Lect. 4
2 Oct. 2001

131.215.142.214

Reading: KKK, pp. 25-39

Last lecture:
- Lyapunov stability
- Lyapunov's direct method
- " indirect method (MLS reading)

This lecture:
- control Lyapunov functions (clfs)
- Sontag's formula
- Integrator backstepping
- ...

Last time: $\underline{\dot{x}} = \underline{f}(\underline{x}, t)$

Now: $\underline{\dot{x}} = \underline{f}(\underline{x}, u)$ $\quad x \in \mathbb{R}^n,\ u \in \mathbb{R},\ \underline{f}(\underline{0}, 0) = \underline{0}$

Desire: (global, if possible) A.S. of $\underline{x} = \underline{0}$

Recall: for $\underline{\dot{x}} = \underline{f}(\underline{x}, t)$,

(G)AS of $\underline{x} = \underline{0}$ $\Longleftrightarrow$ $\exists$ a (global) (strong) Lyap. fn. for the system.

Question: Does $\exists$ a control $u = \alpha(\underline{x})$ s.t. the modified (controlled) system $\underline{\dot{x}} = \underline{f}(\underline{x}, \alpha(\underline{x}))$ is (G)AS ?

Notes:

1. Results can be generalized to $u \in \mathbb{R}^m$ and $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$.
2. We use control to modify the system dynamics

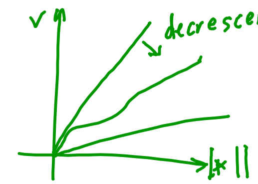
$$\dot{\underline{x}} = \underline{f}(\underline{x}) \longrightarrow \dot{\underline{x}} = \underline{f}(\underline{x}, u) = \underline{f}(\underline{x}, \alpha(\underline{x})) = \underline{\tilde{f}}(\underline{x})$$

3. For our new system $\dot{\underline{x}} = \underline{\tilde{f}}(\underline{x})$, we again seek a Lyap. fn. We call this a <u>control Lyapunov fn</u> (clf)

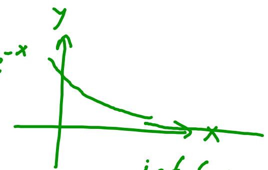
more precisely...

Defn: A (smooth, pos. def., radially unbdd) fn. $V: \mathbb{R}^n \to \mathbb{R}$ is called a control Lyap. fn. (clf) for (*)

(*) $\dot{\underline{x}} = \underline{f}(\underline{x}, u)$



$y = e^{-x}$



$\overset{\Delta}{\iff}$ $\inf\limits_{u \in \mathbb{R}} \left\{ \frac{\partial V}{\partial \underline{x}}(\underline{x}) \cdot \underline{f}(\underline{x}, u) \right\} < 0$, $\forall \underline{x} \neq \underline{0}$

$\inf\limits_{x \in \mathbb{R}} \{y\} = 0$

$\min\limits_{x \in \mathbb{R}} \{y\}$ d.n.e.

Compare to Lyap. fn.:

(i) $V(\underline{0}) = 0$
(ii) $V(\underline{x}) > 0$ in $D - \{\underline{0}\}$ } pos.def.
(iii) $\dot{V}(\underline{x}) < 0$ [in $D - \{\underline{0}\}$ $\overset{D=\mathbb{R}^n}{\iff}$ $\forall \underline{x} \neq \underline{0}$]

Let S be a real-valued set.

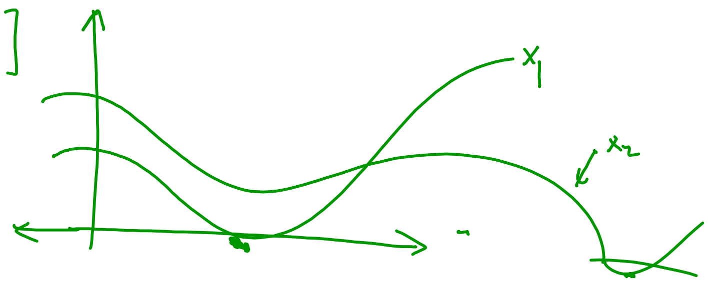
$x = \inf(S)$ $\overset{\Delta}{\iff}$ (i) $\forall x' \in S$, $x \le x'$; and
(ii) $\nexists$ $a > x$ s.t. $\forall x' \in S$, $a \le x'$

