

Basics of Feedback Control

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FUNDAMENTALS

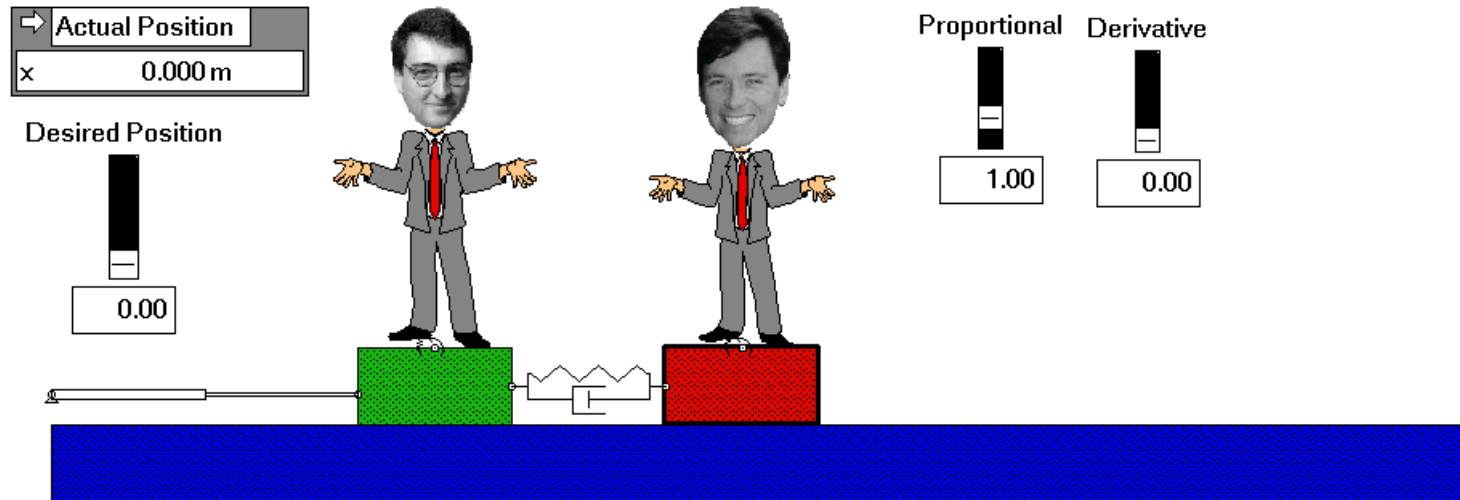
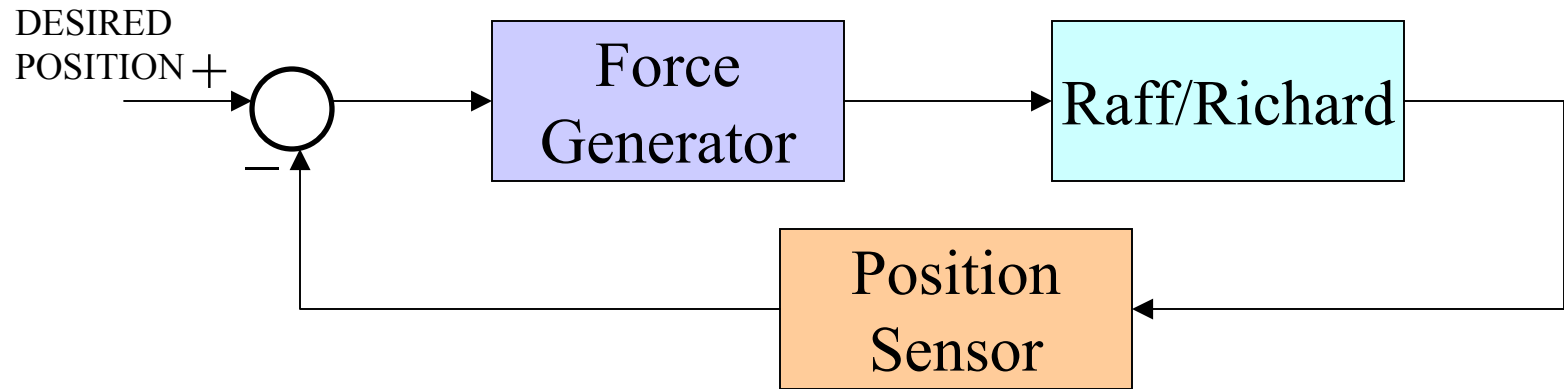
- The importance of understanding dynamics
- Open loop versus closed loop control
- Shifting sensitivity and uncertainty management
- Time scales
- Time delays
- System coupling

SOME BASIC TOOLS

- PID control (proportional, integral, derivative control)
- State feedback and LQR design (linear/quadratic regulator)

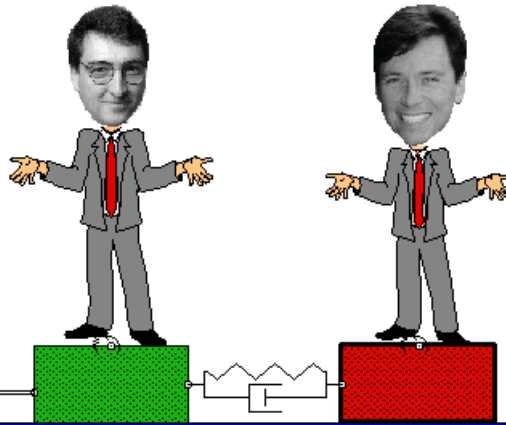
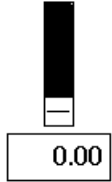
The importance of dynamics...

Isn't feedback control intuitive?

