# 4 Lyapunov Stability Theory

In this section we review the tools of Lyapunov stability theory. These tools will be used in the next section to analyze the stability properties of a robot controller. We present a survey of the results that we shall need in the sequel, with no proofs. The interested reader should consult a standard text, such as Vidyasagar [**?**] or Khalil [**?**], for details.

## 4.1 Basic definitions

Consider a dynamical system which satisfies

$$\dot{x} = f(x,t) \qquad x(t_0) = x_0 \qquad x \in \mathbb{R}^n. \tag{4.31}$$

We will assume that $f(x,t)$ satisfies the standard conditions for the existence and uniqueness of solutions. Such conditions are, for instance, that $f(x,t)$ is Lipschitz continuous with respect to $x$, uniformly in $t$, and piecewise continuous in $t$. A point $x^* \in \mathbb{R}^n$ is an *equilibrium point* of (4.31) if $f(x^*,t) \equiv 0$. Intuitively and somewhat crudely speaking, we say an equilibrium point is *locally stable* if all solutions which start near $x^*$ (meaning that the initial conditions are in a neighborhood of $x^*$) remain near $x^*$ for all time. The equilibrium point $x^*$ is said to be *locally asymptotically stable* if $x^*$ is locally stable and, furthermore, all solutions starting near $x^*$ tend towards $x^*$ as $t \to \infty$. We say somewhat crude because the time-varying nature of equation (4.31) introduces all kinds of additional subtleties. Nonetheless, it is intuitive that a pendulum has a locally stable equilibrium point when the pendulum is hanging straight down and an unstable equilibrium point when it is pointing straight up. If the pendulum is damped, the stable equilibrium point is locally asymptotically stable.

By shifting the origin of the system, we may assume that the equilibrium point of interest occurs at $x^* = 0$. If multiple equilibrium points exist, we will need to study the stability of each by appropriately shifting the origin.

**Definition 4.1. Stability in the sense of Lyapunov**
The equilibrium point $x^* = 0$ of (4.31) is *stable (in the sense of Lyapunov)* at $t = t_0$ if for any $\epsilon > 0$ there exists a $\delta(t_0, \epsilon) > 0$ such that

$$\|x(t_0)\| < \delta \quad \Longrightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq t_0. \tag{4.32}$$

Lyapunov stability is a very mild requirement on equilibrium points. In particular, it does not require that trajectories starting close to the origin tend to the origin asymptotically. Also, stability is defined at a time instant $t_0$. *Uniform stability* is a concept which guarantees that the equilibrium point is not losing stability. We insist that for a uniformly stable equilibrium point $x^*$, $\delta$ in the Definition 4.1 not be a function of $t_0$, so that equation (4.32) may hold for all $t_0$. Asymptotic stability is made precise in the following definition:

**Definition 4.2. Asymptotic stability**
An equilibrium point $x^* = 0$ of (4.31) is *asymptotically stable* at $t = t_0$ if

1. $x^* = 0$ is stable, and
2. $x^* = 0$ is locally attractive; i.e., there exists $\delta(t_0)$ such that

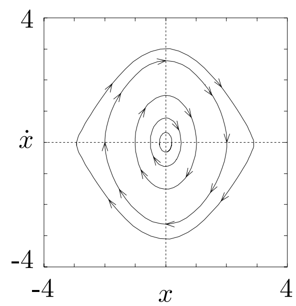
$$\|x(t_0)\| < \delta \quad \Longrightarrow \quad \lim_{t \to \infty} x(t) = 0. \tag{4.33}$$

As in the previous definition, asymptotic stability is defined at $t_0$. *Uniform asymptotic stability* requires:

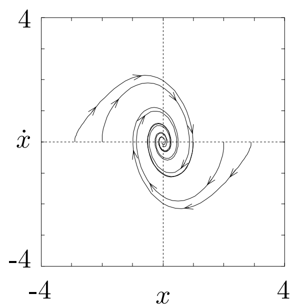
1. $x^* = 0$ is uniformly stable, and
2. $x^* = 0$ is uniformly locally attractive; i.e., there exists $\delta$ independent of $t_0$ for which equation (4.33) holds. Further, it is required that the convergence in equation (4.33) is uniform.

