M. Murray. Polyhedral Cone Invariance Applied to Rendezvous of Multiple Agents. In *IEEE Conference on Decision and Control (submitted)*, Bahamas, 2004.

[10] R. J. Stern and H. Wolkowicz. Exponential Nonnegativity on the Ice Cream Cone. *Siam J. Matrix Anal. Appl.*, 12:160–165, January 1991.

[11] J. A. Yorke. Invariance for Ordinary Differential Equations. *Math Sys. Theory*, 1:353–372, 1967.

[12] H. Schneider and M. Vidyasagar. Cross-Positive Matrices. *SIAM J. Numer. Anal.*, 7(4):508–519, 1970.

[13] D. G. Meyer, T. L. Piatt, H. N. G. Wadley, and R. Vancheeswaran. Nonlinear Invariance: Cross-Positive Vector Fields. In *Proceedings of the American Control Conference*, Philadelphia, 1998.