LPE Reading Group

Controls Pr ture # 5

9 October 2001, 10:30 am PDT

Optimal Control

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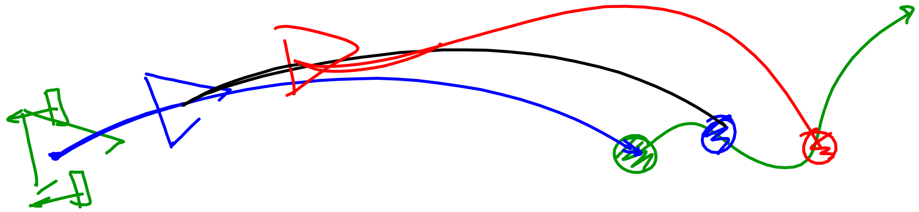
Overview: RMM + RaFF
Lyapunov, CLF
Optimization & optimal control

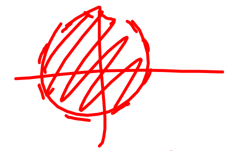
→ Model Predictive Control

Model Predictive Control (or receding horizon control)



Optimal doesn't matter (suboptimal is fine, computable is required)

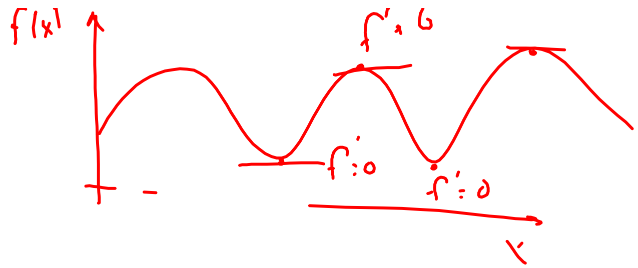
Opt -ion (notes to be posted)



normed linear space. Think $\mathbb{R}^n$

open

Defn Let $f : \Omega \to \mathbb{R}$ be a real-valued function on $\Omega \subseteq X$, a NLS.
A point $x_0 \in \Omega$ is a local minimum or Ω if $\exists$ a nbd $N \subset X$
s.t. $f(x_0) \leq f(x)$ $\forall\, x \in \Omega \cap N$. The pt x_0 is a strict local min
if $f(x_0) < f(x)$ $\forall\, x \in \Omega \cap N$, $x \neq x_0$. The pt x_0 is a global min
if " $\forall\, x \in \Omega$



Example: $f : \mathbb{R}^n \to \mathbb{R}$

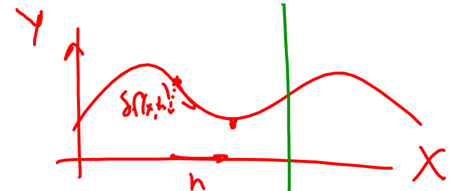
$f'(x_1, \ldots, x_n)$

$$\delta f(x;h) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

$h = (h_1, \ldots, h_n)$

Defn Let X & Y be normed linear spaces and $T : D \subset X \to Y$. The mapping T is Frechet differentiable at $x \in D$ if $\forall\, h \in X$ there exist $\delta T(x;h) \in Y$ which is linear, cts in h and satisfies

$$\lim_{h \to 0} \frac{\| T(x+h) - T(x) - \delta T(x,h) \|}{\|h\|} = 0$$



Defn Extrema = min or max

Thm Let $f: X \to \mathbb{R}$ be Frechet differentiable at x_0. Then f has an extremum at $x_0 \in X$ only if $\delta f(x_0; h) = 0 \quad \forall \; h \in X$

Example ($X = D[t_1, t_2]$) The set $D[a,b]$ of all real-valued, continuously differentiable function on the real interval $[a,b]$ w/ $\|x\| = \max|x(t)| + \max|\dot{x}(t)|$

Example 6 (E-L equations) Let $X = D[t_1, t_2]$

and cost function

$$J(x) = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$$

To solve, compute $\delta J(x;h)$. Let $h \in D[t_1, t_2]$ that vanishes at the endpoints

$$\delta J(x;h) = \left. \frac{d}{d\alpha} \int_{t_1}^{t_2} L(x+\alpha h, \dot{x}+\alpha \dot{h}, t)\, dt \right|_{\alpha=0}$$