Finally, we say that an equilibrium point is *unstable* if it is not stable. This is less of a tautology than it sounds and the reader should be sure he or she can negate the definition of stability in the sense of Lyapunov to get a definition of instability. In robotics, we are almost always interested in uniformly asymptotically stable equilibria. If we wish to move the robot to a point, we would like to actually converge to that point, not merely remain nearby. Figure 4.7 illustrates the difference between stability in the sense of Lyapunov and asymptotic stability.

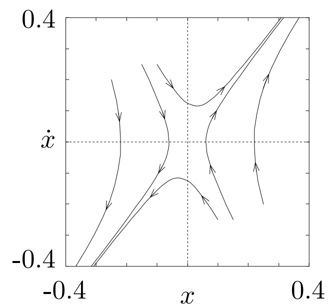
Definitions 4.1 and 4.2 are *local* definitions; they describe the behavior of a system near an equilibrium point. We say an equilibrium point $x^*$ is *globally* stable if it is stable for all initial conditions $x_0 \in \mathbb{R}^n$. Global stability is very desirable, but in many applications it can be difficult to achieve. We will concentrate on local stability theorems and indicate where it is possible to extend the results to the global case. Notions




(a) Stable in the sense of Lyapunov




(b) Asymptotically stable




(c) Unstable (saddle)

Figure 4.7: Phase portraits for stable and unstable equilibrium points.

of uniformity are only important for time-varying systems. Thus, for time-invariant systems, stability implies uniform stability and asymptotic stability implies uniform asymptotic stability.

It is important to note that the definitions of asymptotic stability do not quantify the rate of convergence. There is a strong form of stability which demands an exponential rate of convergence:

**Definition 4.3. Exponential stability, rate of convergence**
The equilibrium point $x^* = 0$ is an *exponentially stable* equilibrium point of (4.31) if there exist constants $m, \alpha > 0$ and $\epsilon > 0$ such that

$$\|x(t)\| \leq m e^{-\alpha(t-t_0)} \|x(t_0)\| \tag{4.34}$$

for all $\|x(t_0)\| \leq \epsilon$ and $t \geq t_0$. The largest constant $\alpha$ which may be utilized in (4.34) is called the *rate of convergence*.

Exponential stability is a strong form of stability; in particular, it implies uniform, asymptotic stability. Exponential convergence is important in applications because it can be shown to be robust to perturbations and is essential for the consideration of more advanced control algorithms, such as adaptive ones. A system is *globally exponentially stable* if the bound in equation (4.34) holds for all $x_0 \in \mathbb{R}^n$. Whenever possible, we shall strive to prove global, exponential stability.

## 4.2 The direct method of Lyapunov

Lyapunov's direct method (also called the second method of Lyapunov) allows us to determine the stability of a system without explicitly integrating the differential equation (4.31). The method is a generalization of the idea that if there is some "measure of energy" in a system, then we can study the rate of change of the energy of the system to ascertain stability. To make this precise, we need to define exactly what one means by a "measure of energy." Let $B_\epsilon$ be a ball of size $\epsilon$ around the origin, $B_\epsilon = \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$.

**Definition 4.4. Locally positive definite functions (lpdf)**
A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a *locally positive definite function* if for some $\epsilon > 0$ and some continuous, strictly increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}$,

$$V(0,t) = 0 \quad \text{and} \quad V(x,t) \geq \alpha(\|x\|) \qquad \forall x \in B_\epsilon, \forall t \geq 0. \tag{4.35}$$

A locally positive definite function is locally like an energy function. Functions which are globally like energy functions are called positive definite functions:

**Definition 4.5. Positive definite functions (pdf)**
A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a *positive definite function* if it satisfies the conditions of Definition 4.4 and, additionally, $\alpha(p) \to \infty$ as $p \to \infty$.

