STABILITY PROOF FOR PD CONTROLLER

FRANCESCO BULLO

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Control and Dynamical Systems
California Institute of Technology

This is a corrected proof of Proposition 4.10 at page 194, Chapter 4, Section 5.3 of (Murray, Li & Sastry 1994).

We adopt the notations as in Section 5.1. The equation of motion of a robot manipulator is given by

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta, \dot{\theta}) = \tau, \tag{4.47}$$

where the various symbols are defined in the book. Let $e = \theta - \theta_{\mathrm{d}}$ be the error; we call *augmented PD control law* the feedback given by:

$$\tau = M(\theta)\ddot{\theta}_{\mathrm{d}} + C(\theta, \dot{\theta})\dot{\theta}_{\mathrm{d}} + N(\theta, \dot{\theta}) - K_v\dot{e} - K_p e. \tag{4.53}$$

Assume the desired trajectory satisfies

$$\sup_t \|\theta_{\mathrm{d}}\| < \rho_0, \qquad \sup_t \|\dot{\theta}_{\mathrm{d}}\| < \rho_1. \tag{Bounds}$$

Then the control law (4.53) applied to (4.47) results in exponential trajectory tracking if $K_v, K_p > 0$.

Proof. The closed–loop system is

$$M(\theta)\ddot{e} + C(\theta, \dot{\theta})\dot{e} + K_v\dot{e} + K_p e = 0$$

or, rearranging the terms,

$$\ddot{e} = -M(\theta)^{-1}\Big(C(\theta, \dot{\theta})\dot{e} + K_v\dot{e} + K_p e\Big).$$

Consider the candidate Lyapunov function

$$V(e, \dot{e}, t) = V_0(e, \dot{e}, t) + V_{\mathrm{cross}}(e, \dot{e}, t),$$

where as usual

$$V_0(e, \dot{e}, t) = \frac{1}{2}e^T K_p e + \frac{1}{2}\dot{e}^T M(\theta)\dot{e}$$

and

$$V_{\mathrm{cross}}(e, \dot{e}, t) = \epsilon\, e^T M(\theta)\dot{e}.$$

Computing the time derivative of V_0, we can easily prove stability:

$$\begin{aligned}\dot V_0(e,\dot e,t) &= e^T K_p \dot e + \dot e^T M(\theta)\ddot e + \frac{1}{2}\dot e^T \dot M(\theta)\dot e\\ &= e^T K_p \dot e + \dot e^T\big(-C(\theta,\dot\theta)\dot e - K_v \dot e - K_p e\big) + \frac{1}{2}\dot e^T \dot M(\theta)\dot e\\ &= -\dot e^T K_v \dot e + \frac{1}{2}\dot e^T\big(\dot M(\theta) - 2C(\theta,\dot\theta)\big)\dot e.\end{aligned}$$

From Lemma 4.2 we know that the matrix $\dot M(\theta) - 2C(\theta,\dot\theta)$ is skew-symmetric, hence

$$\dot V_0(e,\dot e,t) = -\dot e^T K_v \dot e$$

and the closed-loop system is proven to be (at least) Lyapunov stable. Exponential stability is proved by introducing V_{cross}. Its time derivative satisfies

$$\begin{aligned}\dot V_{\text{cross}}(e,\dot e,t) &= \epsilon\big(\dot e^T M(\theta)\dot e + e^T \dot M(\theta)\dot e + e^T M(\theta)\ddot e\big)\\ &= \epsilon\dot e^T M(\theta)\dot e + \epsilon e^T \dot M(\theta)\dot e - \epsilon e^T\big(C(\theta,\dot\theta)\dot e + K_v\dot e + K_p e\big)\end{aligned}$$

and rearranging terms

$$\begin{aligned}\dot V_{\text{cross}}(e,\dot e,t) &= \epsilon\Big(-e^T K_p e + \dot e^T M(\theta)\dot e + e^T\big(\dot M(\theta) - C(\theta,\dot\theta) - K_v\big)\dot e\Big)\\ &\triangleq A_1 + A_2 + A_3.\end{aligned}$$

The first addendum A_1 ensures that $\dot V$ is negative definite. The second addendum sums up with $\dot V_1 = -\dot e^T K_v \dot e$. Since A_2 is of size ϵ, $\dot V$ will remain negative definite. Now, we only need to show that the third addendum preserves this property too. Since A_3 is a a cross term in e and $\dot e$, it suffice to show the boundedness of the operator norm of

$$\dot M(\theta) - C(\theta,\dot\theta) - K_v.$$

Since K_v is constant, it is bounded. Also, using the definitions in the text, we have

$$e^T \dot M(\theta)\dot e = \frac{\partial M_{ij}}{\partial\theta_k}\dot\theta_k e_i \dot e_j$$

and

$$e^T C(\theta,\dot\theta)\dot e = \Gamma_{ijk}\dot\theta_k e_i \dot e_j = \frac{1}{2}\Big(\frac{\partial M_{ij}}{\partial\theta_k} + \frac{\partial M_{ik}}{\partial\theta_j} - \frac{\partial M_{kj}}{\partial\theta_i}\Big)\dot\theta_k e_i \dot e_j$$

so that we have the following bound

$$\|e^T(\dot M(\theta) - C(\theta,\dot\theta))\dot e\| \le \frac{5}{2}\left\|\frac{\partial M}{\partial\theta}\right\| \cdot \|\dot\theta\| \cdot \|e\| \cdot \|\dot e\|.$$

Since $M(\theta)$ is analytic in θ, also $\|\frac{\partial M}{\partial\theta}\|$ is a bounded operator, as long as θ remains in a compact set. This is ensured by the boundedness assumptions in equation (Bounds) coupled with the Lyapunov stability of the closed loop, proven above.

Finally we have

$$A_3 \leq \left(\|K_v\| + \frac{5}{2} \left\| \frac{\partial M}{\partial \theta} \right\| \cdot \|\dot{\theta}\| \right) \|e\| \cdot \|\dot{e}\|.$$

By using the boundedness assumptions, it's easy to conclude that A_3 is a size ϵ bounded operator evaluated on e, $\dot{e}$. Hence, the time derivative of V is negative definite and exponential convergence follows as usual. □

For more details, we refer to (Whitcomb, Rizzi & Koditschek 1993).

References

Murray, R., Li, Z. & Sastry, S. S. (1994), *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Boca Raton, Florida.
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E-mail address: `bullo@indra.caltech.edu`

Caltech 104-44, Padadena, CA 91125