Basics of Feedback Control

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FUNDAMENTALS

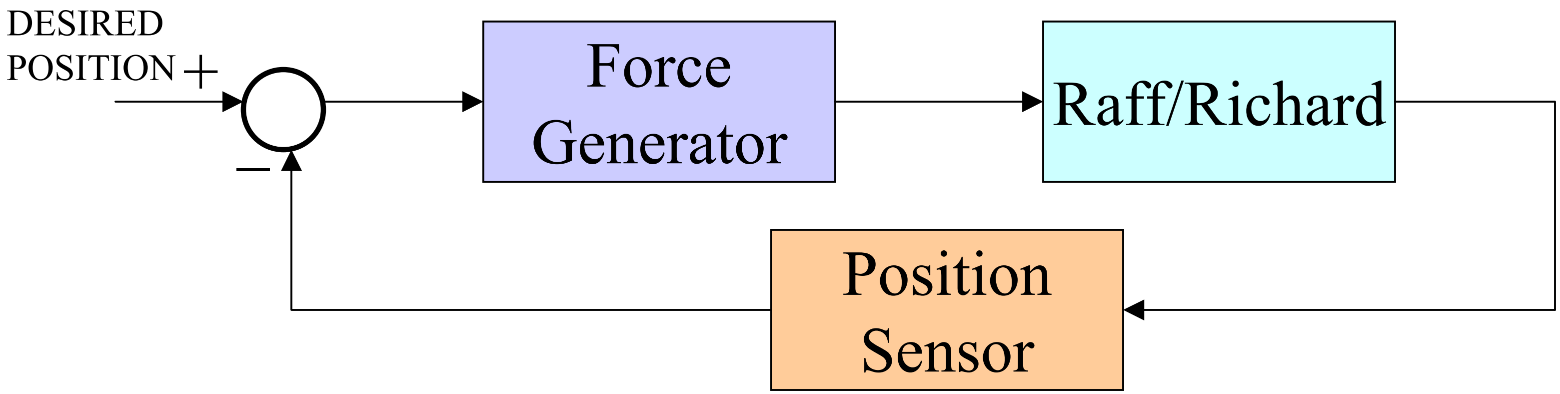
- The importance of understanding dynamics
- Open loop versus closed loop control
- Shifting sensitivity and uncertainty management
- Time scales
- Time delays
- System coupling

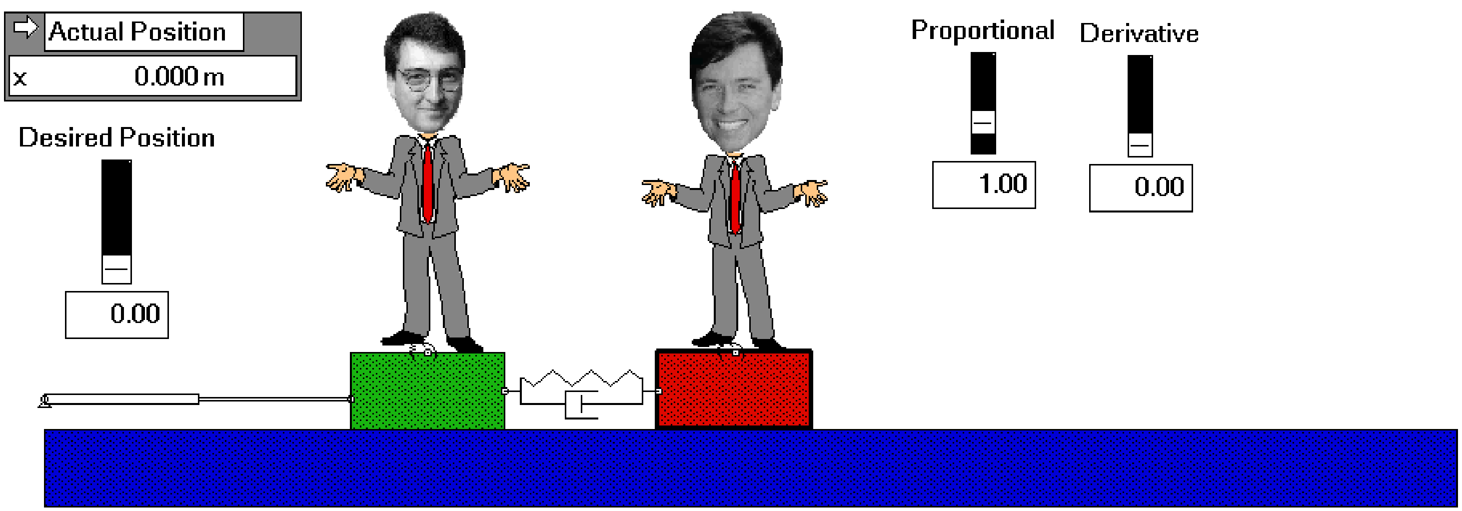
SOME BASIC TOOLS

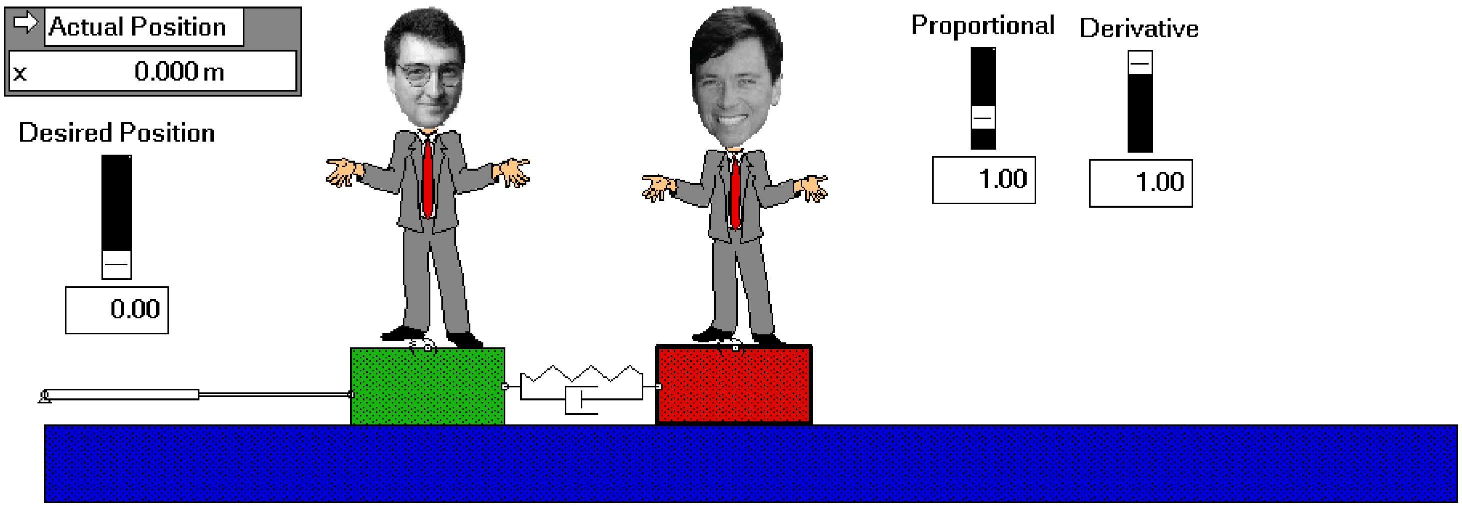
- PID control (proportional, integral, derivative control)
- State feedback and LQR design (linear/quadratic regulator)

