Controls Primer, Lect. 3
25 Sept 2001

So far: – intro. to control thy.
– basic concepts of fbk. control
– several apps. & examples

Outline:
- defn. of stability / classes of stab.
- Lyapunov's direct stab. thm.
- examples
- ...

Types of stab:
- BIBO (bounded inputs ⇒ bounded outputs)
- input-to-state, input-to-output;
- stab. of systems, equilibrium pts., trajectories, ...

Consider:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t), \quad \forall t \geq t_0$$

$$\underline{x}(t_0) = \underline{x}_0, \quad \underline{x} \in \mathbb{R}^n$$

stab. of an eq. pt. of the system, i.e. $\underline{x}^*$ s.t. $\underline{f}(\underline{x}^*, t) = \underline{0}$.

w/o loss of generality, assume $\underline{x}^* = \underline{0}$. To see:

$$\dot{\underline{z}} = g(\underline{z}, t), \quad \forall t \geq t_0$$

$$\underline{z}(t_0) = \underline{z}_0, \quad z \in \mathbb{R}^n \quad \text{where } z^* = z_{eq} \text{ is eq. pt.}$$

Let $\underline{x} = \underline{z} - \underline{z}_{eq}$ $\Rightarrow$ $\dot{\underline{x}} + \dot{\underline{z}}_{eq} = g(\underline{x} + \underline{z}_{eq}, t) \triangleq f(\underline{x}, t)$

$=0$

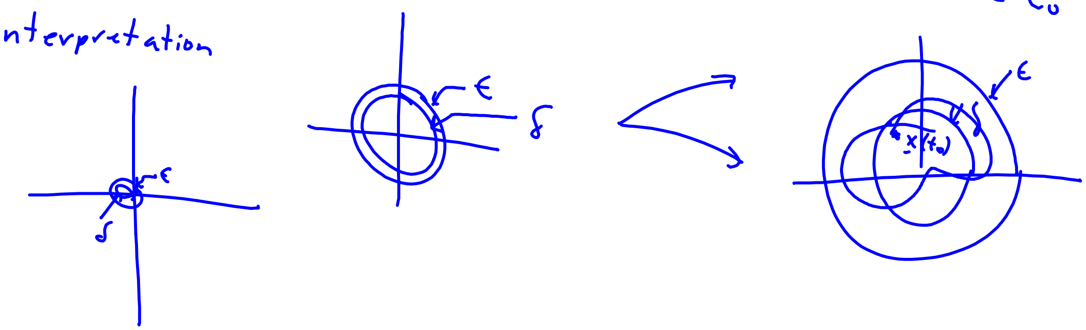
$\dot{\underline{x}} = \underline{f}(x,t) \qquad \underline{x}(t_0) = \underline{x}_0$

Defn 4.1 The eq. pt $\underline{x}^* = \underline{0}$ of (*) is **stable** (in the sense of Lyapunov $\Leftrightarrow$ isL) at $t = t_0$ $\overset{\Delta}{\Longleftrightarrow}$

$$\forall \epsilon > 0 \quad \exists \ \delta(t_0, \epsilon) > 0 \quad \text{s.t.}$$

$$\| \underline{x}(t_0) \| < \delta \implies \| \underline{x}(t) \| < \epsilon \quad \forall t \geq t_0$$

Interpretation



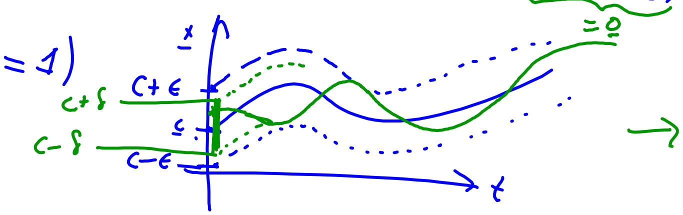
$\underline{x}(t_0)=\underline{c}$

Def'n A: A solution $\underline{x}(t;\underline{c})$ of $(*)$ is (Lyap.) stable at t_0 $\overset{\Delta}{\Longleftrightarrow}$ $\forall \epsilon>0$, $\exists\ \delta=\delta(t_0,\epsilon)$ s.t. $(\underline{c}'\in\mathbb{R}^n)$:

$(**)\rightarrow$

$$\|\underline{c}-\underbrace{\underline{c}'}_{=\underline{0}}\|<\delta(t_0,\epsilon) \Longrightarrow \|\underline{x}(t;\underline{c})-\underbrace{\underline{x}(t;\underline{c}')}_{=\underline{0}}\|<\epsilon \quad \forall t_0$$

Interpretation $(n=1)$



Consider $\underline{c}'=\underline{0}$ $(**)$

Defn. B: The soln. $\underline{x}(t;\underline{c})$ of (x) is ((yap.)) asymptotically stable at $t=t_0$ $\overset{\Delta}{\Longleftrightarrow}$

