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MPC Primer

I. Numerical Methods

- Parametric Opt.

$\mathbb{R}^n \ni u$, $\mathbb{R}^+ \ni \min_u L(x,u)$ s.t. $f(x,u)=0$

- + dynamics

$\longrightarrow \min_u \phi(t_f, x(t_f)) + \int_{t_0}^{t_f} L(x,u,t)\,dt$

- RIOTS

s.t. $\dot{x} = f(x,u)$, $x(t_0)$

II. MPC

A) Problem Statem.

B) Proofs of Stability

Recall: $L: \mathbb{R}^n \to \mathbb{R}^+$, $\Omega \underset{\text{open}}{\subset} \mathbb{R}^n$

Solve $\min_{u \in \Omega} L(u)$

Know u^* is an extremum $\Longrightarrow$

$\frac{\partial L}{\partial u}(u^*) = 0$, $\frac{\partial^2 L}{\partial u^2}(u^*) \geq 0$

$\Longleftarrow$ $\frac{\partial^2 L}{\partial u^2}(u^*) > 0$

Descent Algorithms

idea $\frac{\partial L}{\partial u} \perp \left\{ \begin{matrix} L = \text{const.} \\ \text{curves} \end{matrix} \right\}$

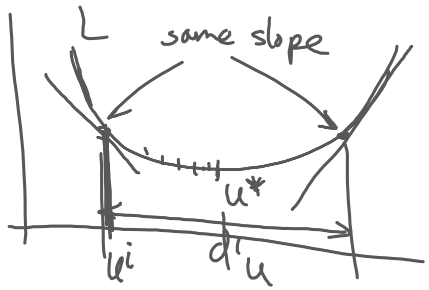
is greatest decrease direction

1) 1st order gradient
2) Conjugate directions (L is quadratic)
3) Successive 2)

1) Steepest Descent (gradient) : improve guess u^* until $\frac{\partial L}{\partial u}$ "small"

<u>alg.</u> a) $i=0$, guess u^i, compute $\frac{\partial L}{\partial u}(u^i)$

b) $u^{i+1} = u^i - d_i\left(\frac{\partial L}{\partial u}(u^i)\right)$ $\quad 0 < d_i < 1$



$$L(u^{i+1}) = L\left(u^i - d_i\left(\frac{\partial L}{\partial u}(u^i)\right)\right) \simeq L(u^i) - \frac{\partial L}{\partial u}(u^i)^T d_i \frac{\partial L}{\partial u}(u^i) \leq L(u^i)$$

$d_{i+1} = d_i$ if $L(u^{i+1}) < L(u^i)$

$d_{i+1} = 1/k$, k smallest integer s.t. •

2) Conjugate directions

solve $\min\{\hat{L} = u^T A u + Bu + c\}$

$A > 0$, get $\min_u$ in n-steps $u \in \mathbb{R}^n$

3) Successive approx.

at u^0, approx L by $\hat{L}$, get u^1 from Conj. direc. $u^0 \leftarrow u^1$. Repeat until $u^0 \approx u^1$

Add Equality Constraints: $\min_u L(x,u)$ s.t. $f(x,u)=0$

$\Leftrightarrow$ $\underline{\min H(x,\lambda,u)}$

$H = L + \lambda^T f$

<u>Result</u>: extremal $z^* \triangleq (x^*, \lambda^*, u^*)$ of H

$\left(\frac{\partial H}{\partial x}\Big|_{z^*} = 0, \frac{\partial H}{\partial \lambda}\Big|_{z^*} = 0, \frac{\partial H}{\partial u}\Big|_{z^*} = 0 \right)$ is extremal of L holding $f=0$

$L_x + \lambda^T f_x = 0$

$\lambda^T = -L_x f_x^{-1}$

$\frac{\partial L}{\partial u}\Big|_{\text{s.t. } f=0} = 0$

$\frac{\partial H}{\partial u} = 0 = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u}$

Revision of 1st order gradient:

(a) guess u^i, $i = 0$

(b) compute x^i from $\underline{f(x^i, u^i) = 0} = \frac{\partial H}{\partial \lambda}$

(c) $\underline{\frac{\partial H}{\partial x} = 0} \longrightarrow \lambda^i = \left[-\left(\frac{\partial L}{\partial x}\right) \frac{\partial f}{\partial x}^{-1} \right]^T \Bigg|_{x^i, u^i}$

(d) $\underline{\left(\frac{\partial H}{\partial u}\right)^i} = \left(\frac{\partial L}{\partial u}\right) + \lambda^{iT} \left(\frac{\partial f}{\partial u}\right)$ ($\neq 0$ in general)

(e) $u^{i+1} = u^i - k\left[\left(\frac{\partial H}{\partial u}\right)^i\right] \Rightarrow J^{i+1} \simeq J^i - k \underbrace{\left(\frac{\partial H}{\partial u}^i\right)^T \left(\frac{\partial H}{\partial u}^i\right)}$

(f) repeat (b)–(e) until small

$$\min_u \left\{ \phi(x(t_f),t_f) + \int_{t_0}^{t_f} L(x,u,t)\,dt \right\} \quad \text{s.t.} \quad \underline{\dot{x} = f(x,u)},\ x(t_0) \text{ given}$$

$(x,u)(\cdot)$ functions over $[t_0,t_f]$

Note: $\delta J\Big|_{\substack{\text{varying} \\ \delta u \text{ only}}} \neq \int_{t_0}^{t_f} L_u\,\delta u\,dt$

$$\underline{\delta J = \int_{t_0}^{t_f} (H_u\,\delta u)\,dt}$$ variation of J w.r.t. u holding $\dot{x}=f(x,u)$

$H = L + \lambda^T(t) f$

"Computing the gradient be the adjoint."

Problem Statement

Nonlinear two-point Boundary-Value Problem

Find: (a) $x(t) \in \mathbb{R}^n$
(b) $\lambda(t) \in \mathbb{R}^n$
(c) $u(t) \in \mathbb{R}^m$

$\left.\right\}\ t \in [t_0, t_f]$

that simultaneously solve:

(i) • $\dot{x} = f(x,u),\ x(t_0) \quad \in \mathbb{R}^n$

(ii) • $\dot{\lambda} = -f_x^T \lambda - L_x,\ \lambda(t_f) = \left[\frac{\partial \phi}{\partial x(t_f)}\right]^T$

(iii) $\frac{\partial H}{\partial u} = 0 \in \mathbb{R}^n$

~~(iv)~~