Controls Primer, Lect. 3
25 Sept 2001

So far: – intro. to control thy.
– basic concepts of fbk. control
– several apps. & examples

Outline:
- defn. of stability / classes of stab.
- Lyapunov's direct stab. thm.
- examples
- ...

Types of stab.:
- BIBO (bounded inputs ⇒ bounded outputs)
- input-to-state, input-to-output;
- stab. of systems, equilibrium pts., trajectories, ...

Consider:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t), \quad \forall t \geq t_0$$
$$\underline{x}(t_0) = \underline{x}_0, \quad \underline{x} \in \mathbb{R}^n$$

stab. of an eq. pt. of the system, i.e. $\underline{x}^*$ s.t. $\underline{f}(\underline{x}^*, t) = \underline{0}$.

w/o loss of generality, assume $\underline{x}^* = \underline{0}$. To see:

$$\dot{\underline{z}} = g(\underline{z}, t), \quad \forall t \geq t_0$$
$$\underline{z}(t_0) = \underline{z}_0, \quad z \in \mathbb{R}^n \quad \text{where } z^* = z_{eq} \text{ is eq. pt.}$$

Let $\underline{x} = \underline{z} - \underline{z}_{eq}$ $\Rightarrow$ $\dot{\underline{x}} + \underbrace{\dot{\underline{z}}_{eq}}_{=0} = g(\underline{x} + \underline{z}_{eq}, t) \triangleq f(\underline{x}, t)$

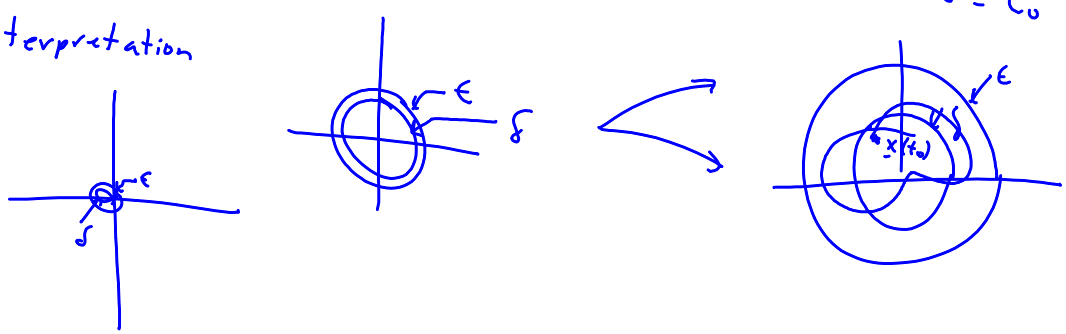
$\dot{\underline{x}} = \underline{f}(x,t) \quad \underline{x}(t_0) = \underline{x}_0$

<u>Defn 4.1</u> The eq. pt $\underline{x}^* = \underline{0}$ ~~isf~~ of $(*)$ is <u>stable</u> (in the sense of Lyapunov $\Leftrightarrow$ isL) at $t = t_0$ $\overset{\Delta}{\Longleftrightarrow}$

$$\forall \epsilon > 0 \quad \exists \ \delta(t_0, \epsilon) > 0 \quad \text{s.t.}$$

$$\| \underline{x}(t_0) \| < \delta \Rightarrow \| \underline{x}(t) \| < \epsilon \quad \forall t \geq t_0$$

Interpretation



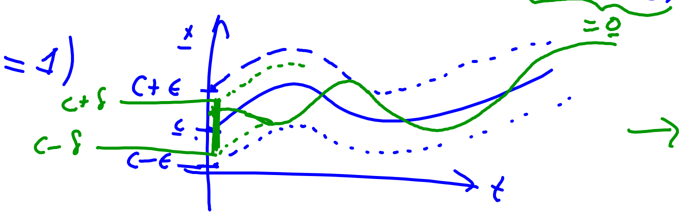
$x(t_0) = c$

<u>Def'n A</u>: A <u>solution</u> $\underline{x}(t;\underline{c})$ of $(*)$ is (Lyap.) stable at t_0 $\overset{\Delta}{\Longleftrightarrow}$ $\forall \epsilon > 0$, $\exists\ \delta = \delta(t_0, \epsilon)$ s.t. $(\underline{c}' \in \mathbb{R}^n)$:

$(**)\rightarrow$

$$\| \underline{c} - \underbrace{\underline{c}'}_{=0} \| < \delta(t_0, \epsilon) \implies \| \underline{x}(t;\underline{c}) - \underbrace{\underline{x}(t;\underline{c}')}_{=0} \| < \epsilon \quad \forall t_0$$

Interpretation $(n=1)$



Consider $\underline{c}' = \underline{0}$ $(**)$

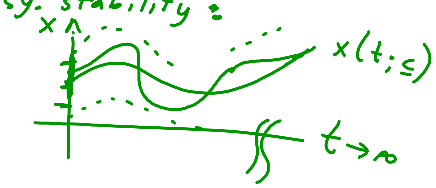
Def'n. B: The sol'n. $\underline{x}(t;\underline{c})$ of (*) is (Lyap.) asymptotically stable at $t=t_0$ $\stackrel{\Delta}{\iff}$