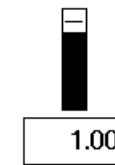


Actual Position
x 0.000 m

Desired Position

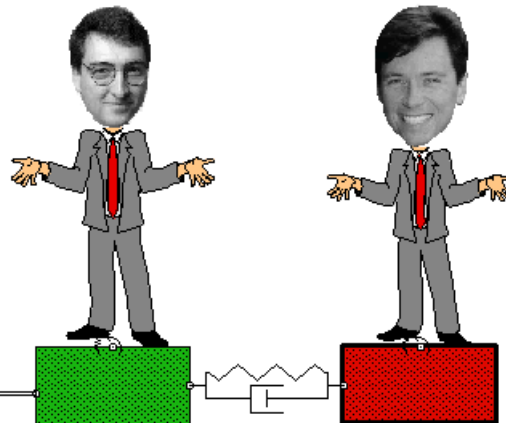
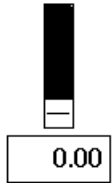


Proportional Derivative

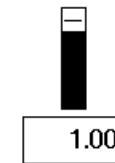


Actual Position
x 0.000 m

Desired Position



Proportional Derivative



... but seriously, even seemingly simple systems can be difficult to control **WITHOUT** a basic understanding of the system dynamics.

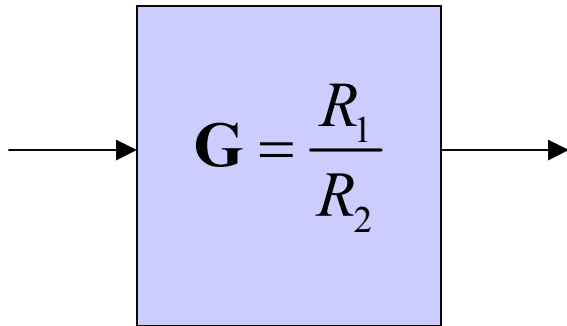
On the flip side, designing a controller for the Raff/Richard system is very easy to do once you have a model **AND** some basic control tools.

Open loop vs. closed loop control...

A Simple Example (No Dynamics!!!)

Given the task of designing a power amplifier, desired gain of 1, given the following components:

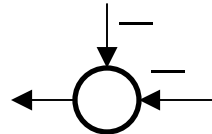
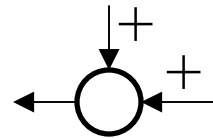
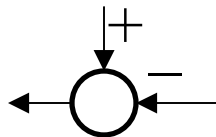
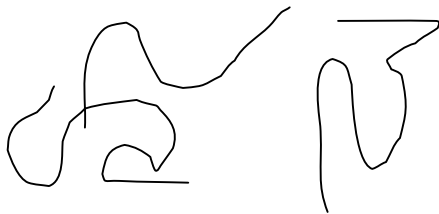
GAIN BLOCK:



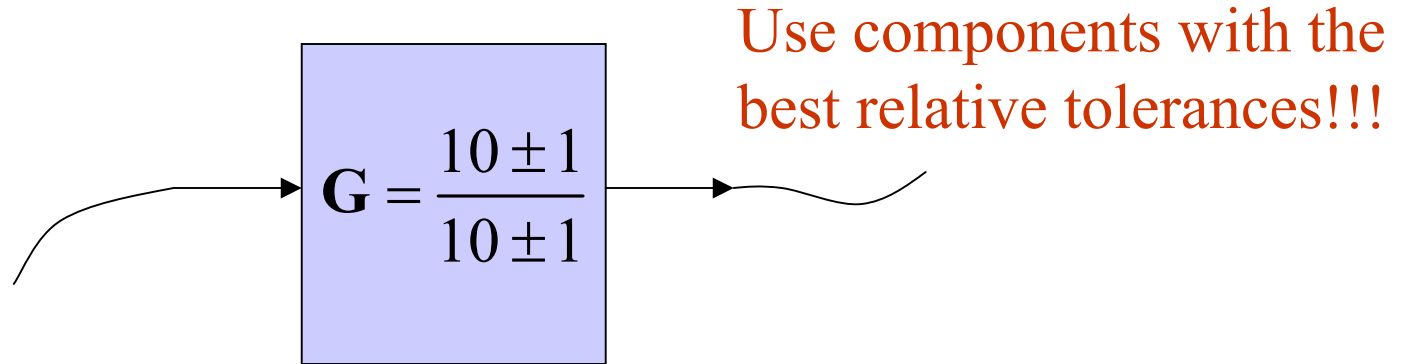
DRAWER FULL OF RESISTORS:

$$\begin{aligned} R &= 10 \pm 1 \\ &= 100 \pm 20 \\ &= 1,000 \pm 300 \\ &= 10,000 \pm 5,000 \end{aligned}$$

DRAWER FULL OF BASIC INTERCONNECTION COMPONENTS:



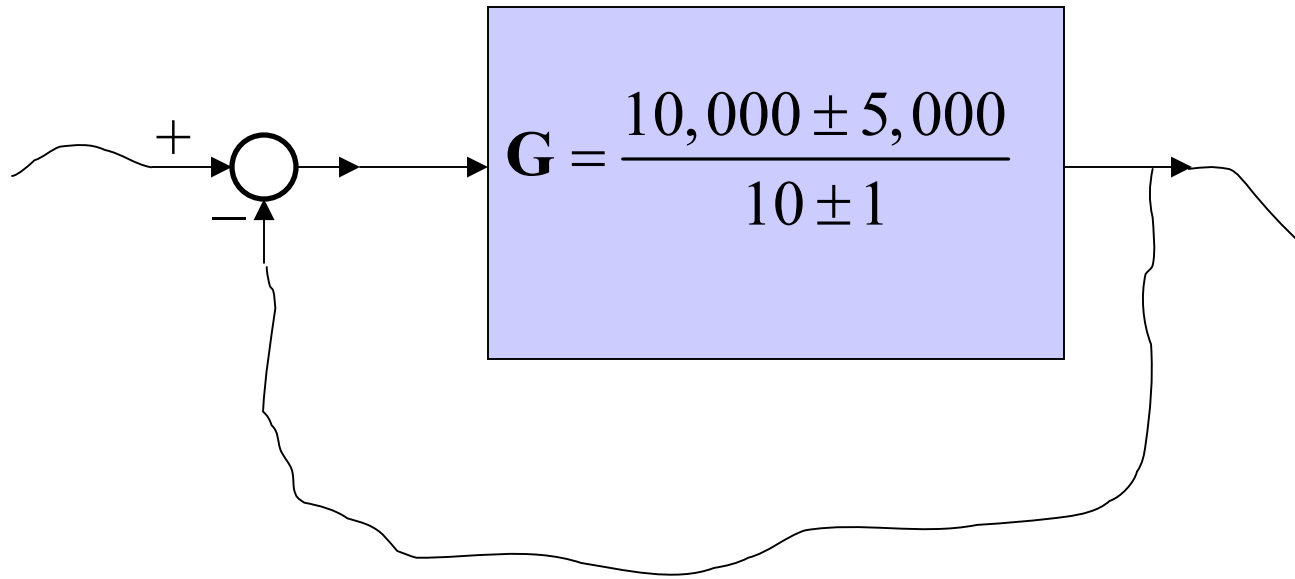
Straight-forward approach:



Amplifier Gain = Input/Output Gain: $0.82 < G < 1.22$

Variation from desired gain: $> 20 \%$

Design based on feedback:



Amplifier Gain: $455 < G < 1,667$

Input/Output Gain: $0.9978 < G/(1+G) < 0.9994$

Variation from desired gain: $< 0.25 \%$

A component with 50 % error can yield a design with 0.25% error!!!

... incidentally, there are other benefits of the feedback design. Assume G has the following frequency dependence:

$$G = G_0 \quad \text{for } 0 \leq f \leq 1,000$$
$$G = \frac{1,000}{f} G_0 \quad \text{for } f > 1,000$$

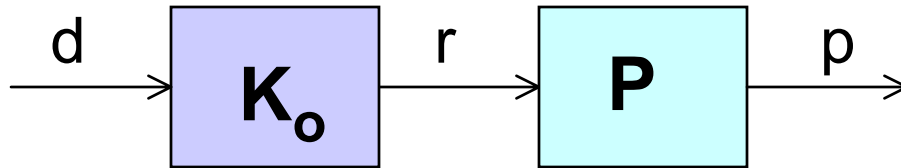
- Without feedback, the gain has dropped off by a factor of 2 when $f=2$ kHz.
- With feedback, the 3dB frequency will occur when

$$\frac{G}{1+G} = 0.5, G = 1.0, \quad f = 1,000 * 455 = 455 \text{ kHz}$$

Shifting Sensitivity and Uncertainty Management...

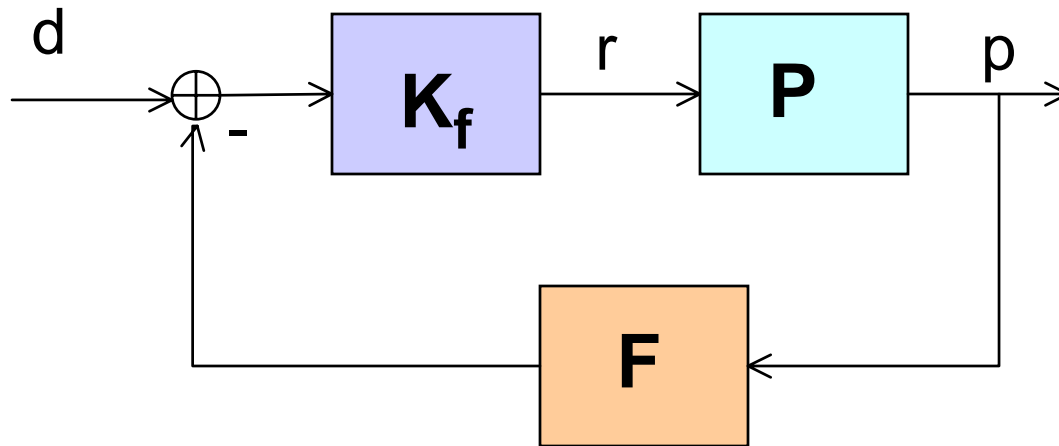
Design a controller so that the input/output gain is close to 1:

OPEN LOOP:



$$\mathbf{T_o = PK_o \approx 1}$$

CLOSED LOOP:

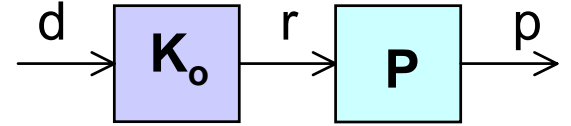


$$\mathbf{T_f = \frac{PK_f}{1 + PK_f F} = T_o}$$

$$\mathbf{K_f = \frac{K_o}{1 - PK_o F}}$$

Open Loop Sensitivity

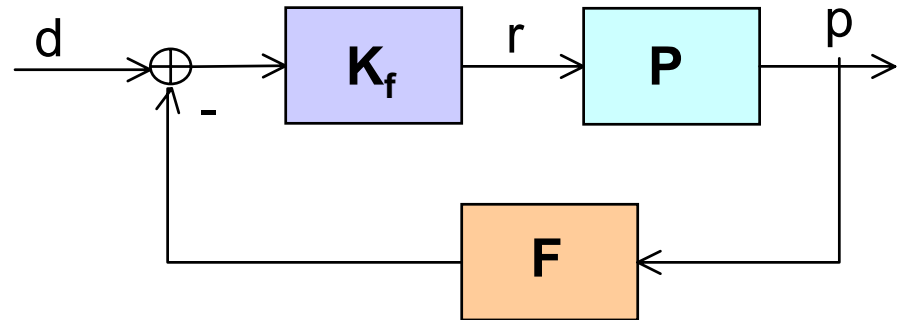
$$\frac{\partial T_o}{T_o} = \frac{\partial P}{P}, \quad \frac{\partial T_o}{T_o} = \frac{\partial K_o}{K_o}$$



Closed Loop Sensitivity

$$\frac{\partial T_f}{T_f} = (1 - F) \frac{\partial P}{P}, \quad \frac{\partial T_f}{T_f} = (1 - F) \frac{\partial K_f}{K_f}$$

$$\frac{\partial T_f}{T_f} = F \frac{\partial F}{F},$$

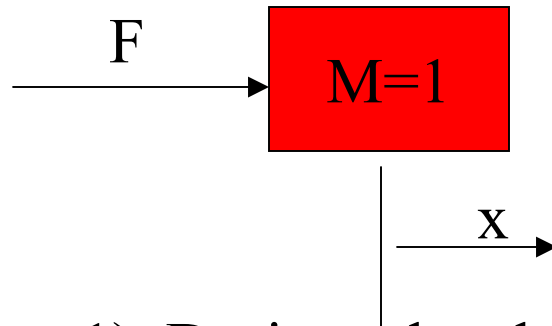


- Sensitivity can be shifted: move to less costly, easier to design components.