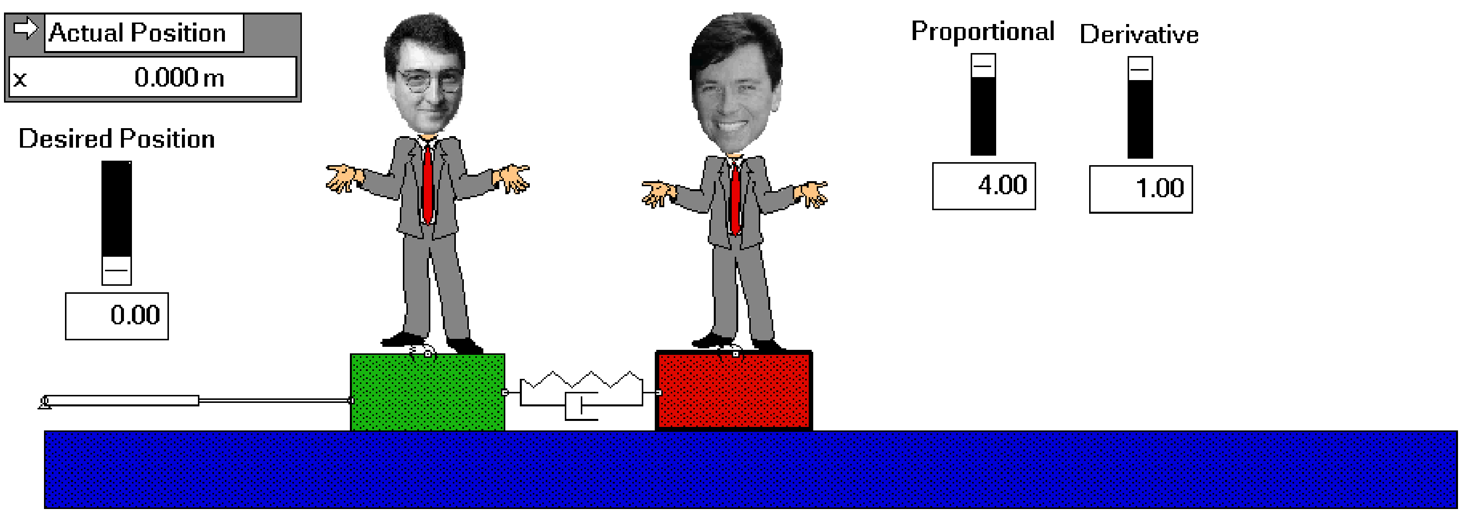
The importance of dynamics...

Isn't feedback control intuitive?






Actual Position
x 0.000 m
Desired Position
0.00
Proportional
Derivative
1.00
1.00


Actual Position
x 0.000 m
Desired Position
0.00
Proportional
Derivative
4.00
1.00

… but seriously, even seemingly simple systems can be difficult to control WITHOUT a basic understanding of the system dynamics.

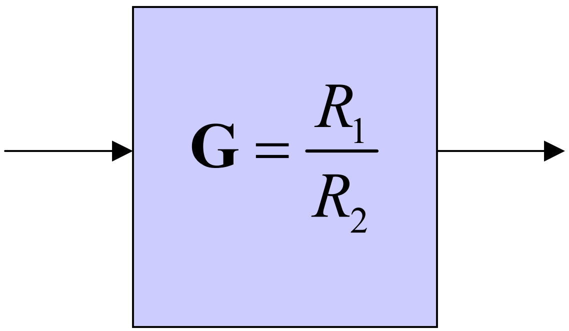
On the flip side, designing a controller for the Raff/Richard system is very easy to do once you have a model AND some basic control tools.

Open loop vs. closed loop control...

A Simple Example (No Dynamics!!!)

Given the task of designing a power amplifier, desired gain of 1, given the following components:

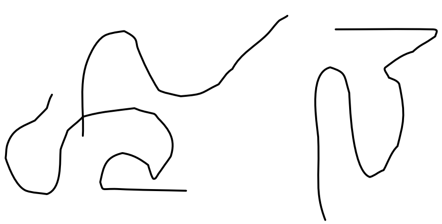
GAIN BLOCK:

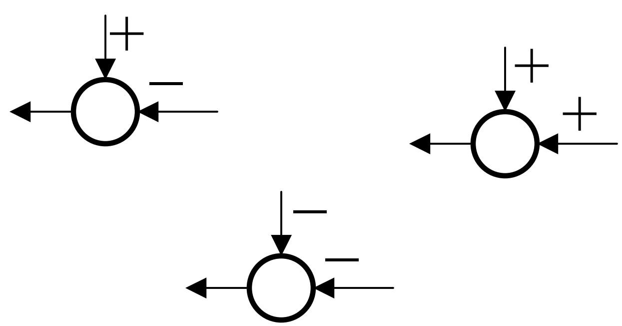


DRAWER FULL OF RESISTORS:

$$\begin{aligned} R &= 10 \pm 1 \\ &= 100 \pm 20 \\ &= 1,000 \pm 300 \\ &= 10,000 \pm 5,000 \end{aligned}$$

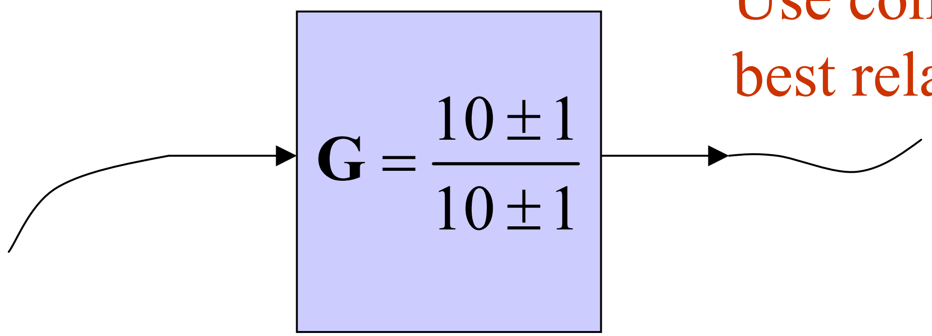
DRAWER FULL OF BASIC INTERCONNECTION COMPONENTS:





Straight-forward approach:

Use components with the best relative tolerances!!!

