LPE Reading Group

Controls Primer, Lecture #6

16 October 2001

Last time

- Optimization (general)
- Euler-Lagrange eqs
- w/ constraints

Today

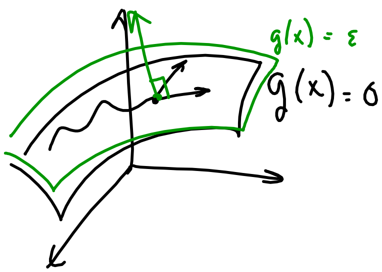
- Finish equality constraints
- Optimal control
- Special cases: LQR, bang-bang

Next week: BILL! MPC + CLF

Can you read this?

Thm 5 If x_0 is an extremum of $f: X \to \mathbb{R}$ subject to $g_i(x_0) = 0$, $i = 1, \dots, m$ and if x_0 is a regular point then

$$\delta f(x_0; h) = 0 \quad \forall h \text{ such that } \delta g_i(x_0; h) = 0 \quad i = 1, \dots, m$$



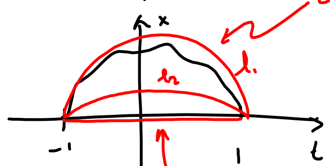
Thm 6. If .... then $\exists$ scalars λ_i, $i = 1, \dots, m$ such that

$$\ell(x) = f(x) + \sum_i \lambda_i g_i(x)$$

has a stationary point at x_0 $\left(\ell'(x_0) = 0\right)$

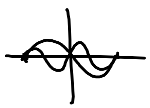
Corollary: $\ell'(x) = f'(x) + \sum \lambda_i g_i'(x)$

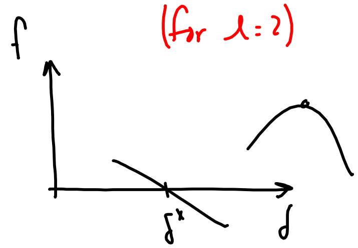
Example



optimal

also a sol'n of $J' = 0$ (for $\lambda = 2$)





$$\max \overbrace{\int_{-1}^{1} x(t)\,dt}^{f} \quad \text{subject to} \quad \int_{-1}^{1} \sqrt{\dot{x}^2+1}\,dt = \ell = \int_{-1}^{1} \frac{\ell}{2}$$

$$J(x) = \int_{-1}^{1} \underbrace{\left(x + \lambda\left(\sqrt{\dot{x}^2+1} - \frac{\ell}{2}\right)\right)}_{L(x,\dot{x})} dt$$

$$J' = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \underbrace{\lambda \frac{d}{dt} \frac{\dot{x}}{\sqrt{\dot{x}^2+1}} - 1 = 0}$$

$$(x - x_1)^2 + (t - t_1)^2 = r^2$$

Optimal control

$\dot{x} = f(x, u, t)$ $t \geq t_0$ $x \in \mathbb{R}^n$ $x(t_0)$ given $u \in \mathbb{R}^m$ (T given) System

$\min J(x(), u()) = \int_{t_0}^{T} L(x,u,t)\,dt + \phi(x(T), T)$ cost fcn

$\psi(x(T), T) = 0$ final constraint

$\xi \in X = D^n[t_0, T] \times C^m[t_0, T] \cancel{\times \mathbb{R}}$

cts diff fcns on $\mathbb{R}^n$ w/ $\|x\| = \max|x(t)| + \max|\dot{x}(t)|$

real-valued ($\mathbb{R}^m$) cts fcns w/ $\|u\| = \max|u(t)|$

Aside:

$f + \sum \lambda_i g_i \rightarrow f + \langle \lambda, g \rangle$

$\lambda \in Y^*$

$g : X \to Y$

$J(\xi) = J(x(), u())$ $g_c(\xi) = \dot{x} - f(x,u,t)$

$g_0(\xi) = x(t_0) - x_0$ $g_f(\xi) = \psi(x(T), T)$

$\tilde{J} = J + \langle \lambda, g \rangle$ $\tilde{J}' = 0$

$$H = \tilde{J} = \phi(x(T),T) + \int_{t_0}^{T} L(x,u)\,dt + \underbrace{\langle \lambda, \dot{x} - f(x,u,t)\rangle}_{\int_{t_0}^{T} \lambda^T(t)(\dot{x} - f(x,u,\tau))\,d\tau} + \nu^T \psi(x(T),T) + \eta^T (x(t_0) - x_0)$$