- There is no free-lunch: sensitivity is in some sense preserved.

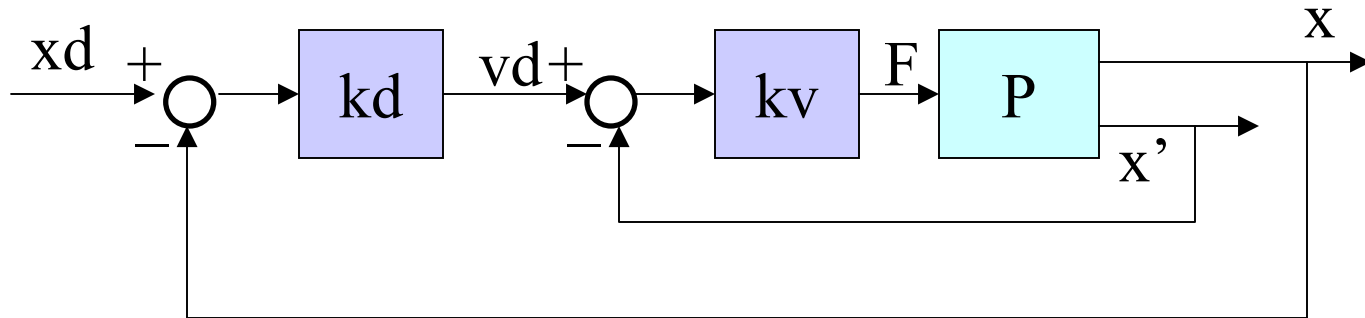
Time Scales...

(simplified version of what is used for RoboCup)



Objective:

- 1) Design a local controller that tracks velocity
- 2) Design a global controller that tracks position



1) $F = \ddot{x} = k_v (v_d - \dot{x}), \quad \text{speed of response} = k_v$

2) $v_d = k_d (x_d - x), \quad \text{"speed of response"} = k_d$

Actual dynamics:

$$\ddot{x} = k_v(k_d(x_d - x) - \dot{x}),$$
$$\ddot{x} + k_v\dot{x} + k_vk_dx = k_vk_dx_d$$

Actual time constants and decay rates:

$$\frac{k_v}{2} \left(-1 \pm \sqrt{1 - 4 \frac{k_d}{k_v}} \right)$$

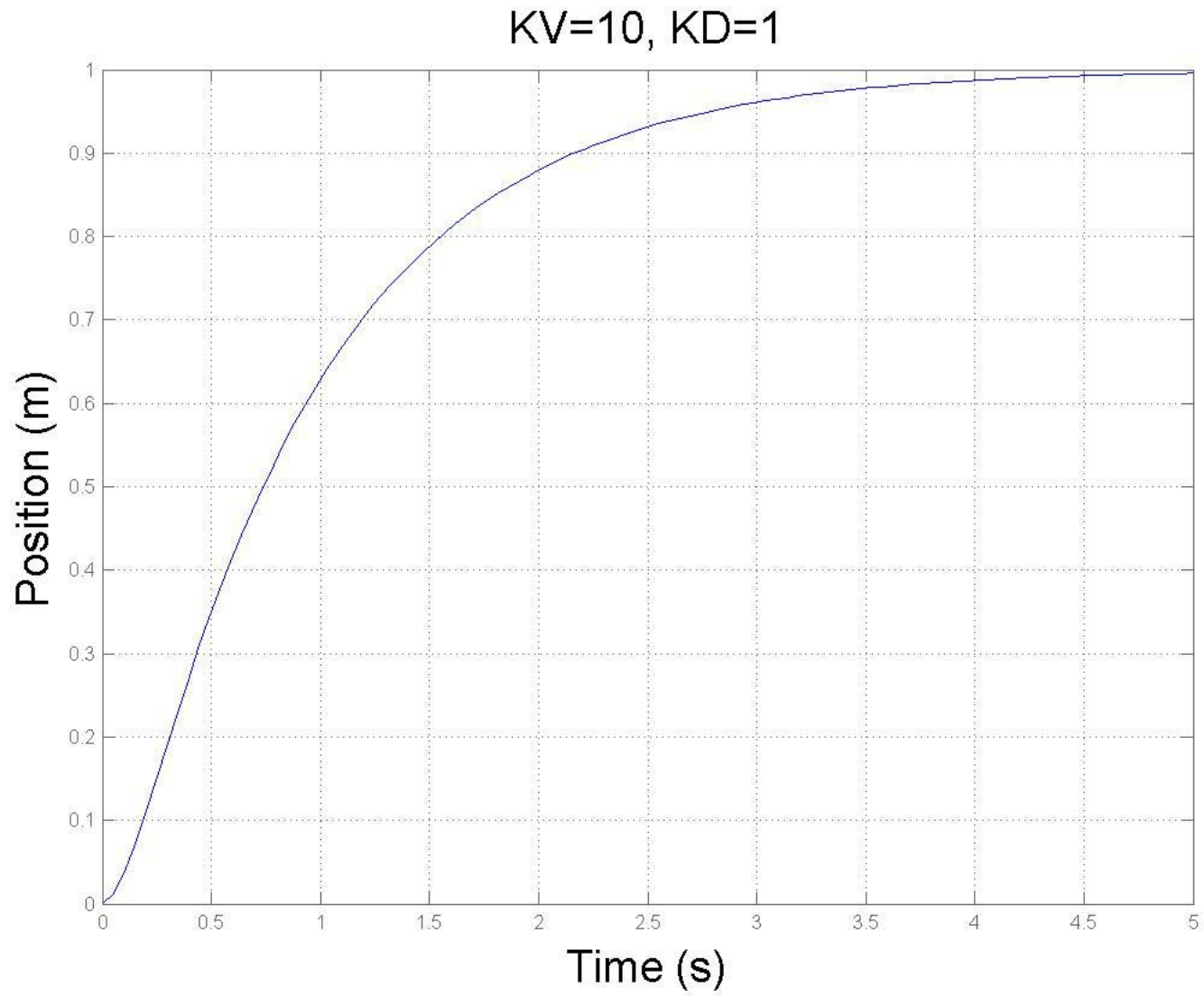
CASE 1, $k_d \ll k_v$:

$$\approx \frac{k_v}{2} \left(-1 \pm \left(1 - 2 \frac{k_d}{k_v}\right) \right) = -k_v, -k_d$$