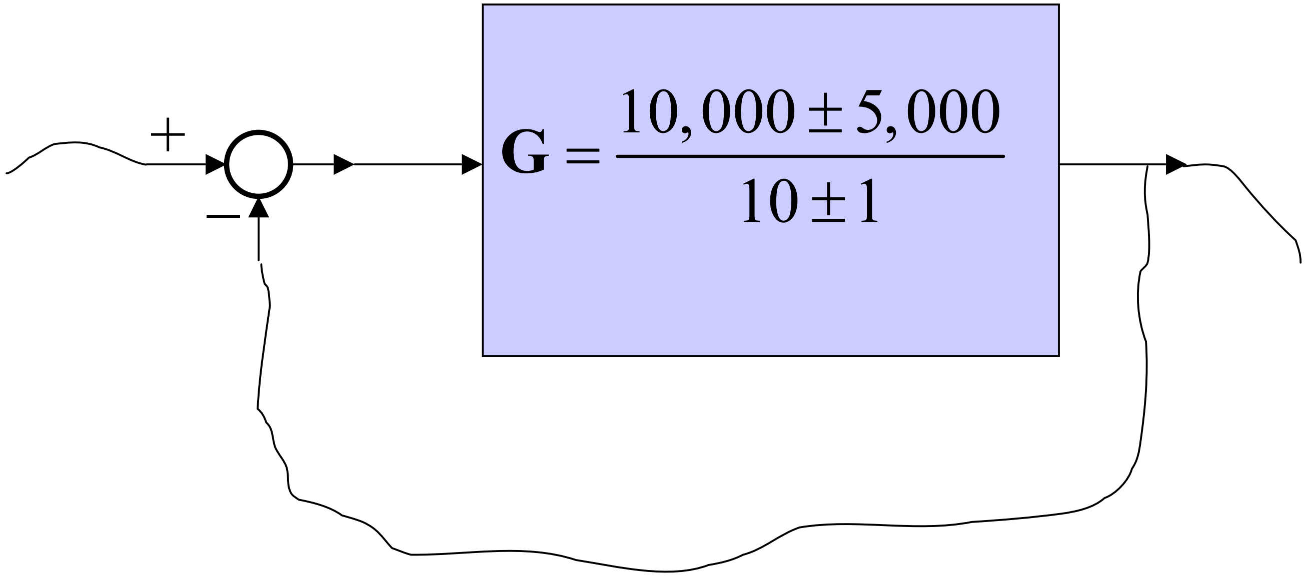


$$G = \frac{10 \pm 1}{10 \pm 1}$$

Amplifier Gain = Input/Output Gain: $0.82 < G < 1.22$

Variation from desired gain: $> 20\ \%$

Design based on feedback:



Amplifier Gain: 455 < G < 1,667

Input/Output Gain: 0.9978 < G/(1+G) < 0.9994

Variation from desired gain: < 0.25 %

A component with 50 % error can yield a design with 0.25% error!!!

… incidentally, there are other benefits of the feedback design. Assume G has the following frequency dependence:

$$G = G_0 \quad \text{for} \quad 0 <= f <= 1,000$$
$$G = \frac{1,000}{f} G_0 \quad \text{for} \quad f > 1,000$$

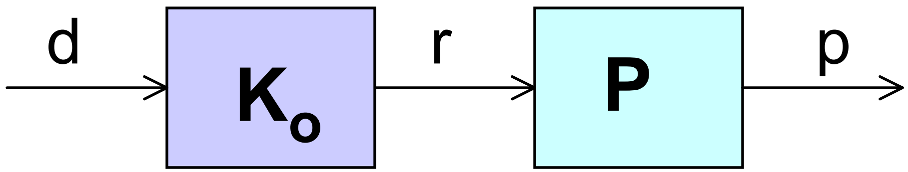
- Without feedback, the gain has dropped off by a factor of 2 when f=2 kHz.
- With feedback, the 3dB frequency will occur when

$$\frac{G}{1+G} = 0.5, G = 1.0, \quad f = 1,000 * 455 = 455\,\text{kHz}$$

Shifting Sensitivity and Uncertainty Management...

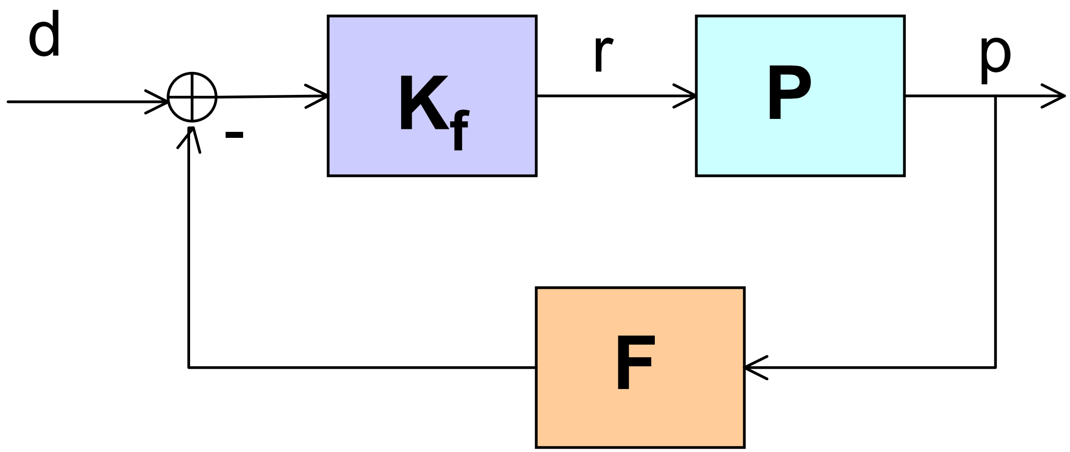
Design a controller so that the input/output gain is close to 1:

OPEN LOOP:



$$T_o = PK_o \approx 1$$

CLOSED LOOP:

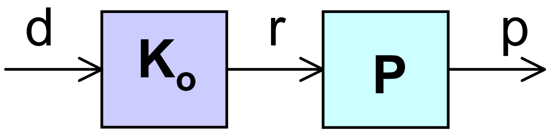


$$T_f = \frac{PK_f}{1 + PK_f F} = T_o$$

$$K_f = \frac{K_o}{1 - PK_o F}$$

Open Loop Sensitivity

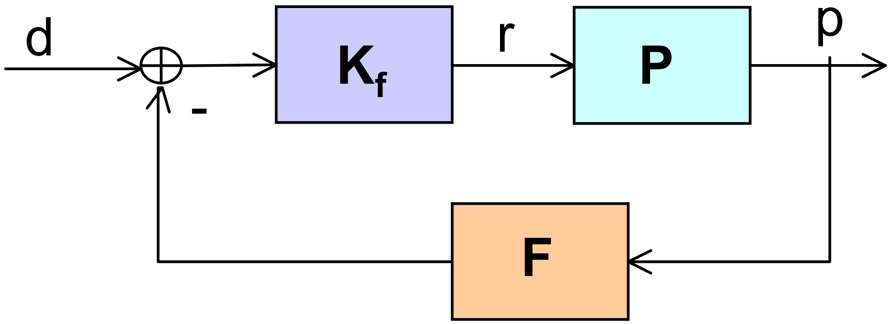
$$\frac{\partial T_o}{T_o} = \frac{\partial P}{P}, \quad \frac{\partial T_o}{T_o} = \frac{\partial K_o}{K_o}$$



Closed Loop Sensitivity

$$\frac{\partial T_f}{T_f} = (1 - F)\frac{\partial P}{P}, \quad \frac{\partial T_f}{T_f} = (1 - F)\frac{\partial K_f}{K_f}$$

$$\frac{\partial T_f}{T_f} = F\frac{\partial F}{F},$$

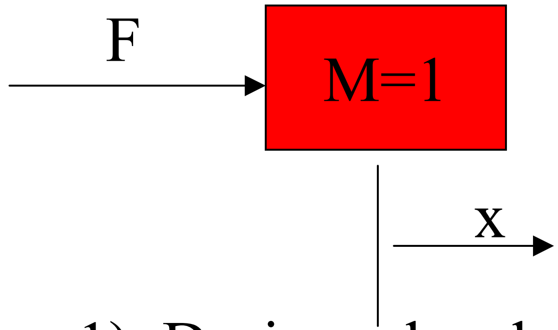


- Sensitivity can be shifted: move to less costly, easier to design components.

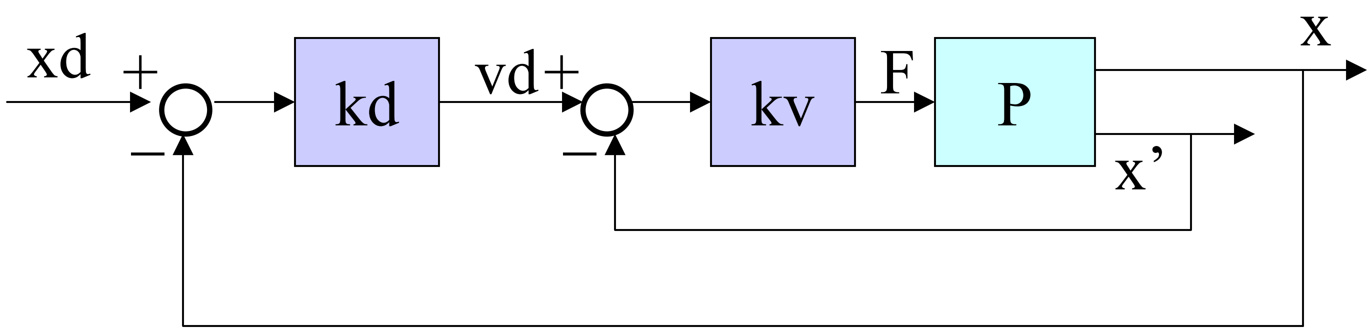
- There is no free-lunch: sensitivity is in some sense preserved.

Time Scales...

(simplified version of what is used for RoboCup)



Objective:
1) Design a local controller that tracks velocity
2) Design a global controller that tracks position



1) $$F = \ddot{x} = k_v(v_d - \dot{x}), \quad \text{speed of response} = k_v$$

2) $$v_d = k_d(x_d - x), \quad \text{"speed of response"} = k_d$$

Actual dynamics:

$$\ddot{x} = k_v(k_d(x_d - x) - \dot{x}),$$
$$\ddot{x} + k_v\dot{x} + k_v k_d x = k_v k_d x_d$$

Actual time constants and decay rates:

$$\frac{k_v}{2}\left(-1 \pm \sqrt{1 - 4\frac{k_d}{k_v}}\right)$$

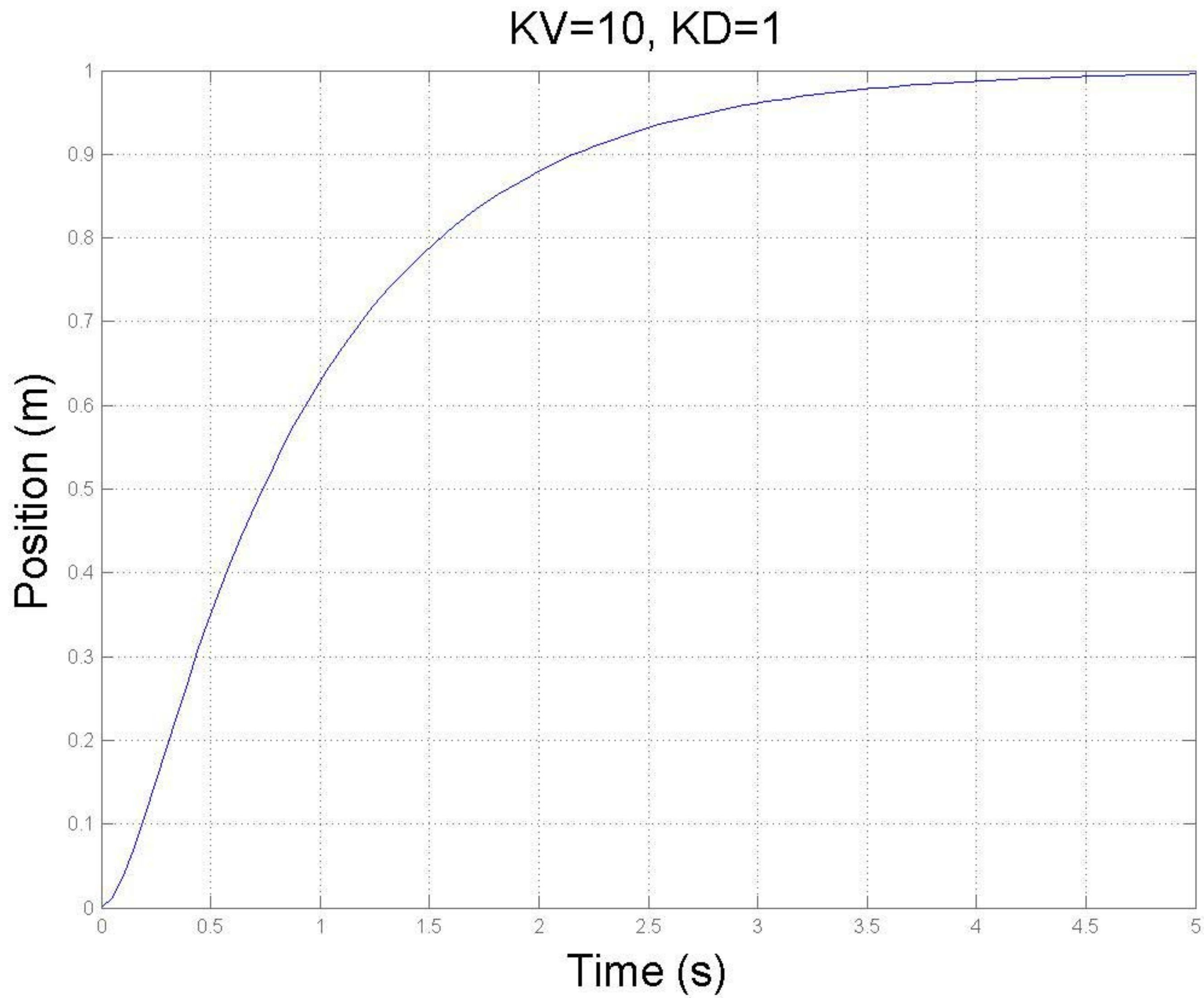
CASE 1, kd<<kv:

$$\approx \frac{k_v}{2}\left(-1 \pm (1 - 2\frac{k_d}{k_v})\right) = -k_v, -k_d$$

CASE 2, kd > kv:

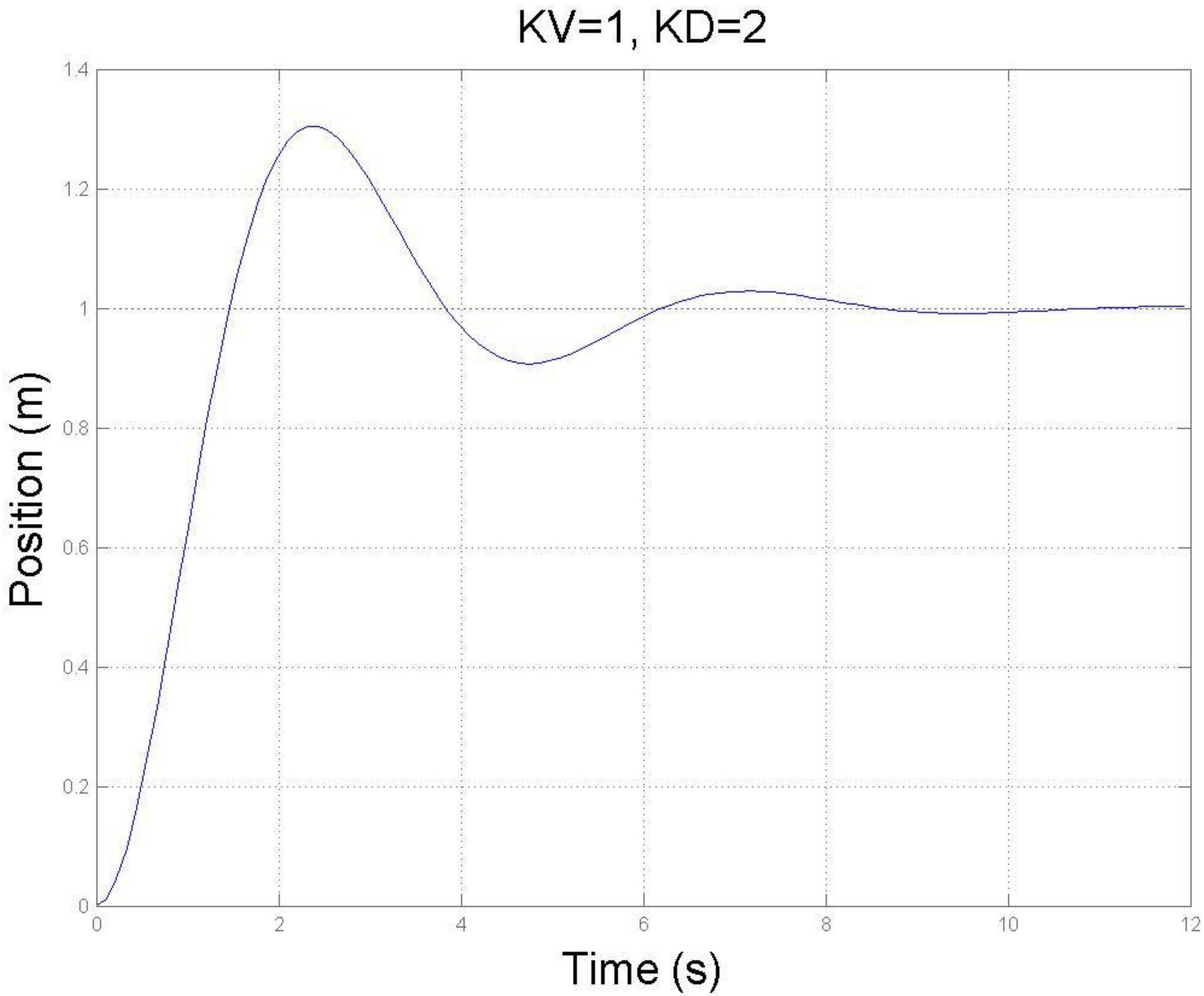
$$\approx \frac{k_v}{2}\left(-1 \pm 2j\sqrt{\frac{k_d}{k_v}}\right)$$