To bound the energy function from above, we define decrescence as follows:

**Definition 4.6. Decrescent functions**
A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is *decrescent* if for some $\epsilon > 0$ and some continuous, strictly increasing function $\beta : \mathbb{R}_+ \to \mathbb{R}$,

$$V(x,t) \leq \beta(\|x\|) \qquad \forall x \in B_\epsilon, \forall t \geq 0 \tag{4.36}$$

Using these definitions, the following theorem allows us to determine stability for a system by studying an appropriate energy function. Roughly, this theorem states that when $V(x,t)$ is a locally positive definite function and $\dot{V}(x,t) \leq 0$ then we can conclude stability of the equilibrium point. The time derivative of $V$ is taken along the trajectories of the system:

$$\dot{V}\Big|_{\dot{x}=f(x,t)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f.$$

Table 4.1: Summary of the basic theorem of Lyapunov.

| | Conditions on $V(x,t)$ | Conditions on $-\dot{V}(x,t)$ | Conclusions |
|---|---|---|---|
| 1 | lpdf | $\geq 0$ locally | Stable |
| 2 | lpdf, decrescent | $\geq 0$ locally | Uniformly stable |
| 3 | lpdf, decrescent | lpdf | Uniformly asymptotically stable |
| 4 | pdf, decrescent | pdf | Globally uniformly asymptotically stable |

In what follows, by $\dot{V}$ we will mean $\dot{V}|_{\dot{x}=f(x,t)}$.

**Theorem 4.4. Basic theorem of Lyapunov**
*Let $V(x,t)$ be a non-negative function with derivative $\dot{V}$ along the trajectories of the system.*

1. *If $V(x,t)$ is locally positive definite and $\dot{V}(x,t) \leq 0$ locally in $x$ and for all $t$, then the origin of the system is locally stable (in the sense of Lyapunov).*

2. *If $V(x,t)$ is locally positive definite and decrescent, and $\dot{V}(x,t) \leq 0$ locally in $x$ and for all $t$, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).*

3. *If $V(x,t)$ is locally positive definite and decrescent, and $-\dot{V}(x,t)$ is locally positive definite, then the origin of the system is uniformly locally asymptotically stable.*

4. *If $V(x,t)$ is positive definite and decrescent, and $-\dot{V}(x,t)$ is positive definite, then the origin of the system is globally uniformly asymptotically stable.*

The conditions in the theorem are summarized in Table 4.1.

Theorem 4.4 gives sufficient conditions for the stability of the origin of a system. It does not, however, give a prescription for determining the Lyapunov function $V(x,t)$. Since the theorem only gives sufficient conditions, the search for a Lyapunov function establishing stability of an equilibrium point could be arduous. However, it is a remarkable fact that the converse of Theorem 4.4 also exists: if an equilibrium point is stable, then there exists a function $V(x,t)$ satisfying the conditions of the theorem. However, the utility of this and other converse theorems is limited by the lack of a computable technique for generating Lyapunov functions.

Theorem 4.4 also stops short of giving explicit rates of convergence of solutions to the equilibrium. It may be modified to do so in the case of exponentially stable equilibria.

**Theorem 4.5. Exponential stability theorem**
*$x^* = 0$ is an exponentially stable equilibrium point of $\dot{x} = f(x,t)$ if and only if there exists an $\epsilon > 0$ and a function $V(x,t)$ which satisfies*

$$\begin{aligned}
\alpha_1\|x\|^2 \le V(x,t) \le \alpha_2\|x\|^2 \\
\dot{V}|_{\dot{x}=f(x,t)} \le -\alpha_3\|x\|^2 \\
\|\frac{\partial V}{\partial x}(x,t)\| \le \alpha_4\|x\|
\end{aligned}$$

*for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and $\|x\| \le \epsilon$.*

The rate of convergence for a system satisfying the conditions of Theorem 4.5 can be determined from the proof of the theorem [**?**]. It can be shown that

$$m \le \left(\frac{\alpha_2}{\alpha_1}\right)^{1/2} \qquad \alpha \ge \frac{\alpha_3}{2\alpha_2}$$

are bounds in equation (4.34). The equilibrium point $x^* = 0$ is globally exponentially stable if the bounds in Theorem 4.5 hold for all $x$.