CASE 2, $k_d > k_v$:

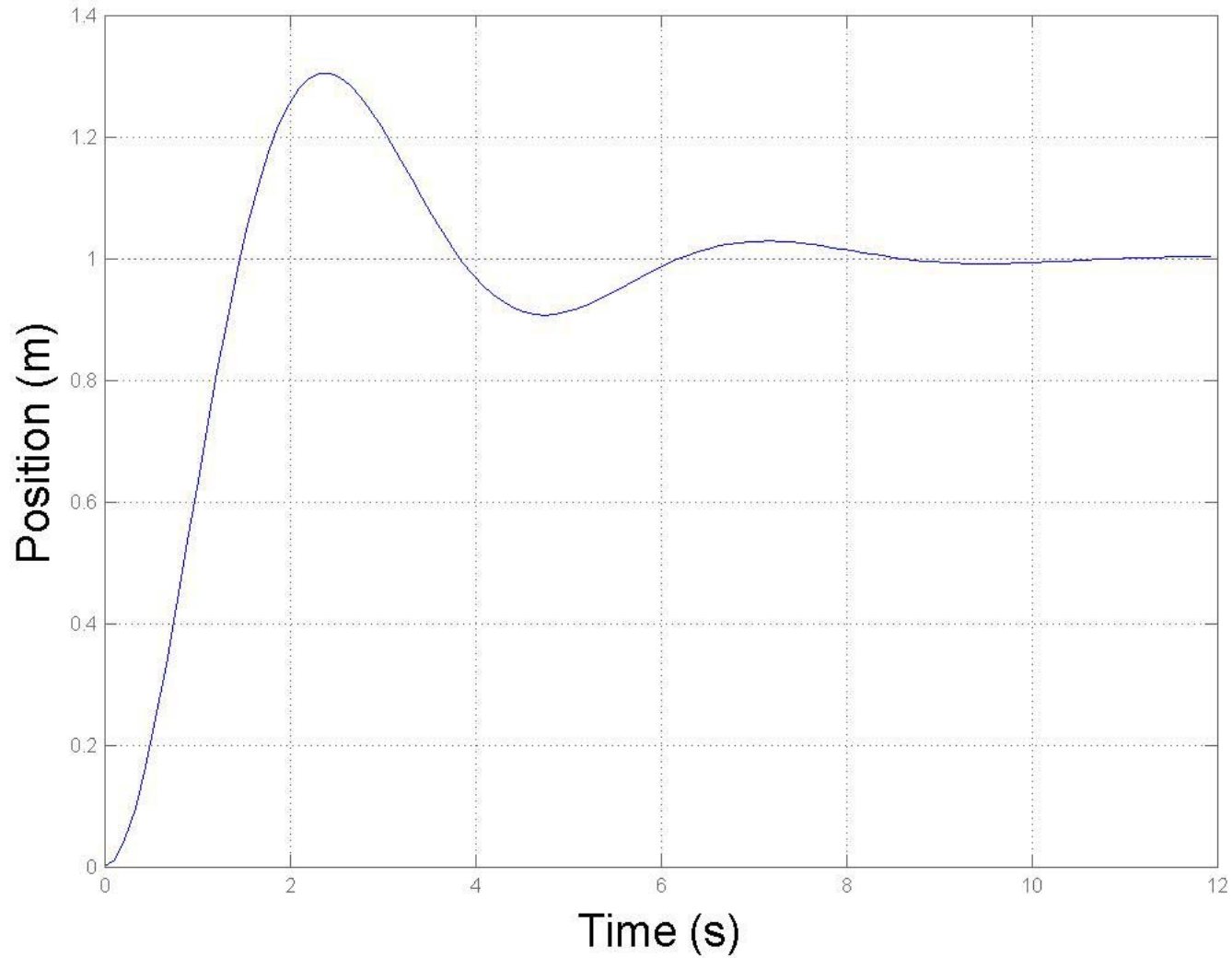
$$\approx \frac{k_v}{2} \left(-1 \pm 2j \sqrt{\frac{k_d}{k_v}} \right)$$

CASE 1:



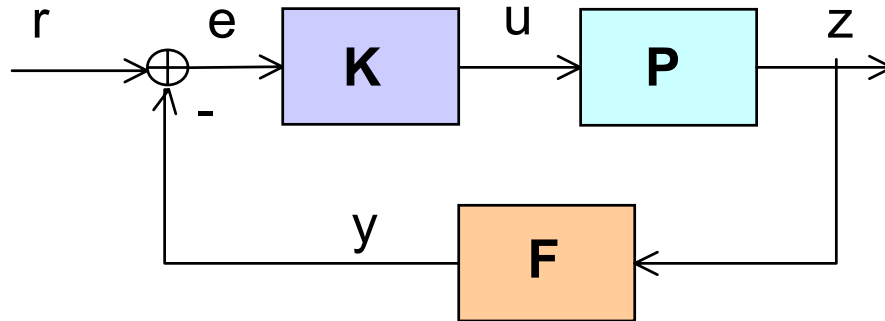
CASE 2:

KV=1, KD=2



Must keep time-scales in mind when designing control systems for complex systems.

Time Delays...