CASE 1:



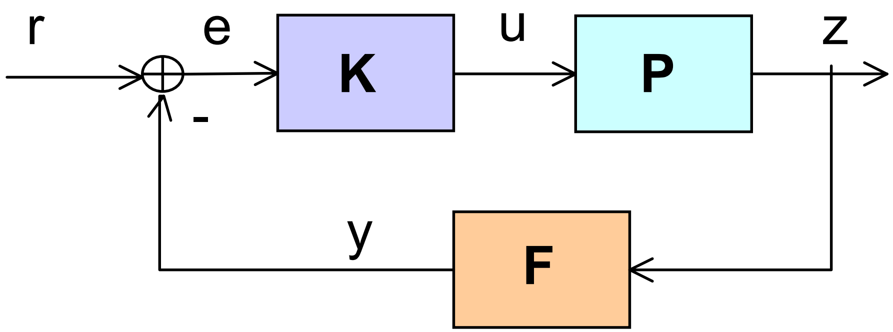
KV=10, KD=1
Position (m)
Time (s)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

CASE 2:



Must keep time-scales in mind when designing control systems for complex systems.

Time Delays...



Given:

$$P: \quad \dot{z}(t) = u(t)$$

$$F: \quad y(t) = z(t-0.1) \quad \text{(delay of 0.1 seconds)}$$

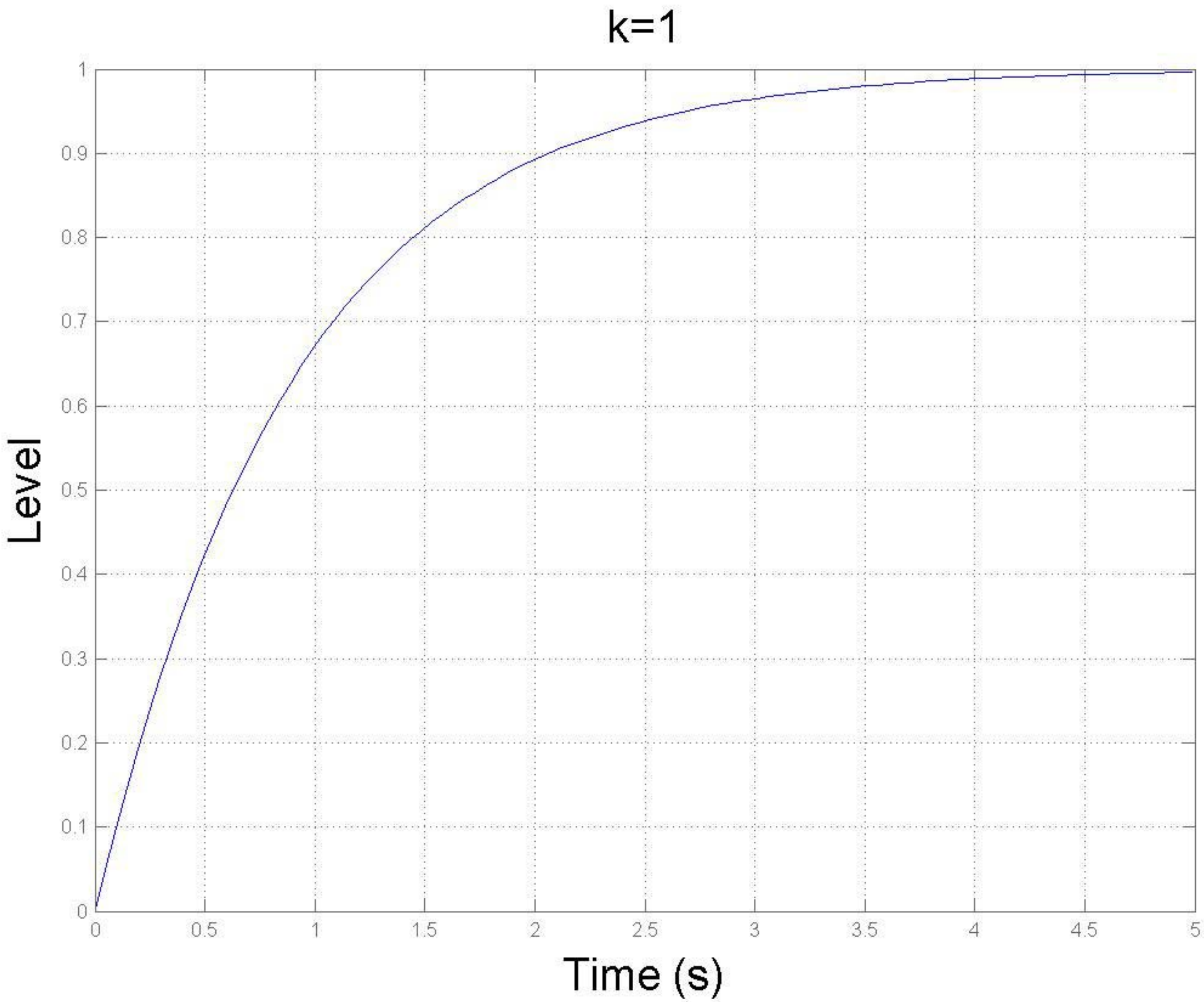
Controller: K=constant.