Claim. This is a Frechet derivative. (Thm: Frechet der is unique)

$$\delta J(x;h) = \int_{t_1}^{t_2} \frac{\partial L}{\partial x}(x,\dot{x},t)\,h(t)\,dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}}(x,\dot{x},t)\,\dot{h}(t)\,dt$$

min $\int_{t_1}^{t_2} L(x,\dot{x},t)\,dt$

curve $x(t)$ satisfying this must also satisfy

$$\frac{\partial L}{\partial x}(x,\dot{x},t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x,\dot{x},t) = 0$$

$L = \frac{1}{2}x^2 - \frac{1}{2}\dot{x}^2$

$\rightarrow x + \ddot{x} = 0$

$$= \int_{t_1}^{t_2} \cdots h(t) + \frac{\partial L}{\partial \dot{x}}(x,\dot{x},t)\,h(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x,\dot{x},t)\,h(t)\,dt$$

(boundary term $\rightarrow 0$)

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x}(x,\dot{x},t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x,\dot{x},t)\right] h(t)\,dt$$

(bracket) $= 0$

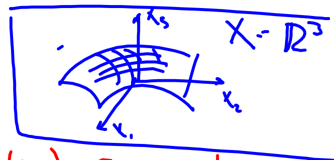
Optimization w/ equality constraints

Lin indep on $D[a,b]$

g_1, g_2, g_3 are lin dep if

$g_3^{(t)} = \alpha g_1^{(t)} + \beta g_2^{(t)}$, $t \in [a,b]$

constraints



$f : X \to \mathbb{R}$ cost fcn

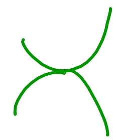
$g_i : X \to \mathbb{R}$ $\quad g_i(x) = 0 \quad i = 1, \ldots, m$

f, g_i are Frechet differentiable

<u>Defn</u> A point $x_0 \in X$ satisfying $g_i(x_0) = 0$, $i = 1, \ldots, m$ is a <u>regular point</u> if $g_1'(x_0), \ldots, g_m'(x_0)$ are lin. indep.

$\delta g(x; h) = g'(x)\, h$, $\quad g : X \to Y$

Thm If x_0 is an extremum of $f: X \to \mathbb{R}$ subject to $g_i(x_0) = 0$, $i = 1, \dots, m$ and if x_0 is a regular pt of $\{g_i\}$ then



$$\delta f(x_0; h) = 0 \qquad \forall h \text{ s.t. } \delta g_i(x_0; h) = 0 \qquad i = 1, \dots, m$$

Revisit E-L example

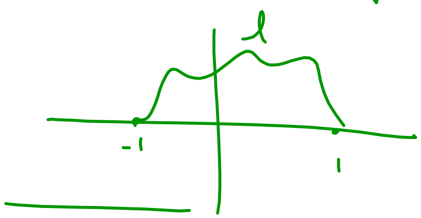
$X: \mathbb{R}^3$

$$g_1(x) = x(t_1) - x_0 = 0$$

$$g_2(x) = x(t_2) - x_f = 0$$

$\Rightarrow$ h vanishes at t_1, t_2

<u>Example</u> $x(t) \in \mathbb{R}$ a curve on $[-1, 1]$ which is zero at endpoints. Maximize $\int x$ subject to a fixed length constraint



$$\max \int_{-1}^{1} x(t)\,dt$$

subject to

$$\int_{-1}^{1} \sqrt{\dot{x}^2 + 1}\,dt = \ell$$

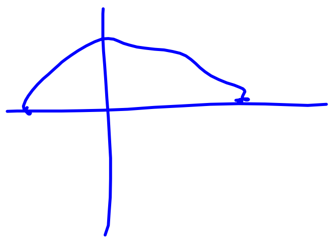
Aside: Thm 6 If x_0 is extremum of f subject to g_i then $\exists$ scalars $\lambda_i \in \mathbb{R}$ s.t.

$$\tilde{h}(x) = f(x) + \sum \lambda_i g_i(x)$$

has a stationary pt ($\delta \tilde{h}(x; h) = 0$)

f subject to g

minimize $H = f + \lambda^T g$



Optimal control

$g_1: \dot{x} = f(x,u)$ ← constraint

& $\min J = \int_0^T L(x,u)\,dt$

$L(x,u) = x^T Q x + u^T R u$

$\delta J = 0 \quad \forall h \quad \delta g(x;h) = 0$