Given:

$$P : \dot{z}(t) = u(t)$$

$$F : y(t) = z(t - 0.1) \quad (\text{delay of 0.1 seconds})$$

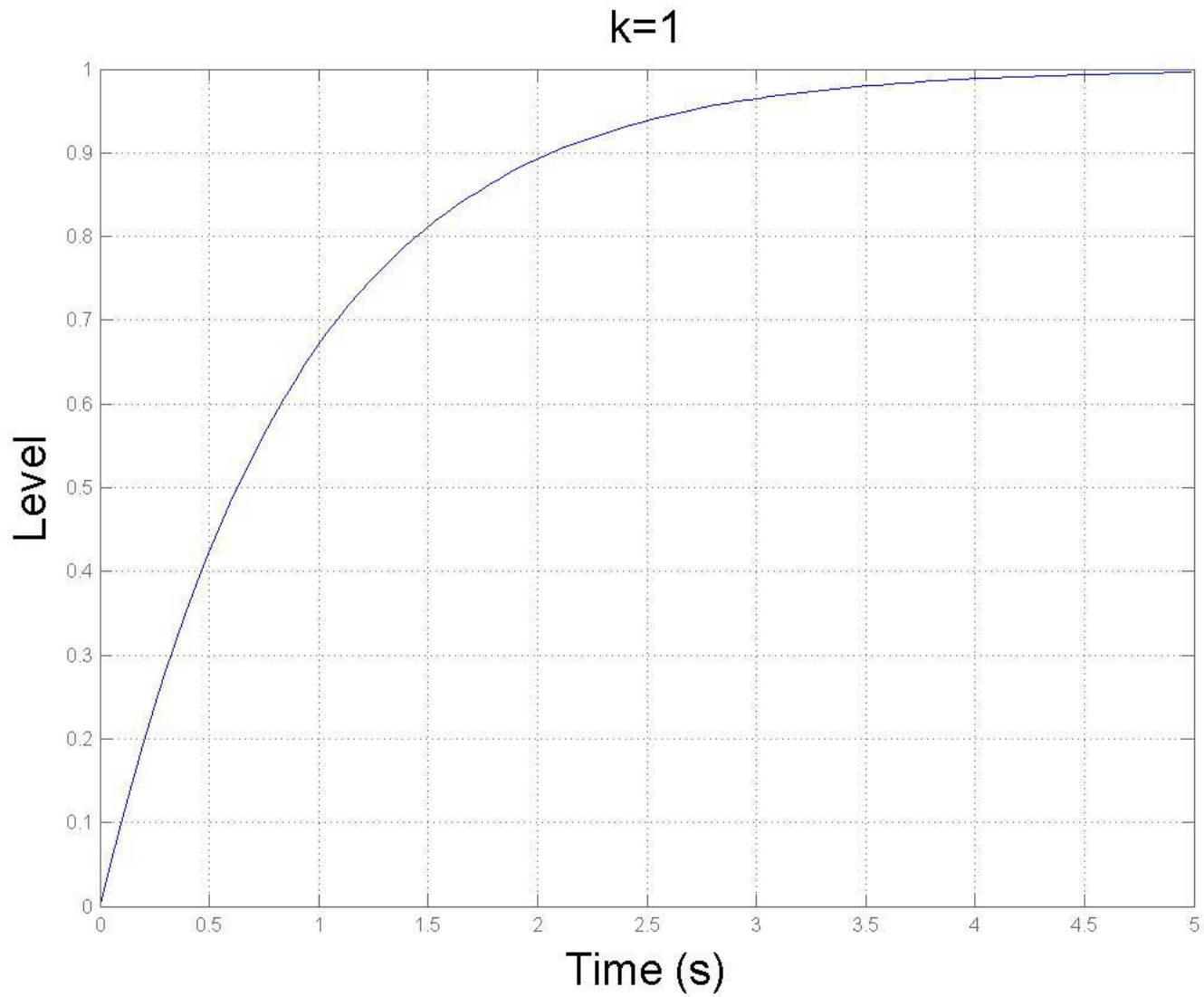
Controller: $K = \text{constant}$.

OBJECTIVE: Make z track r

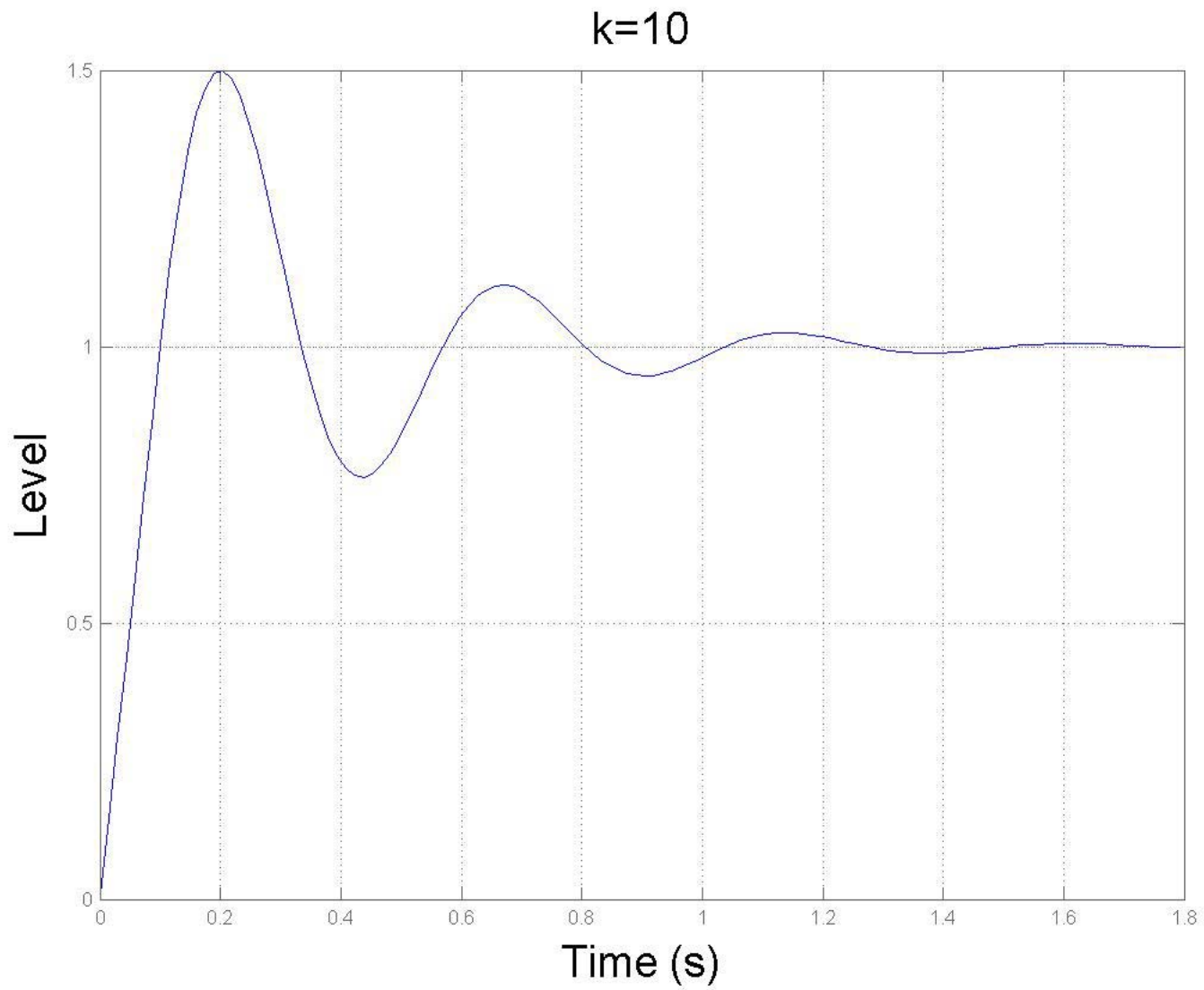
Without delay: $\dot{z}(t) + kz(t) = kr(t)$, response speed = k