OBJECTIVE: Make z track r

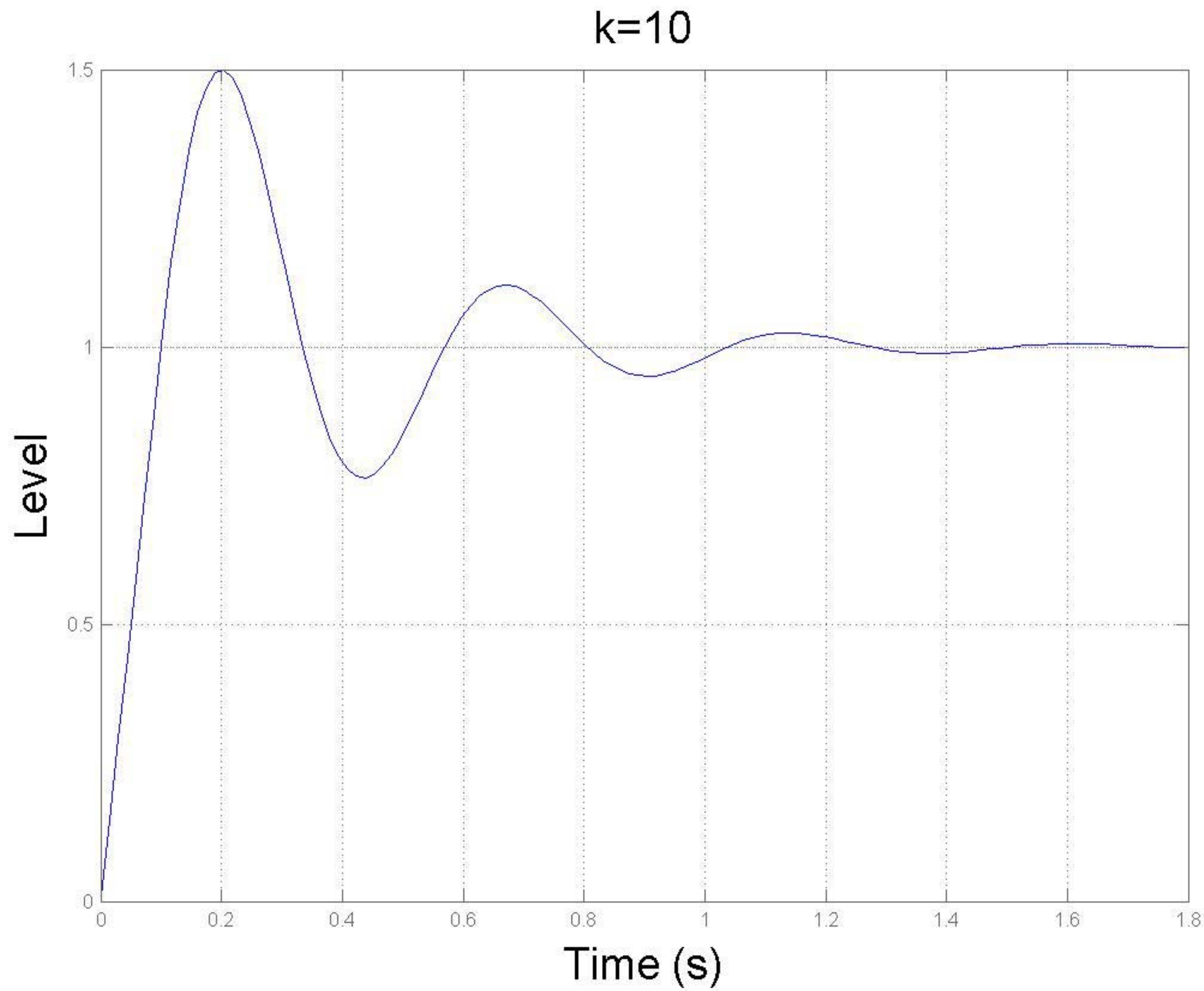
Without delay: $\dot{z}(t) + kz(t) = kr(t), \text{ response speed} = \text{k}$

With delay: $\dot{z}(t) + kz(t-0.1) = kr(t)$

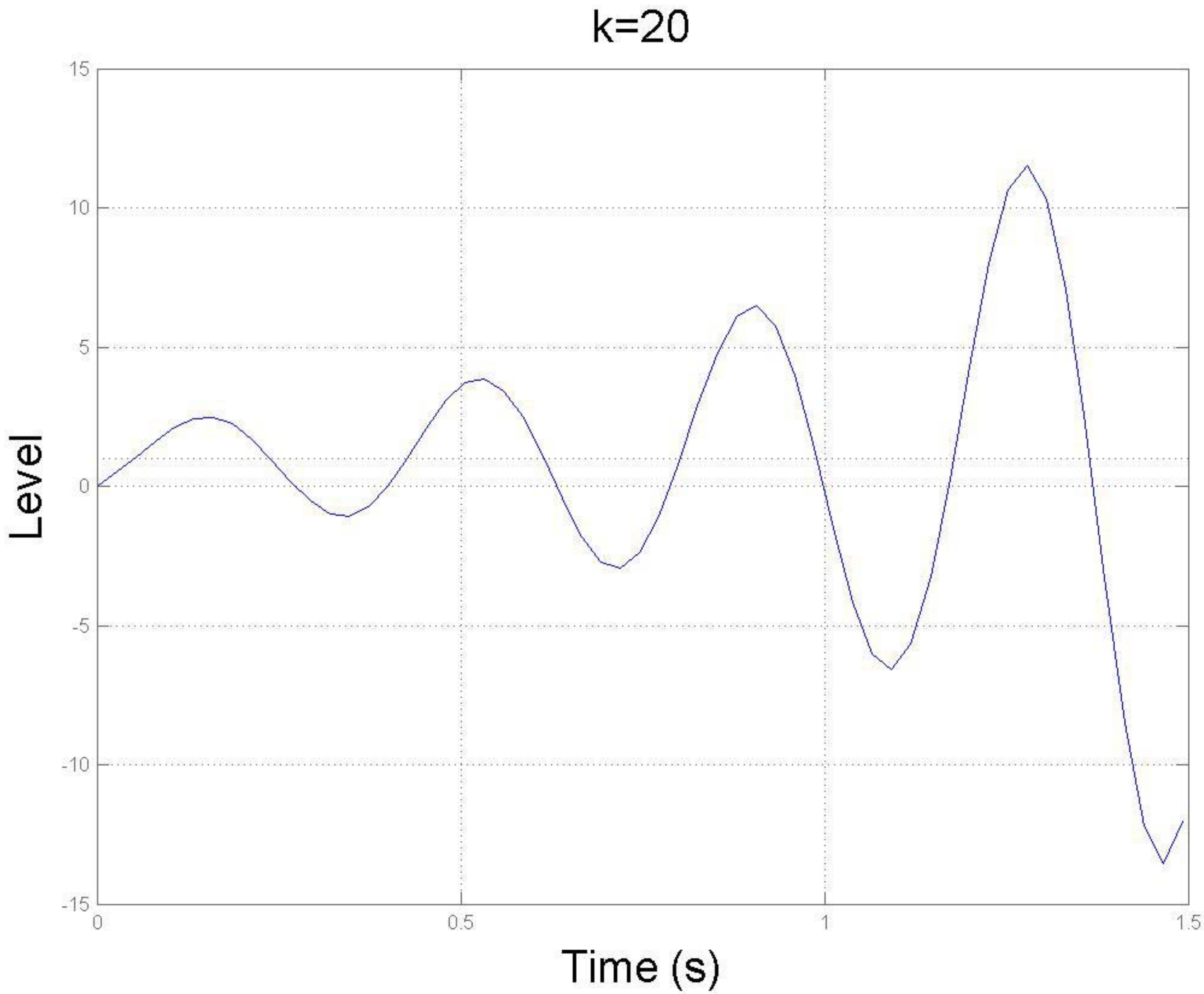
CASE 1 (k=1):



CASE 2 (k=10):



CASE 3 (k=20):



Delayed information has the effect of limiting how quickly we can control a system.