<u>Thm 7</u> If x_0 is an extremum of f subject to $g(x) = 0$ where $g : X \to Y$ then $\exists\ \lambda_0 \in Y^*$ such that (x_0, λ_0) is a stationary point for $H : X \times Y^* \to \mathbb{R}$

$$H(x,\lambda) = f(x) + \langle \lambda, g(x)\rangle$$

(x,u) (λ, ν, η)

$$\delta H(\xi, \gamma) = (\ \underset{0}{\overset{\|}{}}\ )\delta x + (\ \underset{0}{\overset{\|}{}}\ )\delta u + (\ \underset{0}{\overset{\|}{}}\ )\delta \lambda + (\ \underset{0}{\overset{\|}{}}\ )\delta \nu + (\ \underset{0}{\overset{\|}{}}\ )\delta \eta$$

Working out...

$\delta x: \quad \dot{\lambda} = -\frac{\partial f}{\partial x}\lambda + \frac{\partial L}{\partial x}$

$\delta u: \quad 0 = \frac{\partial f}{\partial u}\lambda + \frac{\partial L}{\partial u}$ → $L: \cdots u^T R u \quad \frac{\partial L}{\partial u} : Ru$

$\delta \lambda: \quad \dot{x} = f(x, u, t)$

$\delta \nu \quad 0 = \psi(x(T), T)$ → $\delta x(T): \left[\frac{\partial \phi}{\partial x} + \frac{\partial \psi^T}{\partial x}\nu - \lambda\right]^T \Big|_{t=T} = 0$

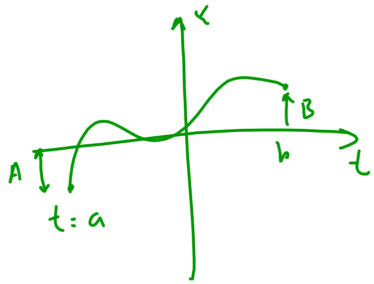
$\partial \eta \quad 0 = x(t_0) - x_0$

$H = L(x,u,t) + \lambda^T f(x,u,t)$

$\dot{x} = \frac{\partial H}{\partial \lambda}$ state

$\dot{\lambda} = -\frac{\partial H}{\partial x}$ costate + bdy conditions

$0 = \frac{\partial H}{\partial u}$ stationarity



Time-optimal

T free $\Rightarrow$ $\int T$

$$J = \int_0^T 1\, dt$$

$\dot{x} = u \qquad x \in \mathbb{R},\ u \in \mathbb{R}$

$$J = \int_a^b \sqrt{1+u^2}\, dt \qquad H(x,u,\lambda) = \sqrt{1+u^2} + \lambda u$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} = u$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \Rightarrow \lambda \text{ constant}$$

$$0 = \frac{\partial H}{\partial u} = 0 = \lambda + \frac{u}{\sqrt{1+u^2}} \qquad u = \frac{-\lambda}{\sqrt{1-\lambda^2}} = \text{constant}$$

$$x(t) = \frac{(A-B)t + (aB - bA)}{a-b} \qquad u(t) = \frac{A-B}{a-b}$$

Special case $\dot{x} = Ax + Bu$

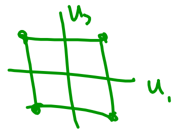
1. LQR $\int_0^T (x^T Q x + u^T R u)\,dt + x^T(T) P_1 x(T)$

Riccati

Optimal: $u(t) = -R^{-1} B^T P(t) x$ $\qquad -\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q$

If $T = \infty$ then .... $\qquad P(t_1) = P_1 \qquad P \in \mathbb{R}^{n \times n}$

$A^T P + PA - PBR^{-1}B^T P = -Q$ ← Algebraic Riccati

2. $\dot{x} = Ax + Bu$ $\quad u \in$ 

J = min time $\Rightarrow$ u bounces between corners