## 4.3 The indirect method of Lyapunov

The indirect method of Lyapunov uses the linearization of a system to determine the local stability of the original system. Consider the system

$$\dot{x} = f(x,t) \tag{4.37}$$

with $f(0,t) = 0$ for all $t \ge 0$. Define

$$A(t) = \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=0} \tag{4.38}$$

to be the Jacobian matrix of $f(x,t)$ with respect to $x$, evaluated at the origin. It follows that for each fixed $t$, the remainder

$$f_1(x,t) = f(x,t) - A(t)x$$

approaches zero as $x$ approaches zero. However, the remainder may not approach zero *uniformly*. For this to be true, we require the stronger condition that

$$\lim_{\|x\|\to 0} \sup_{t\ge 0} \frac{\|f_1(x,t)\|}{\|x\|} = 0. \tag{4.39}$$

If equation (4.39) holds, then the system

$$\dot{z} = A(t)z \tag{4.40}$$

is referred to as the (uniform) *linearization* of equation (4.31) about the origin. When the linearization exists, its stability determines the local stability of the original nonlinear equation.

**Theorem 4.6. Stability by linearization**
*Consider the system* (4.37) *and assume*

$$\lim_{\|x\|\to 0} \sup_{t\geq 0} \frac{\|f_1(x,t)\|}{\|x\|} = 0.$$

*Further, let* $A(\cdot)$ *defined in equation* (4.38) *be bounded. If 0 is a uniformly asymptotically stable equilibrium point of* (4.40) *then it is a* locally *uniformly asymptotically stable equilibrium point of* (4.37).

The preceding theorem requires *uniform* asymptotic stability of the linearized system to prove uniform asymptotic stability of the nonlinear system. Counterexamples to the theorem exist if the linearized system is not uniformly asymptotically stable.

If the system (4.37) is time-invariant, then the indirect method says that if the eigenvalues of

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$$

are in the open left half complex plane, then the origin is asymptotically stable.

This theorem proves that *global* uniform asymptotic stability of the linearization implies *local* uniform asymptotic stability of the original nonlinear system. The estimates provided by the proof of the theorem can be used to give a (conservative) bound on the domain of attraction of the origin. Systematic techniques for estimating the bounds on the regions of attraction of equilibrium points of nonlinear systems is an important area of research and involves searching for the "best" Lyapunov functions.

## 4.4 Examples

We now illustrate the use of the stability theorems given above on a few examples.

**Example 4.5. Linear harmonic oscillator**
Consider a damped harmonic oscillator, as shown in Figure 4.8. The dynamics of the system are given by the equation

$$M\ddot{q} + B\dot{q} + Kq = 0, \tag{4.41}$$

where $M$, $B$, and $K$ are all positive quantities. As a state space equation we rewrite equation (4.41) as

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -(K/M)q - (B/M)\dot{q} \end{bmatrix}. \tag{4.42}$$

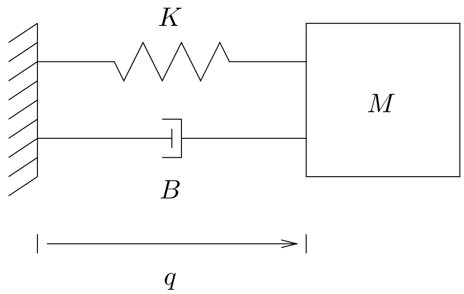



Figure 4.8: Damped harmonic oscillator.

Define $x = (q, \dot{q})$ as the state of the system.

Since this system is a linear system, we can determine stability by examining the poles of the system. The Jacobian matrix for the system is

$$A = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix},$$

which has a characteristic equation

$$\lambda^2 + (B/M)\lambda + (K/M) = 0.$$

The solutions of the characteristic equation are

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4KM}}{2M},$$

which always have negative real parts, and hence the system is (globally) exponentially stable.

We now try to apply Lyapunov's direct method to determine exponential stability. The "obvious" Lyapunov function to use in this context is the energy of the system,

$$V(x,t) = \frac{1}{2}M\dot{q}^2 + \frac{1}{2}Kq^2. \tag{4.43}$$

Taking the derivative of $V$ along trajectories of the system (4.41) gives

$$\dot{V} = M\dot{q}\ddot{q} + Kq\dot{q} = -B\dot{q}^2. \tag{4.44}$$

The function $-\dot{V}$ is quadratic but not locally positive definite, since it does not depend on $q$, and hence we cannot conclude exponential stability. It is still possible to conclude *asymptotic* stability using Lasalle's invariance principle (described in the next section), but this is obviously conservative since we already know that the system is exponentially stable.