System Coupling...

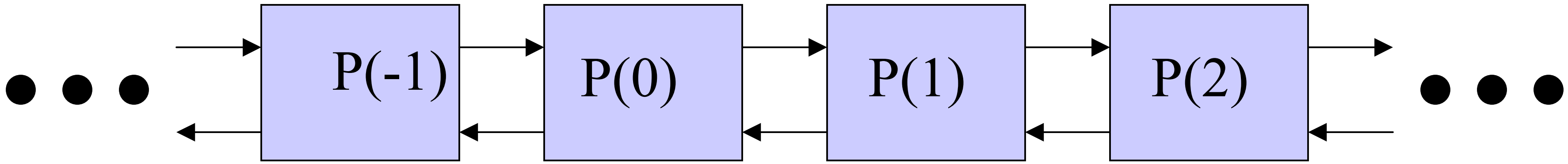
CONSIDER THE FOLLOWING COUPLED EQUATIONS:

$$\dot{x}_1(t,s) = 0.9x_1(t,s) - 0.5x_2(t,s) + w_1(t,s)$$
$$\dot{x}_2(t,s) = 0.5x_1(t,s) - x_2(t,s) + w_2(t,s)$$

$$w_1(t,s) = 0.75x_2(t,s+1) + 0.5w_1(t,s+1)$$
$$w_2(t,s) = -0.75x_1(t,s-1) + 0.5w_2(t,s+1)$$

t=time (continuous), s=space (discrete)

IMPLEMENTATION:



Decoupled:

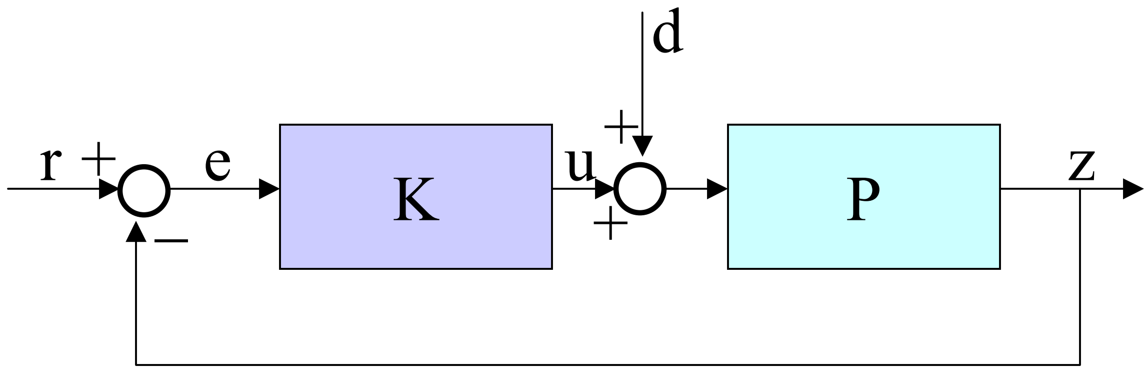
$$\dot{x}_1(t,s) = 0.9x_1(t,s) - 0.5x_2(t,s)$$
$$\dot{x}_2(t,s) = 0.5x_1(t,s) - x_2(t,s)$$

Eigs=0.76, -0.86

…turns out that if you have at least 10 of the systems connected, the overall system will be stable.

SOME BASIC TOOLS

PID Control...



$$u(t) = k_P e(t) + k_I \int_0^t e(\tau)\,d\tau + k_D \dot{e}(t)$$

ROUGHLY:

SPEED

STEADY STATE PERFORMANCE

STABILITY

k_P: the larger the error, the larger the control effort.

k_I: if system is stable, e(t) must go to zero for constant d(t) and u(t).

k_D: apply more control effort if error is getting larger.

These interpretations are only rules of thumb: in general, the effects of the gains are dictated by the plant dynamics.

LQR Control...

BACKGROUND

Many systems can be captured by sets of ordinary differential equations:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

- x(t): State of the system, an n-valued vector $(x(t)=(x_1(t),\ldots x_n(t))$
- u(t): The input to the system, an m-valued vector
- y(t): The output of the system, a p-valued vector

If we want to control the system about an operating point (x_E,u_E), and we can measure all the states, we can linearize about (x_E,u_E) to obtain

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = x(t)$$

CONTROL PROBLEM:

Given system dynamics $\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t), \quad x(0) = x_0$

Find control input u(t) which minimizes

$$J(u) = \int_0^\infty \left(x^T(t)\mathbf{Q}x(t) + u^T(t)\mathbf{R}u(t) \right) dt$$

where $Q=Q^T$ and $R=R^T$ have strictly positive eigenvalues,

- we are penalizing the state AND the control effort.
- x(t) and u(t) must eventually go to zero for cost to be finite.
- expect u to be a function of x(0)...

Scalar Case:

$$\dot{x}(t) = \mathbf{a}x(t) + \mathbf{b}u(t), \quad x(0) = x_0,$$

$$J(u) = \int_0^\infty \left(\mathbf{q}x^2(t) + \mathbf{r}u^2(t) \right) dt$$

Look for solutions of the form u(t)=kx(t):

$$x(t) = \exp((\mathbf{a} + \mathbf{bk})t)x_0,$$

$$J = -\frac{x_0^2}{2}\left(\frac{\mathbf{q} + \mathbf{r}k^2}{\mathbf{a} + \mathbf{b}k} \right) \text{ (assuming } \mathbf{a} + \mathbf{b}k < 0)$$

Minimize J(k):

$$-2k\frac{\mathbf{ar}}{\mathbf{b}} - k^2\mathbf{r} + \mathbf{q} = 0$$

Substitute k=-(b/r)s:

$$2\mathbf{a}s - \frac{\mathbf{b}s^2}{\mathbf{r}} + \mathbf{q} = 0$$

General Case:

Algebraic Riccati Equation

$$S\mathbf{A} + \mathbf{A}^T S - S\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T S + \mathbf{Q} = 0$$

NOTE:

- We restricted our search to u(t)=Kx(t). No obvious reason why this should be the optimal u(t). In fact, we can prove that it is!!!
- Unlike most optimal control strategies, the LQR solution is a feedback solution.