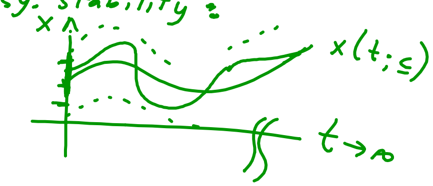
(i) it is stable (isL) (at t_0), and

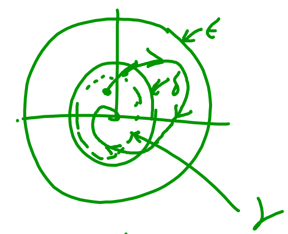
(ii) $\exists\ \gamma = \gamma(t_0)$ s.t. $\|\underline{c}-\underline{c}'\| \le \gamma \Rightarrow$

$\|\underline{x}(t;\underline{c}) - \underline{x}(t;\underline{c}')\| \to 0$ as $t \to \infty$

$\dot{\underline{x}} = \underline{f}(\underline{x}, t)$

Solution asy. stability:





eq. pt. asy. stability

Def'n Stable soln's of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, $\underline{x}(t_0) = \underline{x}_0$ are called <u>uniformly stable</u> if δ, γ can be chosen which do not depend on t_0.

" $\dot{X} = -\underbrace{e^{-t}} X$ "

$\dot{x} = -t$

4.1, A	instead of $\delta(t_0, \epsilon)$,	have $\delta(\epsilon)$
B	" " $\gamma(t_0)$,	" γ
4.2	" " $\delta(t_0)$,	" δ

Note: $\dot{\underline{x}} = \underline{f}(\underline{x})$ (not $\underline{f}(\underline{x}, t)$)
$\Rightarrow$ all forms of Lyap. stab. are uniform

at x_0

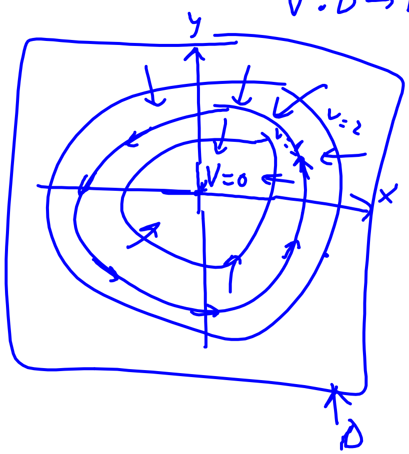
Defn: soln globally asy. stable (GAS) $\overset{\Delta}{\Longleftrightarrow}$

soln is asy. stable $\forall\ \gamma(t_0) \subset \mathbb{R}^+$

$\underline{x}(t_0) \in \mathbb{R}^n$

Thm (Lyapunov's direct method, aut. systems)

Let $\underline{x}^* = \underline{0}$ be an eq. pt. for $\dot{\underline{x}} = \underline{f}(\underline{x})$, $\underline{0} \in D \subset \mathbb{R}^n$, and $V: D \to \mathbb{R}$ be a ctsly. diff'able fn.



If $\exists$ a $V(\underline{x})$ satisfying

(i) $V(\underline{0}) = 0$, and

(ii) $V(\underline{x}) > 0$ in $D - \{\underline{0}\}$, and

(iii) $\dot{V}(\underline{x}) \leq 0$ in D,

then $\underline{x} = \underline{0}$ is stable (isL).

$\dot{(\ )} = \frac{d}{dt}(\ )$

$\dot{V}(\underline{x}(t)) = \frac{\partial V}{\partial \underline{x}} \cdot$,

If $\exists$ a $V(\underline{x})$ satisfying (i), (ii), (iii), and

(iv) $\dot{V}(\underline{x}) < 0$ in $D - \{\underline{0}\}$,

then $\underline{x} = \underline{0}$ is asymptotically stable.

Example:

$\dot{\underline{x}} = \underline{f}(\underline{x})$

$\dot{x} = Ax = f(x)$

$V(x) = x^T P x$

$\dot{V} = x^T(A^T P + PA)x$

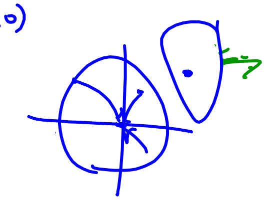
If $\lambda(A)$ stable then $\exists\, P > 0$ s.t. $A^T P + PA < 0$

If $\dot{x} = f(x)$ NL w/ $\left.\frac{\partial f}{\partial x}\right|_{x=0} = A$ try $V = x^T P x$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x - 2y^2 \\ xy - y^3 \end{bmatrix} \xrightarrow[\text{about } \underline{x} = (0,0)]{\text{linearized}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = e^{-t} x_0$

$y = c$



Try: $V(\underline{x}) = \frac{1}{2}(x^2 + ay^2)$

(i) ✓

(ii) ✓ if $a > 0$

(iii) $\dot{V}(\underline{x}) = \frac{d}{dt}(V(\underline{x})) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$

$= x(-x - 2y^2) + ay(xy - y^3)$

$= -x^2 + (a-2)xy^2 - ay^4$

Let $a = 2 \Rightarrow \dot{V} = -(x^2 + 2y^4)$ ✓

(iii) ✓

(iv) ✓

D.O.A. $\triangleq \{ \underline{c} \mid \lim_{t \to \infty} \underline{x}(t; \underline{c}) = \underline{0} \}$