Back to defn. of clf:

For each $\underline{0} \neq \underline{x} \in \mathbb{R}^n$, does $\exists$ a $u \in \mathbb{R}$ (depending on x)

s.t. $\inf_{u \in \mathbb{R}} \{ \dots \} < 0$?

$$\left[\frac{\partial V}{\partial \underline{x}} \cdot f(\underline{x}, u)\right]_x = \left[\dot{V}\Big|_x\right]$$



Notes: 1. $\exists$ clf $\not\Rightarrow$ $\exists$ smooth associated $u(\underline{x})$.

2. However, $\exists$ clf $\overset{\text{Artstein}}{\Longleftrightarrow}$ global asymptotic stabilizability,

i.e. the existence of a control law u that renders the origin of $\dot{\underline{x}} = \underline{\tilde{f}}(\underline{x})$ GAS.

3. This does not imply that $\alpha(\underline{x})$ (in 2) is a cts. fn., but in some cases we can always find a cts. $\alpha(\underline{x})$ ($\underline{x} \neq \underline{0}$) associated with our clf.

Consider the class of systems $(\ast)$

recall (MLS reading), a strict way of saying $\dot{V} < 0$ is

$$\dot{V} \leq -\underset{\text{pos. def.}}{W(\underline{x})} < 0$$

$$(\ast) \quad \dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})\, u$$

V: if pos. def. and decrescent satisfying above, origin is GUAS.

Choose a W:

$$W(\underline{x}) = \sqrt{\left(\frac{\partial V}{\partial \underline{x}}\cdot \underline{f}\right)^2 + \left(\frac{\partial V}{\partial \underline{x}}\cdot \underline{g}\right)^4} \underset{\text{want}}{>} 0 \quad \forall\, \underline{x} \neq 0$$

to satisfy, assume/require

$$\frac{\partial V}{\partial \underline{x}}\cdot \underline{g} = 0 \Rightarrow \frac{\partial V}{\partial \underline{x}}\cdot \underline{f} < 0 \qquad (2.20)$$

$$\dot{V} \leq -W(x) \overset{(\ast)}{\Rightarrow} \frac{\partial V}{\partial \underline{x}}\cdot \underline{f} + \frac{\partial V}{\partial \underline{x}}\cdot g(x)\, \alpha(\underline{x}) \leq -W(\underline{x})$$

Given a clf $V(\underline{x})$, a GAS control law is given by Sontag's formula:

$$\alpha_S(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial \underline{x}}\cdot \underline{f} + W(\underline{x})}{\frac{\partial V}{\partial \underline{x}}\cdot \underline{g}} & \frac{\partial V}{\partial \underline{x}}\cdot \underline{g} \neq 0 \quad \checkmark \\ 0 & \frac{\partial V}{\partial \underline{x}}\cdot \underline{g} = 0 \quad \checkmark \end{cases}$$

Sontag

$$\begin{cases} \dot{x} = f(x,u) \\ \dot{u} = v \end{cases}$$

$$z = \begin{bmatrix} x \\ u \end{bmatrix} \rightarrow \dot{z} = \tilde{f}(\cdot) + \tilde{g}(\cdot)\, v$$

$$W(\underline{x}) = \sqrt{\left(\frac{\partial V}{\partial \underline{x}}\cdot \underline{f}\right)^2 + \underbrace{\left(\frac{\partial V}{\partial \underline{x}}\cdot \underline{g}\right)^4}_{=0}} = \frac{\partial V}{\partial \underline{x}}\cdot \underline{f}$$

$$\overbrace{\frac{\partial V}{\partial \underline{x}}\cdot \underline{f}}^{<0} + \frac{\partial V}{\partial \underline{x}}\cdot \underline{g}(\underline{x})\,\alpha(\underline{x}) \leq -W(\underline{x}) = -\overbrace{\frac{\partial V}{\partial \underline{x}}\cdot \underline{f}}^{\neq 0}$$

Tore notes : 1. No procedure for general systems for finding clf or GAS controller, but for certain classes there are sytematic techniques.