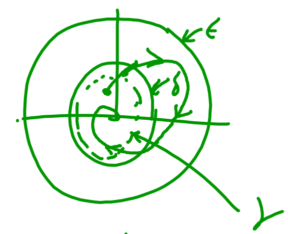
(i) it is stable (isL) (at t_0), and

(ii) $\exists\ \gamma = \gamma(t_0)$ s.t. $\|\underline{c} - \underline{c}'\| \le \gamma \Rightarrow$ $\|\underline{x}(t;\underline{c}) - \underline{x}(t;\underline{c}')\| \to 0$ as $t \to \infty$

$\dot{\underline{x}} = \underline{f}(\underline{x}, t)$

Solution asy. stability:





eq. pt. asy. stability

Def'n Stable soln's of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, $\underline{x}(t_0) = \underline{x}_0$ are called uniformly stable if δ, γ can be chosen which do not depend on t_0.

" $\dot{X} = -\underbrace{e^{-t}}\, X$ "

$\dot{x} = -t$

4.1, A	instead of	$\delta(t_0, \epsilon)$,	have $\delta(\epsilon)$
B	" "	$\gamma(t_0)$,	" γ
4.2	" "	$\delta(t_0)$,	" δ

Note: $\dot{\underline{x}} = \underline{f}(\underline{x})$ (not $\underline{f}(\underline{x}, t)$)

$\Rightarrow$ all forms of Lyap. stab. are uniform

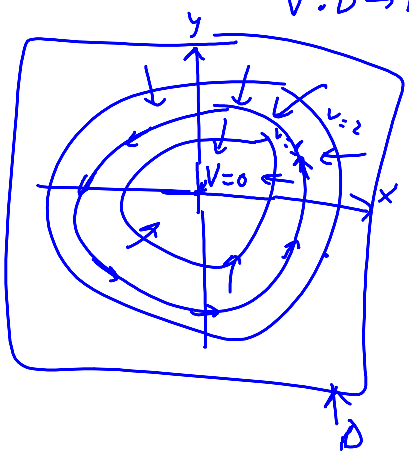
<u>Defn</u> : soln <u>globally asy. stable</u> (GAS) at $\underline{x}_0$ $\overset{\Delta}{\Longleftrightarrow}$

soln is asy. stable $\forall\ \underline{x}(t_0) \in \mathbb{R}^+$

$\underline{x}(t_0) \in \mathbb{R}^n$

Thm (Lyapunov's direct method, aut. systems)

Let $\underline{x}^* = \underline{0}$ be an eq. pt. for $\dot{\underline{x}} = \underline{f}(\underline{x})$, $\underline{0} \in D \subset \mathbb{R}^n$, and
$V: D \to \mathbb{R}$ be a ctsly. diff'able fn.



If $\exists$ a $V(\underline{x})$ satisfying

(i) $V(\underline{0}) = 0$, and

(ii) $V(\underline{x}) > 0$ in $D - \{\underline{0}\}$, and

(iii) $\dot{V}(\underline{x}) \leq 0$ in D,

then $\underline{x} = \underline{0}$ is stable (isL).

$\dot{(\ )} = \frac{d}{dt}(\ )$

$\dot{V}(\underline{x}(t)) = \frac{\partial V}{\partial \underline{x}} \cdot$,

If $\exists$ a $V(\underline{x})$ satisfying (i), (ii), (iii), and

(iv) $\dot{V}(\underline{x}) < 0$ in $D - \{\underline{0}\}$,

then $\underline{x} = \underline{0}$ is asymptotically stable.

Example:

$\dot{x} = Ax = f(x)$

$V(x) = x^T P x$

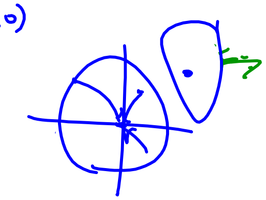
$\dot{V} = x^T(A^T P + PA)x$

If $\lambda(A)$ stable then $\exists\, P>0$ s.t. $A^T P + PA < 0$

If $\dot{x} = f(x)$ NL w/ $\left.\frac{\partial f}{\partial x}\right|_{x=0} = A$ try $V = x^T P x$

$\dot{\underline{x}} = \underline{f}(\underline{x}) \iff$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x - 2y^2 \\ xy - y^3 \end{bmatrix} \xrightarrow[\text{about } \underline{x} = (0,0)]{\text{linearized}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$x = e^{-t} x_0$

$y = c$

Try: $V(\underline{x}) = \frac{1}{2}(x^2 + a y^2)$

(i) ✓

(ii) ✓ if $a > 0$

(iii) $\dot{V}(\underline{x}) = \frac{d}{dt}(V(\underline{x})) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$

$= x(-x - 2y^2) + ay(xy - y^3)$

$= -x^2 + (a-2)xy^2 - ay^4$

Let $a = 2 \Rightarrow \dot{V} = -(x^2 + 2y^4)$ ✓

(iii) ✓

(iv) ✓

D.O.A. $\triangleq$ $\{\underline{c} \mid \lim_{t\to\infty} \underline{x}(t;\underline{c}) = \underline{0}\}$