With delay: $\dot{z}(t) + kz(t - 0.1) = kr(t)$

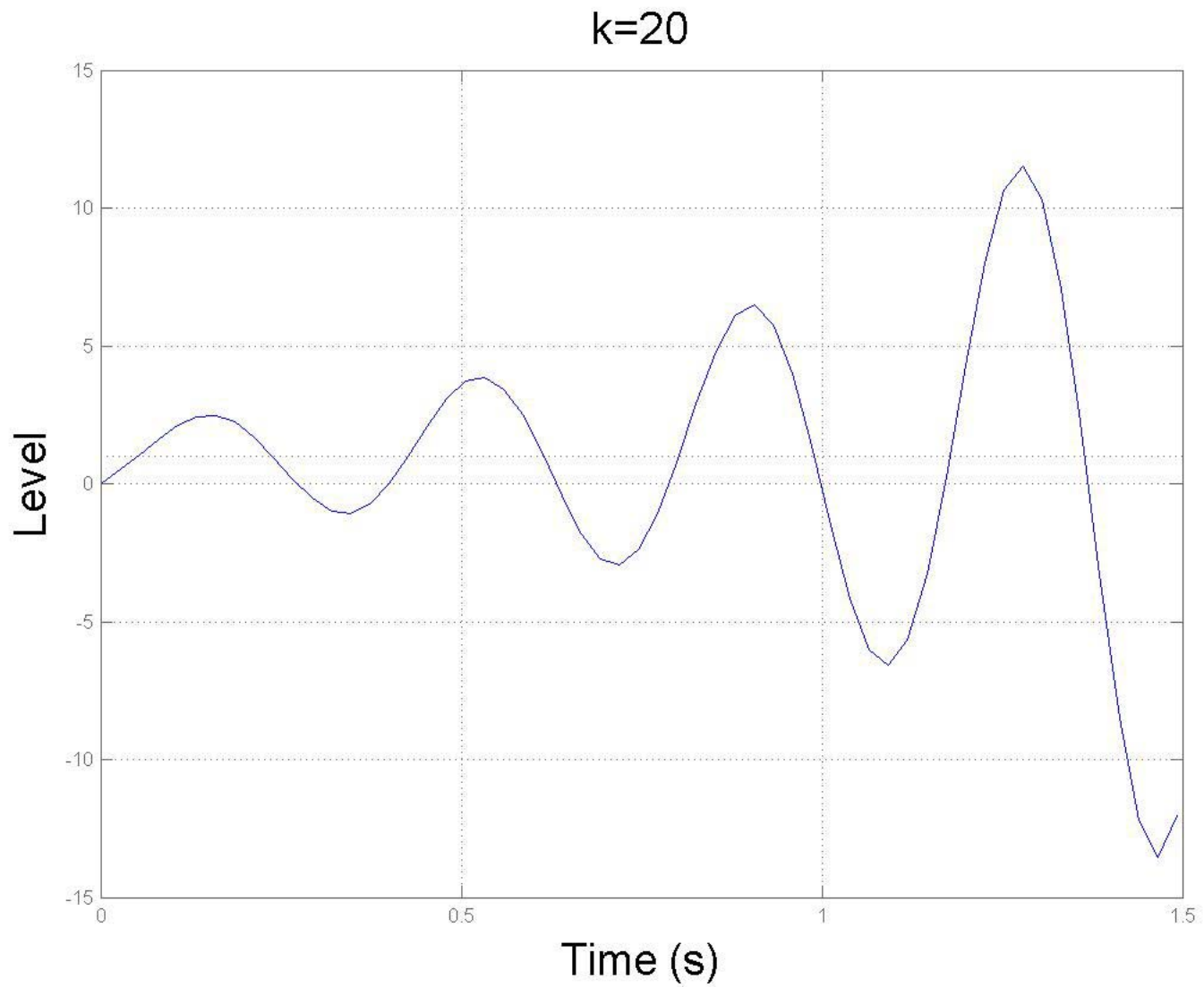
CASE 1 (k=1):



CASE 2 (k=10):



CASE 3 (k=20):



Delayed information has the effect of limiting how quickly we can control a system.

System Coupling...