$$\alpha_s(\underline{x}) = - \frac{\frac{\partial V}{\partial \underline{x}} \cdot \underline{f} + \sqrt{\left(\frac{\partial V}{\partial \underline{x}} \cdot \underline{f}\right)^2 + \left(\frac{\partial V}{\partial \underline{x}} \cdot \underline{g}\right)^4}}{\frac{\partial V}{\partial \underline{x}} \cdot \underline{g}}$$

Example: LQR optimal control "$Q, R > 0$"

$$\rightarrow \min \left\{ J(u) = \int_0^\infty \{ \underline{x}^T(t)\, Q\, \underline{x}(t) + \underline{u}^T(t)\, R\, \underline{u}(t) \}\, dt \right\}$$

subject to $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

is solved by finding the (pos. def.) soln. P to

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 .$$

The optimal control is given by

$$\underline{u} = -R^{-1}B^T P \underline{x}$$

and a clf is given by $V(\underline{x}) = \underline{x}^T P \underline{x}$

Notes, cont'd.

$\dot{x}_1 = f_1(x_1, x_2)$

$\dot{x}_2 = f_2'(x_1, x_2, x_3)$

$\vdots$

$\dot{x}_n = f_n(x_1, \ldots, x_n, u)$

2. For certain classes, can find clf & design a stabilizing controller simultaneously, using a technique called <u>(integrator) backstepping</u>.

Let $\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,u$ $\quad (f(0)=0)$

where $\exists$ a (smooth pos.def. radially unbdd.) fn $V(\underline{x})$ s.t. for $u = \alpha(\underline{x})$ $(\alpha \in C^1(\underline{x}), \alpha(0)=0)$ $\quad \dot{V} \le -W(\underline{x}) < 0 \quad \forall \underline{x} \in \mathbb{R}^n$ $\quad (W(\underline{x})$ pos. def.$)$.

<u>Lemma:</u> Under these cond'ns., the <u>augmented system</u>

$$\begin{aligned} \dot{\underline{x}} &= \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\xi \\ \dot{\xi} &= u \end{aligned} \qquad (2.52)$$

has a clf $\quad V_a(\underline{x}, \xi) = V(\underline{x}) + \frac{1}{2}\left[\xi - \alpha(\underline{x})\right]^2$

and $\exists$ a fblc. controller $\quad \alpha_a(\underline{x}, \xi)$ which renders $(\underline{x}, \xi) = (\underline{0}, 0)$ of (2.52) GAS.

Ex.

$\dot{x} = \cos x - x^3 + \xi$

$\dot{\xi} = u$

(cf. $\sim$ 29)

— eq. at $(0,-1)$

For the "original" system

$\dot{x} = \underbrace{\cos x - x^3}_{f(x)} + u$, $g(x) = 1$

$[u = -\cos x] + \cancel{x^3}$

$\Rightarrow \dot{x} = -x^3$

$V(x) = \frac{1}{2}x^2 \Rightarrow \dot{V}(x) = x\dot{x} = -x^4$ ← GAS

Knowing this, pick

$\xi_{des} = -\cos x \triangleq \alpha(x)$

define $z = \xi - \xi_{des} \Rightarrow$

$= \xi - \alpha(x) = \xi + \cos x$

$\boxed{\dot{x} = -x^3 + z}$

$\dot{x} = [\cancel{\cos x} - x^3 + \cancel{-\cos x}] + z$

$[f(x) + g(x)\alpha(x)] + g(x)[\xi - \alpha(x)]$

$\dot{z} = \dot{\xi} - \dot{\xi}_{des} =$

$$\boxed{\dot{z} = u - \dot{\alpha} = u - \underbrace{(\sin x)}_{\frac{\partial \alpha}{\partial x}}\underbrace{(-x^3 + z)}_{\frac{\partial x}{\partial t}}}$$

eq. at $(x, z) = (0,0)$

Try $V_a(x) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}(\xi + \cos x)^2$, $[\xi - \xi_{des}]$

Check $\dot{V}_a = x(-x^3 + z) + z(u + x^3 \sin x - z \sin x)$

$= -x^4 + z(\underbrace{u + x + x^3 \sin x - z \sin x}_{\text{Let this } = -z})$

$\Rightarrow \dot{V}_a = -x^4 - z^2 \Leftarrow$ GAS of $(x,z) = (0,0)$