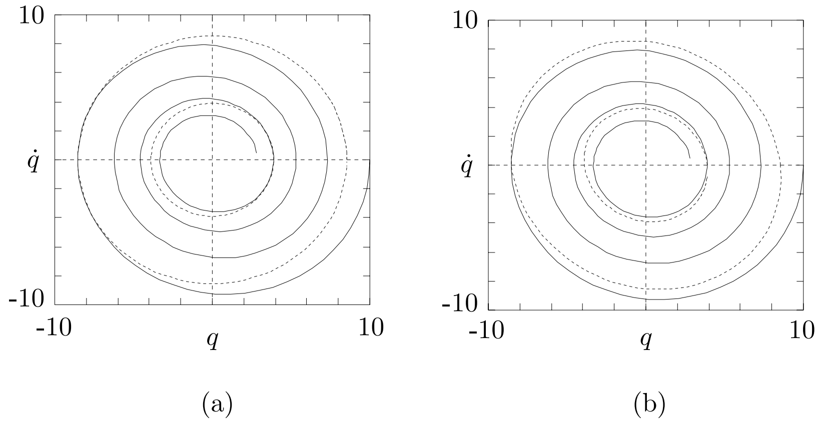



Figure 4.9: Flow of damped harmonic oscillator. The dashed lines are the level sets of the Lyapunov function defined by (a) the total energy and (b) a skewed modification of the energy.

The reason that Lyapunov's direct method fails is illustrated in Figure 4.9a, which shows the flow of the system superimposed with the level sets of the Lyapunov function. The level sets of the Lyapunov function become tangent to the flow when $\dot q = 0$, and hence it is not a valid Lyapunov function for determining exponential stability.

To fix this problem, we skew the level sets slightly, so that the flow of the system crosses the level surfaces transversely. Define

$$V(x,t) = \frac{1}{2}\begin{bmatrix} q \\ \dot q \end{bmatrix}^T \begin{bmatrix} K & \epsilon M \\ \epsilon M & M \end{bmatrix} \begin{bmatrix} q \\ \dot q \end{bmatrix} = \frac{1}{2}\dot q M \dot q + \frac{1}{2} q K q + \epsilon \dot q M q,$$

where $\epsilon$ is a small positive constant such that $V$ is still positive definite. The derivative of the Lyapunov function becomes

$$\begin{aligned} \dot V &= \dot q M \ddot q + q K \dot q + \epsilon M \dot q^2 + \epsilon q M \ddot q \\ &= (-B + \epsilon M)\dot q^2 + \epsilon(-K q^2 - B q \dot q) = -\begin{bmatrix} q \\ \dot q \end{bmatrix}^T \begin{bmatrix} \epsilon K & \frac{1}{2}\epsilon B \\ \frac{1}{2}\epsilon B & B - \epsilon M \end{bmatrix} \begin{bmatrix} q \\ \dot q \end{bmatrix}. \end{aligned}$$

The function $\dot V$ can be made negative definite for $\epsilon$ chosen sufficiently small (see Exercise 11) and hence we can conclude *exponential* stability. The level sets of this Lyapunov function are shown in Figure 4.9b.

This same technique is used in the stability proofs for the robot control laws contained in the next section.