CONSIDER THE FOLLOWING COUPLED EQUATIONS:

$$\dot{x}_1(t, s) = 0.9x_1(t, s) - 0.5x_2(t, s) + w_1(t, s)$$

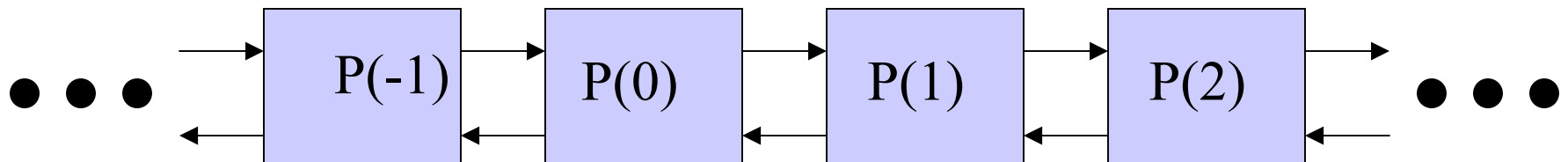
$$\dot{x}_2(t, s) = 0.5x_1(t, s) - x_2(t, s) + w_2(t, s)$$

$$w_1(t, s) = 0.75x_2(t, s + 1) + 0.5w_1(t, s + 1)$$

$$w_2(t, s) = -0.75x_1(t, s - 1) + 0.5w_2(t, s + 1)$$

t=time (continuous), s=space (discrete)

IMPLEMENTATION:



Decoupled:

$$\dot{x}_1(t, s) = 0.9x_1(t, s) - 0.5x_2(t, s)$$

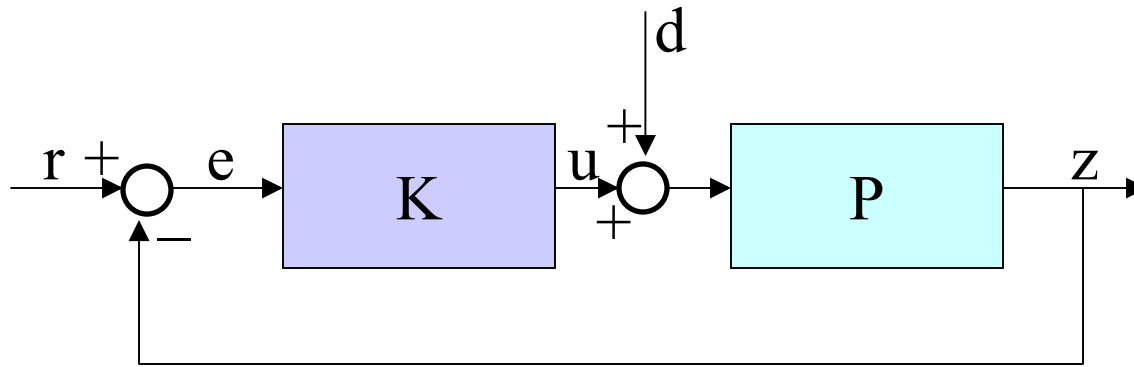
$$\dot{x}_2(t, s) = 0.5x_1(t, s) - x_2(t, s)$$

Eigs=0.76, -0.86

...turns out that if you have at least 10 of the systems connected, the overall system will be stable.

SOME BASIC TOOLS

PID Control...



$$u(t) = k_P e(t) + k_I \int_0^t e(\tau) d\tau + k_D \dot{e}(t)$$

ROUGHLY:

SPEED

STEADY STATE
PERFORMANCE

STABILITY

k_P : the larger the error, the larger the control effort.

k_I : if system is stable, $e(t)$ must go to zero for constant $d(t)$ and $u(t)$.

k_D : apply more control effort if error is getting larger.

These interpretations are only rules of thumb: in general, the effects of the gains are dictated by the plant dynamics.

LQR Control...

BACKGROUND

Many systems can be captured by sets of ordinary differential equations:

$$\dot{x}(t) = f(x(t), u(t))$$

$$y(t) = h(x(t), u(t))$$

- $x(t)$: State of the system, an n -valued vector ($x(t) = (x_1(t), \dots, x_n(t))$)
- $u(t)$: The input to the system, an m -valued vector
- $y(t)$: The output of the system, a p -valued vector

If we want to control the system about an operating point (x_E, u_E) , and we can measure all the states, we can linearize about (x_E, u_E) to obtain

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$

$$y(t) = x(t)$$

CONTROL PROBLEM:

Given system dynamics $\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t), \quad x(0) = x_0$

Find control input $u(t)$ which minimizes

$$J(u) = \int_0^{\infty} \left(x^T(t) \mathbf{Q} x(t) + u^T(t) \mathbf{R} u(t) \right) dt$$

where $\mathbf{Q} = \mathbf{Q}^T$ and $\mathbf{R} = \mathbf{R}^T$ have strictly positive eigenvalues,

- we are penalizing the state AND the control effort.
- $x(t)$ and $u(t)$ must eventually go to zero for cost to be finite.
- expect u to be a function of $x(0)$...

Scalar Case:

$$\dot{x}(t) = \mathbf{a}x(t) + \mathbf{b}u(t), \quad x(0) = x_0,$$

$$J(u) = \int_0^{\infty} (\mathbf{q}x^2(t) + \mathbf{r}u^2(t))dt$$

Look for solutions of the form $u(t)=kx(t)$:

$$x(t) = \exp((\mathbf{a} + \mathbf{b}k)t)x_0,$$

$$J = -\frac{x_0^2}{2} \left(\frac{\mathbf{q} + \mathbf{r}k^2}{\mathbf{a} + \mathbf{b}k} \right) \quad (\text{assuming } \mathbf{a} + \mathbf{b}k < 0)$$

Minimize $J(k)$:

$$-2k \frac{\mathbf{a}\mathbf{r}}{\mathbf{b}} - k^2\mathbf{r} + \mathbf{q} = 0$$

Substitute $k=-(b/r)s$:

$$2as - \frac{bs^2}{r} + q = 0$$

General Case:

Algebraic Riccati Equation

$$SA + A^T S - SBR^{-1}B^T S + Q = 0$$

NOTE:

- We restricted our search to $u(t)=Kx(t)$. No obvious reason why this should be the optimal $u(t)$. In fact, we can prove that it is!!!
- Unlike most optimal control strategies, the LQR solution is a feedback solution.