**Example 4.6. Nonlinear spring mass system with damper**
Consider a mechanical system consisting of a unit mass attached to a

*nonlinear* spring with a velocity-dependent damper. If $x_1$ stands for the position of the mass and $x_2$ its velocity, then the equations describing the system are:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -f(x_2) - g(x_1). \end{aligned} \tag{4.45}$$

Here $f$ and $g$ are smooth functions modeling the friction in the damper and restoring force of the spring, respectively. We will assume that $f, g$ are both passive; that is,

$$\begin{aligned} \sigma f(\sigma) \geq 0 \quad \forall \sigma \in [-\sigma_0, \sigma_0] \\ \sigma g(\sigma) \geq 0 \quad \forall \sigma \in [-\sigma_0, \sigma_0] \end{aligned}$$

and equality is only achieved when $\sigma = 0$. The candidate for the Lyapunov function is

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma)\, d\sigma.$$

The passivity of $g$ guarantees that $V(x)$ is a locally positive definite function. A short calculation verifies that

$$\dot{V}(x) = -x_2 f(x_2) \leq 0 \quad \text{when } |x_2| \leq \sigma_0.$$

This establishes the stability, but not the asymptotic stability of the origin. Actually, the origin is asymptotically stable, but this needs Lasalle's principle, which is discussed in the next section.

## 4.5 Lasalle's invariance principle

Lasalle's theorem enables one to conclude asymptotic stability of an equilibrium point even when $-\dot{V}(x,t)$ is not locally positive definite. However, it applies only to autonomous or periodic systems. We will deal with the autonomous case and begin by introducing a few more definitions. We denote the solution trajectories of the autonomous system

$$\dot{x} = f(x) \tag{4.46}$$

as $s(t, x_0, t_0)$, which is the solution of equation (4.46) at time $t$ starting from $x_0$ at $t_0$.

**Definition 4.7. $\omega$ limit set**
The set $S \subset \mathbb{R}^n$ is the $\omega$ *limit set* of a trajectory $s(\cdot, x_0, t_0)$ if for every $y \in S$, there exists a strictly increasing sequence of times $t_n$ such that

$$s(t_n, x_0, t_0) \to y$$

as $t_n \to \infty$.

**Definition 4.8. Invariant set**
The set $M \subset \mathbb{R}^n$ is said to be an (positively) *invariant set* if for all $y \in M$ and $t_0 \geq 0$, we have

$$s(t, y, t_0) \in M \quad \forall t \geq t_0.$$

It may be proved that the $\omega$ limit set of every trajectory is closed and invariant. We may now state Lasalle's principle.

**Theorem 4.7. Lasalle's principle**
*Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally positive definite function such that on the compact set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ we have $\dot{V}(x) \leq 0$. Define*

$$S = \{x \in \Omega_c : \dot{V}(x) = 0\}.$$

*As $t \to \infty$, the trajectory tends to the largest invariant set inside S; i.e., its $\omega$ limit set is contained inside the largest invariant set in S. In particular, if S contains no invariant sets other than $x = 0$, then 0 is asymptotically stable.*

A global version of the preceding theorem may also be stated. An application of Lasalle's principle is as follows:

**Example 4.7. Nonlinear spring mass system with damper**
Consider the same example as in equation (4.45), where we saw that with

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma)\, d\sigma,$$

we obtained

$$\dot{V}(x) = -x_2 f(x_2).$$

Choosing $c = \min(V(-\sigma_0, 0), V(\sigma_0, 0))$ so as to apply Lasalle's principle, we see that

$$\dot{V}(x) \leq 0 \quad \text{for } x \in \Omega_c := \{x : V(x) \leq c\}.$$

As a consequence of Lasalle's principle, the trajectory enters the largest invariant set in $\Omega_c \cap \{x_1, x_2 : \dot{V} = 0\} = \Omega_c \cap \{x_1, 0\}$. To obtain the largest invariant set in this region, note that

$$x_2(t) \equiv 0 \quad \Longrightarrow \quad x_1(t) \equiv x_{10} \quad \Longrightarrow \quad \dot{x}_2(t) = 0 = -f(0) - g(x_{10}),$$

where $x_{10}$ is some constant. Consequently, we have that

$$g(x_{10}) = 0 \quad \Longrightarrow \quad x_{10} = 0.$$

Thus, the largest invariant set inside $\Omega_c \cap \{x_1, x_2 : \dot{V} = 0\}$ is the origin and, by Lasalle's principle, the origin is locally asymptotically stable.

There is a version of Lasalle's theorem which holds for periodic systems as well. However, there are no significant generalizations for nonperiodic systems and this restricts the utility of Lasalle's principle in applications.